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Dynamics of Neural Networks

By

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ABSTRACT. We propose to study the dynamics of McCulloch-Pitts' neural network and general Boolean networks at the most fundamental level. We propose to study Hopfield's theorem for chaotic iteration and its application to pattern recognition. We propose to furnish a mathematical model of Hebb's postulate of learning. We propose to study the Jacobian problem for Boolean networks.

1. Introductory remarks

In 1943, the neurophysiologist W. McCulloch and a Mathematician W. Pitts[6] claimed that the brain could be modeled as a network of logical operations such as *and, or, not,* and so forth. It had been a revolutionary idea at the time, and had proved to be immensely influential. McCulloch-Pitts model was the first example of what now call a *neural network*. It was the first attempt to understand mental activity as a form of information processing—an insight that provided the inspiration for *artificial intelligence* and *cognitive psychology*. McCulloch-Pitts model was the first indication that a network of very simple logic gates could perform exceedingly complex computation—an insight that was soon incorporated into the general theory of computing machines. McCulloch-Pitts' paper influenced von Neumann to use idealized switch-delay elements derived from the McCulloch-Pitts neuron in the construction of the EDVAC(Electronic Discrete Variable Automatic Computer) (see Aspray and Burks[1]).

In this note, we propose to study the dynamics of McCulloch-Pitts' neural network and general Boolean networks at the most fundamental level.

2. McCulloch-Pitts neural network and its dynamics

In a nervous system, each neuron exhibits an impulse of one electric state, called *action potential*. The state of each neuron can be distinguished by the existence and nonexistence of an action potential. Suppose

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that the nervous system consists of n neurons, we can identify each neuron with an element of $\{1, 2, \cdots, n\}$. The state of the nervous system at time t is expressed by a point $x(t) = (x_1(t), \cdots, x_n(t))$ in $\{0,1\}^n$, the set of all 01-strings of length n. The neural network of McCulloch and Pitts is formulated as follows.

A Boolean function $f : \{0,1\}^n \to \{0,1\}$ is called a *threshold function* if there exist $a \equiv (a_1, \cdots, a_n) \in \mathbb{R}^n$ and a threshold value $\alpha \in \mathbb{R}$ such that $f(x) = Hev(\langle a, x\rangle - \alpha)$, where $Hev(u)$ is the Heaviside function. Thus f is a threshold function if the sets $f^{-1}(1)$, $f^{-1}(0)$ can be separated by a hyperplane in $\mathbb{R}^n$. A Boolean function $F : \{0,1\}^n \to \{0,1\}^n$ is a *threshold function* if each f_i is threshold, i.e. there exist $A \in M_n(\mathbb{R})$ and $b \in \mathbb{R}^n$ such that

$$F(x) = Hev(Ax - b).$$

The finite state space $\{0,1\}^n$ is given a metric structure by the *Hamming metric* $\rho_H(\cdot,\cdot)$, i.e.

$$\rho_H(x,y) \equiv \#\{i;\ x_i \neq y_i\}.$$

Let (i,j) denote a *synapse*, where $i, j \in \{1, \ldots, n\}$, neuron i being the *postsynaptic* neuron and neuron j being the *presynaptic* neuron. Each entry w_{ij} expresses the *efficiency* of the synapse (i,j) and b_i expresses the *threshold value* for the action potential of neuron i. The matrix $A = (a_{ij})$ is called the *synaptic matrix*. Thus the McCulloch-Pitts neural network ($\equiv$ threshold automata) is described by

$$F(x(t)) = Hev(Ax(t) - b) \quad (t = 0, 1, \ldots).$$

There are two important iteration modes to *implement* the updating of the states on the network: *parallel* and *sequential*.

For parallel iteration mode (*synchronous iteration mode*)

$$x_i(t+1) = Hev[\sum_{j=1}^{n} a_{ij}x_j(t) - b_i\ ] \quad (i = 1, \ldots, n) \quad (t = 0, 1, \ldots),$$

Gole[2] proved the following:

Theorem (Gole). *For parallel iteration mode, the attractors of a symmetrical network of threshold automata are cycles with length* ≤ 2.

For sequential iteration mode (*asynchronous iteration mode*):

$$x_i(t+1) = Hev[\sum_{j=1}^{i-1} a_{ij}x_j(t+1) + \sum_{j=i}^{n} a_{ij}x_j(t) - b_i\ ] \quad (i = 1, 2, \ldots, n) \quad (t = 0, 1, \ldots),$$

Hopfield[4] proved the following:

Theorem (Hopfield). *For Gauss-Seidel iteration, the attractors of a symmetrical network of threshold automata are cycles with nonnegative diagonal entries are fixed points.*

Hopfield's theorem is seminal not only for his result about the asynchronous iteration of McCulloch-Pitts' neural network but open a window onto the essence of *Pattern Recognition* and lead to a solution of *Traveling Salesman Problem*[5].

3. Hopfield's theorem for Chaotic iteration

We now study Hopfield's theorem for chaotic iteration, an asynchronous iteration scheme in *Discrete iteration* may go back to 1995 work of Robert[7]. To formulate the problem, let $F : \{0,1\}^n \to \{0,1\}^n$, $F = (f_1, \cdots, f_n)$. For $x \in \{0,1\}^n$, $i = 1, \ldots, n$, let

$$F_i(x) \equiv \begin{pmatrix} x_1 \\ \vdots \\ f_i(x) \\ \vdots \\ x_n \end{pmatrix}.$$

Let $F(x) \equiv Hev(Ax - \theta)$ be a threshold map from $\{0,1\}^n$ into itself, and $N \equiv \{1, \ldots, n\}$. For $m = 1, 2, \ldots$, let

$$N^m \equiv \{i_1, \ldots, i_m \; ; \quad i_j \in N, \; j = 1, \ldots, m\}$$

be the set of array of m integers in N. Let

$$S \equiv \{h_0; h_1; h_2; \ldots\},$$

where $h_j \equiv j_1, j_2, \ldots, j_{l(j)} \in N^{l(j)} \;\; (j = 0, 1, \ldots)$. We call S a *strategy-set*. The *chaotic iteration* for F with strategy-set $S = \{h_0; h_1; h_2; \ldots\}$ is defined by

$$H_t(x(t)) = x(t+1) \;\; (t = 0, 1, \ldots), \tag{$*$}$$

where

$$H_t = F_{t_{l(t)}} \circ F_{t_{l(t)-1}} \circ \ldots \circ F_{t_2} \circ F_{t_1}.$$

If $H_t = F_n \circ F_{n-1} \circ \ldots \circ F_1 \;\; (t = 0, 1, \ldots)$, then $(*)$ becomes the Gauss-Seidel iteration. If π is a permutation on N and $H_t = F_{\pi(n)} \circ F_{\pi(n-1)} \circ \ldots \circ F_{\pi(1)} \;\; (t = 0, 1, \ldots)$, then $(*)$ is called Gauss-Seidel type iteration. The chaotic iteration with strategy-set can produce behaviors that are essentially unpredictable. A point ξ is said to be a *point attractor* for a chaotic iteration with a strategy-set if there exists an initial point $x(0) \in \{0,1\}^n$ and $T \geq 0$ such that $H_t(x(t)) = x(t+1) = \xi$ for all $t \geq T$. For chaotic iteration, a point attractor could not be a fixed point and a cycle might be a non-simple cycle, as the following example shows.

Example. Define a threshold map and a strategy-set as follows:

$$F(x) \equiv Hev\left(\begin{pmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} x - \begin{pmatrix} -0.5 \\ -0.5 \\ -0.5 \end{pmatrix}\right),$$

$S_1 = \{1,2;3,3;1,1,2,3,3;3,3,1,1;2;2,2;2,2,2;2,2,2,2;\ldots\}$. Then $F_2 \circ F_1(010) = 110, F_3 \circ F_3(110) = 110, F_3 \circ F_3 \circ F_2 \circ F_1 \circ F_1(110) = 110, F_1 \circ F_1 \circ F_3 \circ F_3(110) = 110, F_2(110) = 110, \ldots$. Thus 110 is a point attractor for this chaotic iteration with strategy-set S_1. However, $F(110) \neq 110$. On the other hand, with another strategy-set $S_2 = \{1;3;3;1;1;3;3;1;1;3;3;1;1;3;3;1;1;3;3;1;\ldots\}$, we have $F_1(010) = 110, F_3(110) = 111, F_3(111) = 110, F_1(110) = 010, F_1(010) = 110, F_3(110) = 111, F_3(111) = 110, F_1(110) = 010$. The question now under consideration is how to get order out of disorder, complexity from simplicity. Now, let us introduce the following notion. The strategy-set is said to be *regular* if there exists a subset $\{k_0, k_1, \ldots, k_{2^n}\} \subset \mathbb{N} \cup \{0\}$ with

$$0 = k_0 < k_1 < \ldots < k_{2^n}$$

such that the resulting set $\{h_{k_j}, h_{k_j+1}, \ldots, h_{k_{j+1}-1}\}$ is equal to $\{1,2,\ldots,n\}$ $(j = 0,1,\ldots,2^n-1)$. Thus $S_1 = \{1,2,3;1,2,3;1,2,3;1,2,3;1,2,3;\ldots\}$ and $S_2 = \{1,2,3;3;2,1;1,1;3;2;2,1,3;1,2,2;3;2;2,1;3;3,2,1;1,2;3;2;2;2;2;2;2;\ldots\}$ are regular strategy-sets, and $S_3 = \{1;2;3;1,2,3;1,2,3;1;2;1;2;1,2;1,2;1;1;1;1;1;\ldots\}$ is an irregular strategy-set.

Both parallel and sequential iteration are powerful for both brains and artificial networks. The difference between Gauss-Seidel iteration and chaotic iteration is like the difference between the regular repetition of a pattern and the rich, coherent variation.

With the notations and notions stated above, we have the following result of Shih and Tsai[10].

Theorem 1 . *If $A = (a_{ij})$ is symmetric with $a_{ii} \geq 0$ $(i = 1,2,\ldots,n)$ and $S = \{h_0; h_1; \ldots\}$ is a regular strategy-set with $h_i \in N^{l(i)}$ $(i = 0,1,\ldots)$, then the sequence $\{x(t)\}$ generated by $(*)$ converges to a fixed point of F for any initial point $x(0) \in \{0,1\}^n$.*

Problem : *Determine the optimality of the length of regular strategy-set.*

By Hopfield's theorem, we have the following existence of fixed points.

Theorem 2 . *If $A = (a_{ij})$ is symmetric with $a_{ii} \geq 0$ $(i = 1, 2, \ldots, n)$, then $F(x) \equiv Hev(Ax - \theta)$ has a fixed point.*

Problem: *Does there exist a combinatorial lemma which serves as a basis for the direct proof of Theorem 2 ?*

4. Inverse Dynamics Problem

Let us begin with the following general *inverse problem*[11]:

"*Given an arbitrary set of configurations in $\{0,1\}^n$, is it possible to construct a network of automata for which this set of configurations is the set of attractors ?*"

Applying Theorem 1, we have the following partial solution of above problem.

Theorem 3 . *Let $M \equiv \{\xi^{(1)}, \ldots, \xi^{(m)}\}$ be a set of configurations in $\{0,1\}^n$ such that M is orthogonal (i.e. $\langle \xi^{(i)}, \xi^{(j)} \rangle = 0$ $(i \neq j)$, $\xi^{(i)} \neq 0$ $\forall i$). Let $A \equiv \sum_{i=1}^{m} \xi^{(i)} {\xi^{(i)}}^T$, $b \equiv (\frac{1}{2}, \ldots, \frac{1}{2})^T$ and $F(x) \equiv Hev(Ax - b)$. Then $\xi^{(i)}$ $(i = 1, \ldots, m)$ are fixed points of F and $\xi^{(i)}$ $(i = 1, \ldots, m)$ are point attractors under chaotic iteration with regular strategy-set.*

Theorem 3 has two defects. First, we do not know the domains of attractions. Second, orthogonality of M is required. To overcome these defects, let us propose another solution.

Our new solution is based on Hebb's postulate of learning[3]. Quoting from Hebb's book ([3], p.62):
When an axon of cell A is near enough to excite a cell B and repeatedly or persistently takes part in firing it, some growth process or metabolic changes take place in one or both cells such that A's efficiency as one of the cells firing B, is increased.

We now propose to furnish a mathematical model which may be viewed as a mathematical model of Hebb's postulate of learning by the following construction[11].

Let M $\equiv \{\xi^{(1)}, \ldots, \xi^{(m)}\}$ be an orthogonal set in $\{0,1\}^n$, A $\equiv \sum_{i=1}^{m} \xi^{(i)} {\xi^{(i)}}^T$, b$\equiv (\frac{1}{2}, \ldots, \frac{1}{2})^T$, and $F(x) \equiv Hev(Ax - b)$. Define

$$(0,1)\text{-span}(M) \equiv \{\sum_{i=1}^{m} \alpha_i \xi^{(i)} \; ; \quad \alpha_i \in \{0,1\}\},$$

$$U_i \equiv \{x \in \{0,1\}^n \; ; \; 0 < x \leq \xi^{(i)}\} (i = 1, \ldots, m),$$

$$x \leq y \iff x_i \leq y_i \quad (i = 1, \ldots, n),$$

$x < y \Longleftrightarrow x \leq y$ *and* $x \neq y$,

$\mathrm{U}_0 \equiv$ (0,1)-span$\{e_i\ ;\ \langle e_i, \xi^{(j)}\rangle = 0\ \ (j = 1, \ldots, m)$,i=1,2,...,n$\}$; (0,1)-span$\{\emptyset\} = \{0\}$.

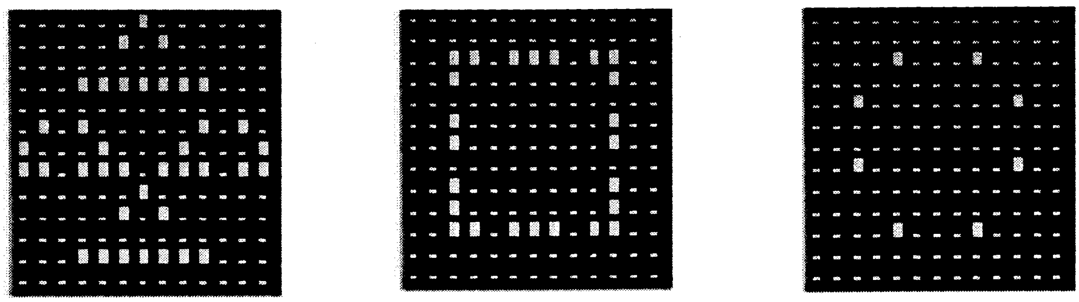
We list some elementary results.

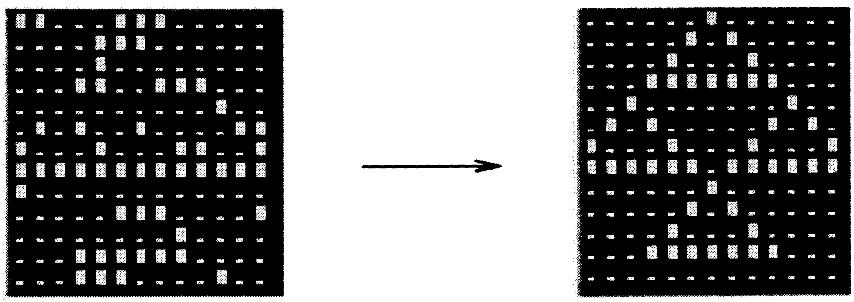
Fact.

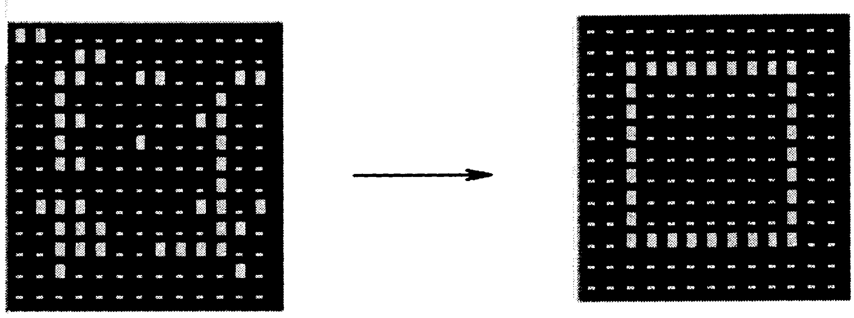
1. $\sharp$(0,1)-span(M)=2^m,
2. $\{\{\sum_{i=0}^{m} \alpha_i \mathrm{U}_i\}\ ;\ \alpha_0 = 1, \alpha_i \in \{0,1\}, i = 1, \ldots, m\}$ is a partition of $\{0,1\}^n$,
3. For all $x \in$ (0,1)-span(M), $F(x) = x$,
4. F has a unique fixed point in $\sum_{i=0}^{m} \alpha_i \mathrm{U}_i$ $(\alpha_0 = 1, \alpha_i \in \{0,1\}, i = 1, \ldots, m)$.

Theorem 4 . *For each* $x \in U_0 + \sum_{i=1}^{m} \alpha_i U_i \quad (\alpha_i \in \{0,1\}, i = 1, \ldots, m)$, $F(x) = \sum_{i=1}^{m} \alpha_i \xi^{(i)}$.

As an illustration of Theorem 4, we have the following computer experiments.







5. Dynamics of Boolean networks

Let us begin with some notions and notations. Let $F : \{0,1\}^n \to \{0,1\}^n$. For $x \in \{0,1\}^n$, denote by $F'(x)$ and $\rho(F'(x))$ the Boolean derivative of F evaluated at x (i.e. $F'(x) = (f_{ij}(x))$, where $f_{ij}(x) = 1$ if $f_i(x) \neq f_i(\widetilde{x}^j)$, $f_{ij}(x) = 0$ otherwise, here $\widetilde{x}^j = x_1 \dots \overline{x_j} \dots x_n$) and the spectral radius of $F'(x)$, respectively.

Motivated by the well-known Markus-Yamabe problem about global stability of dynamical system, Shih and Ho[8] proved that

Theorem 5 . *Let $F : \{0,1\}^n \to \{0,1\}^n$. If $\rho(F'(x)) = 0$ for all $x \in \{0,1\}^n$, and $F'(x)$ has at most one 1 in each column for all $x \in \{0,1\}^n$, then F has a unique fixed point ξ and there exists $p \leq 2^n$ such that $F^p(x) = \xi$ for any $x \in \{0,1\}^n$.*

Motivated by the long-standing Jacobian conjecture, Shih and Ho[8] made the following conjecture.

Combinatorial Fixed Point Conjecture. *If the cube mapping $F : \{0,1\}^n \to \{0,1\}^n$ is such that $\rho(F'(x)) = 0$ for all $x \in \{0,1\}^n$, then F has a unique fixed point.*

Recently Dong and Shih[9] proved this conjecture. Our approach to this conjecture is to make a coherent behavior in the whole cube by way of understanding collective behavior in the subcubes. Now let us introduce the following notation. Let $x \in \{0,1\}^n$. For each $k = 1, 2, \dots, n-1$ and for each choice of $k+1$ distinct integers $i_1, \dots, i_{k+1}$ (which are arranged in any order) from $\{1, \dots, n\}$, we define

$$x\,[\,\{i_1, \dots, i_k\} \mid i_{k+1}\,] \equiv \{\, y \in \{0,1\}^n;\ y_{i_{k+1}} = x_{i_{k+1}},\ y_j = x_j \text{ for all } j \neq i_1, \dots, i_k \,\}.$$

The following lemma will play a prominent role in the proof of the conjecture. And the kernel of the lemma reveals an unexpected regularity hidden in the spectral condition.

Lemma. *Let $F : \{0,1\}^n \to \{0,1\}^n$. If $\rho(F'(x)) = 0$ for all $x \in \{0,1\}^n$, then for each $x \in \{0,1\}^n$ and for each $k = 1, \dots, n-1$ and for each choice of $k+1$ distinct integers $i_1, \dots, i_{k+1}$ (which are arranged in any order) from $\{1, \dots, n\}$, there exists a unique point $\alpha \in x\,[\,\{i_1, \dots, i_k\} \,|\, i_{k+1}\,]$ such that $f_j(\alpha) = \alpha_j$ for all $j = i_1, \dots, i_k$.*

For a long time we intend to prove that such p in Theorem 5 should be lesser or equal to n (i.e. the optimal transient length of ξ is n) by establishing the following conjecture : *Conditions* "$\rho(F'(x)) = 0$ for all $x \in \{0,1\}^n$", *and* "$F'(x)$ has at most one 1 in each column for all $x \in \{0,1\}^n$" *together imply that* $\sup_{x \in \{0,1\}^n} \{F'(x)\}$ *contains a zero row.* Note that the condition "$F'(x)$ has at most one 1 in each column for all $x \in \{0,1\}^n$" is equivalent to the condition "F is a nonexpansive map with respect to Hamming metric, i.e.

$\rho_H(F(x), F(y)) \leq \rho_H(x, y)$ for all $x, y \in \{0,1\}^n$." Recently Dong and Shih[9] proved that if $n \leq 4$ the answer to the conjecture is affirmative. The proof is based on the above lemma. The case $n \geq 5$ of the conjecture remains open.

Let us mention in conclusion that the spectral condition "$\rho(F'(x)) = 0$ *for all* $x \in \{0,1\}^n$" implies that F leaves a unique point invariant. And on toward microscopic perspectives: the spectral condition also implies that for each $k = 1, \ldots, n-1$ and for each k-subcube the boolean function F leaves a unique point in the k-subcube having k components invariant in a very regular pattern indeed. This phenomenon is of exceptional interesting feature, perhaps because we can easily find a boolean function $F : \{0,1\}^n \to \{0,1\}^n$ leaves a unique point invariant but there exists a k-subcube for which F does not leave a point in the k-subcube having k components invariant.

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