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A PROBLEM CONCERNING MAPPINGS WITH CONSTANT DISPLACEMENT.

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ABSTRACT. We present here an open problem concerning Lipschitzian self mappings of closed convex subsets of Banach spaces.

Let $X$ be a Banach space with norm $\| \cdot \|$ and let $C$ be a nonempty, convex closed and bounded subset of $X$. A lot of attention has been focused recently on the behavior of Lipschitzian self mappings of such sets $C$. Let us recall that the mapping $T : C \to C$ is Lipschitzian (satisfies Lipschitz condition) if there exists $k \geq 0$ such that

\[(1) \quad \|Tx - Ty\| \leq k \|x - y\|,\]

for all $x, y \in C$. The smallest $k$ for which (1) holds is said to be the Lipschitz constant for $T$ and is denoted by $k(T)$. If (1) holds we also say that $T$ is $k$-Lipschitzian or that $T$ is of class $L(k), T \in L(k)$.

If $C$ is compact then due to the Schauder Fixed Point Theorem any continuous (thus also any Lipschitzian) mapping $T : C \to C$ has a point $x$ satisfying $x = Tx$, a fixed point of $T$. If $C$ is not compact, it is no longer true. The strongest known result due to P. K. Lin and Y. Sternfeld [6] states:

- **If $C$ is not compact then for any $k > 1$ there exists a mapping $T : C \to C$ of class $L(k)$ such that,**

\[(2) \quad d(T) = \inf \{\|x - Tx\| : x \in C\} > 0.\]

The number $d(T)$ defined by (2) is called the minimal displacement of $T$ and mappings $T$ which satisfy (2) are called mappings with positive displacement.

Once we have a Lipschitzian mapping $T$ with positive displacement $d = d(T) > 0$ we can define a modified mapping $\tilde{T} : C \to C$ by

$$\tilde{T}x = x + d\frac{Tx - x}{\|Tx - x\|}.$$  

It is easy to observe that $\tilde{T}$ is also Lipschitzian but the Lipschitz constant $k(\tilde{T})$ is not necessarily the same as $k(T)$.

This modified mapping has constant positive displacement equal $d$, which means that for all $x \in C$ we have

$$\|x - \tilde{T}x\| = d = d(T) > 0.$$
Now we can observe that for any \( c \in (0, 1] \) the convex combination of the mapping \( \tilde{T} \) with the identity mapping \( I \),
\[
\tilde{T}_c = (1 - c) I + c \tilde{T},
\]
is also of positive displacement equal \( cd \). Moreover, we have
\[
k \left( \tilde{T}_c \right) = k \left( (1 - c) I + c \tilde{T} \right) \leq 1 - c + ck \left( \tilde{T} \right)
\]
and consequently, \( \lim_{c \to 1} k \left( \tilde{T}_c \right) = 1 \).

The above allows us to formulate an equivalent modification of the Lin Sternfeld result.

- **If** \( C \) **is not compact then for any** \( k > 1 \) **there exists a mapping** \( T : C \to C \) **of class** \( L(k) \) **with constant positive displacement.**

From now on we shall discuss only mappings with constant positive displacement. Suppose \( T : C \to C \) is such a mapping with \( d(T) = d > 0 \). The iterated mapping \( T^2 = T \circ T : C \to C \) is not necessarily of constant displacement. For any \( x \in C \) we have an obvious inequality
\[
0 \leq \|T^2 x - x\| \leq \|T^2 x - Tx\| + \|Tx - x\| = 2d.
\]
If \( \|T^2 x - x\| = 2d \) then the line consisting of two linear segments \([x, Tx]\) and \([Tx, T^2 x]\) is isometric to the segment \([x, T^2 x]\) and consequently to the interval \([0, 2d]\). If \( \|T^2 x - x\| < 2d \) it means that the vector \( T^2 x - Tx \) is in some metric sense “rotated” with respect to the vector \( Tx - x \). For this reason it is natural to introduce two coefficients
\[
a_-(T) = \inf \left\{ \frac{1}{d} \|T^2 x - x\| : x \in C \right\}
\]
and
\[
a_+(T) = \sup \left\{ \frac{1}{d} \|T^2 x - x\| : x \in C \right\}.
\]
Intuitively, they represent the **minimal metric rotation** and **global metric rotation**. It is understood in the sense that if \( a_-(T) < 2 \) then for any \( \varepsilon > 0 \) there are some \( x \in C \) for which the vector \( T^2 x - Tx \) of length \( d \) is rotated with respect to the vector \( Tx - x \) of the same length in such a way that
\[
\|(T^2 x - Tx) + (Tx - x)\| < (a_-(T) + \varepsilon) d.
\]
If \( a_+(T) < 2 \), then (3) holds for all \( x \in C \). Especially if \( a_+(T) = 0 \) then \( T \) is an **involution**, \( T^2 = I \) on \( C \).

There are several open problems and questions concerning mutual relations between constants \( k(T), a_-(T) \) and \( a_+(T) \). Here is the first observation.
Let \( T : C \rightarrow C \) be a mapping of class \( \mathcal{L}(k) \) with constant displacement \( d(T) = d \). Take any point \( x \in C \) and put \( u = \frac{1}{2} (Tx + T^2x), v = \frac{1}{2} (x + Tx) \). Then we have

\[
d = \|u - Tu\| = \left\| \frac{1}{2} (Tx + T^2x) - Tu \right\| \leq \\
\leq \frac{1}{2} \|Tx - Tu\| + \frac{1}{2} \|T^2x - Tu\| \leq \\
\leq \frac{k}{2} \|x - u\| + \frac{k}{2} \|Tx - u\| \leq \\
\leq \frac{k}{2} \|x - v\| + \frac{k}{2} \|v - u\| + \frac{k}{2} \|Tx - u\| = \\
= \frac{k}{4} d + \frac{k}{4} \|x - T^2x\| + \frac{k}{4} d = \\
= \frac{k}{2} d + \frac{k}{4} \|x - T^2x\|.
\]

The conclusion of it can be written in the form

\[
\frac{\|x - T^2x\|}{d} \geq 2 \left( \frac{2}{k} - 1 \right)
\]

and this shows that the rotation constants and Lipschitz constant of \( T \) must satisfy

(4) \[ a_+ (T) \geq a_- (T) \geq 2 \left( \frac{2}{k(T)} - 1 \right). \]

In other words we have

- If \( T : C \rightarrow C \) is a lipschitzian mapping with constant positive displacement then

(5) \[ k(T) \geq \frac{4}{a_- (T) + 2}. \]

The above evaluation is probably not sharp. The main open problem connected with mappings of constant displacement can be described as follows.

**Problem 1.** For any \( a \in [0, 2] \) find the value \( \kappa(a) = \inf \{ k : \text{there exists a mapping } T : C \rightarrow C \text{ with } k(T) = k \text{ and } a_-(T) = a \} \)

The evaluation (5) shows that

(6) \[ \kappa(a) \geq \frac{4}{a + 2}. \]

The above has been shown without taking into account any geometrical properties of the set \( C \). One can restrict himself to some particular situation of a given set \( C \) and define relative function \( \kappa_C (a) \). We shall stay here with the general case. However to estimate \( \kappa(a) \) from above we have to discuss a concrete construction.

Let \( X = C [0, 1] \) with the usual uniform norm and let the set \( K \) be defined by

\[ K = \{ x \in C [0, 1] : 0 = x(0) \leq x(t) \leq x(1) = 1 \}. \]

Let \( e \) be the identity function on \([0, 1], e(t) \equiv t \). Any function \( \alpha \in K \) generates a mapping \( T_\alpha : K \rightarrow K \) defined for \( x \in K \) by

\[ (T_\alpha x)(t) = (\alpha \circ x)(t) = \alpha(x(t)). \]
If $\alpha$ is lipschitzian, so is $T_{\alpha}$ and we have
\[ k(T_{\alpha}) = k(\alpha) = \sup \left\{ \frac{|\alpha(t) - \alpha(s)|}{|t - s|} : t, s \in [0, 1], t \neq s \right\}. \]

Moreover, since any $x \in K$ takes all the values between 0 and 1, we have
\[ ||x - T_{\alpha}x|| = \max_{t \in [0, 1]} |x(t) - \alpha(x(t))| = \max_{s \in [0, 1]} |s - \alpha(s)| = ||e - \alpha||. \]

Thus for $\alpha \neq e$, $T_{\alpha}$ has constant positive displacement $d(T_{\alpha}) = ||e - \alpha|| > 0$. The iterated mapping $T_{\alpha}^2 = T_{\alpha} \circ T_{\alpha} = T_{\alpha \circ \alpha}$ is of the same type with $k(T_{\alpha}^2) = k(\alpha \circ \alpha) \leq (k(\alpha))^2$ and $d(T_{\alpha}^2) = ||e - \alpha \circ \alpha|| > 0$. In this case
\[ a_+(T_{\alpha}) = a_-(T_{\alpha}) = \frac{k(T_{\alpha}^2)}{k(T_{\alpha})} = \frac{||e - \alpha \circ \alpha||}{||e - \alpha||}. \]

The relation between Lipschitz and rotation constants in this case can be evaluated as follows. There exists at least one point $t \in [0, 1]$ such that
\[ |\alpha(\alpha(t)) - \alpha(t)| = ||e - \alpha|| = d(T_{\alpha}) > 0. \]

Let us assume that at this point $\alpha(\alpha(t)) > \alpha(t)$. The case with converse inequality can be treated the same way. Let $t_0$ be the minimal point for which the above holds. It means that
\[ t_0 = \min \{ t : \alpha(\alpha(t)) - \alpha(t) = ||e - \alpha|| = d(T_{\alpha}) \}. \]

Obviously
\[ \alpha(\alpha(t_0)) - \alpha(t_0) = ||e - \alpha||. \]

Observe that at $t_0$ we have $\alpha(t_0) \geq t_0$. Indeed, since $\alpha(0) = 0$, if $t_1 = \alpha(t_0) < t_0$ then $\alpha(t_1) = \alpha(\alpha(t_0)) > \alpha(t_0)$ implies existence of a point $t_2 < t_0$ for which $\alpha(t_2) = \alpha(t_0)$. For this point we would have $\alpha(\alpha(t_2)) - \alpha(t_2) = ||e - \alpha||$ which contradicts (8).

Now, assume that $\alpha$ is lipschitzian with $k(\alpha) = k$. Then, we have
\[ \alpha(\alpha(t_0)) - \alpha(t_0) \leq ||\alpha \circ \alpha - \alpha|| = ||T_{\alpha} \alpha - T_{\alpha} e|| \leq k ||\alpha - e|| = k(\alpha(t_0) - t_0). \]

Consequently
\[ ||\alpha \circ \alpha - e|| \geq \alpha(\alpha(t_0)) - t_0 = [\alpha(\alpha(t_0)) - \alpha(t_0)] + [\alpha(t_0) - t_0] \geq \left(1 + \frac{1}{k}\right) ||\alpha - e|| \]
and finally, in view of (7)
\[ 2 \geq a_+(T_{\alpha}) = a_-(T_{\alpha}) \geq 1 + \frac{1}{k}. \]

Both inequalities in (9) are sharp. The case $a_+(T_{\alpha}) = 2$ occurs for any function $\alpha$ of the form
\[ \alpha(t) = \begin{cases} \left(1 + \frac{b}{b} \right) t & \text{for } 0 \leq t \leq b < 1 \\ t + \varepsilon & \text{for } b < t \leq 1 - \varepsilon \\ 1 & \text{for } 1 - \varepsilon < t \leq 1 \end{cases}, \]
where $b \in (0, 1)$ is arbitrary and $\varepsilon$ is sufficiently small.
The equalities $k(\alpha) = k > 1$ and $a_-(T_\alpha) = 1 + \frac{1}{k}$ are satisfied for specially chosen family of functions

$$\alpha_k(t) = \begin{cases} kt & \text{for } 0 \leq t \leq \frac{1}{k} \\ 1 & \text{for } \frac{1}{k} < t \leq 1 \end{cases}.$$ 

In this setting the family of mappings $T_k = T_{\alpha_k}, k \geq 1$ fulfills the conditions

$$k(T_k) = k, \text{ with } d(T_k) = 1 - \frac{1}{k},$$

$$k(T^2_k) = k, \text{ with } d(T_k) = 1 - \frac{1}{k^2},$$

(10)  

$$a_+(T_\alpha) = a_-(T_\alpha) = 1 + \frac{1}{k}.$$  

Comparing (10) with the definition of the function $\kappa(a)$ (see 1) and substituting $a = 1 + \frac{1}{k}$ we get

(11)  

$$\kappa(a) \leq \frac{1}{a-1}$$

for all $a \in (1,2]$.  

Summing up the estimates (6) and (11) we conclude with

(12)  

$$\frac{4}{a+2} \leq \kappa(a) \leq \begin{cases} +\infty & \text{if } a \in [0,1] \\ \frac{1}{a-1} & \text{if } a \in (1,2] \end{cases}$$

The gap between the lower and upper bound given by (12) is large. Both inequalities are probably not sharp. The main problem of finding exact formula for $\kappa(a)$ (see Problem 1) leads to some more specific, seemingly simpler, but still remaining without answer partial questions.

**Problem 2.** Find better then (12) estimate for $\kappa(a)$.

**Problem 3.** Is $\kappa(a) < \infty$ on $[0,1]$? If not, then for which $a \in [0,1]$, $\kappa(a) < \infty$?

In other words. Can one construct an example of a bounded closed convex set $C$ and a lipschitzian mapping with constant positive displacement $T : C \rightarrow C$ such that $a_-(T) \leq 1$? The same with $a_+(T) \leq 1$?

**Problem 4.** Is $\kappa(0) < \infty$?

More specifically, does there exist a bounded closed and convex set $C$ and a lipschitzian mapping $T : C \rightarrow C$ of constant minimal displacement and such that $a_-(T) = 0$? Replacing $a_-(T)$ in the last question to $a_+(T)$ we obtain the last "exotic" question.

**Problem 5.** Does there exists a bounded, closed and convex set $C$ for which there is a lipschitzian involution $(T : C \rightarrow C, T^2 = I)$ having constant positive displacement?

The notion of rotation constant and rotative mappings has been introduced by K. Goebel and M. Koter in [2] and [3]. More informations about these notions can be found in an expository article [4] and books [5] and [1].
REFERENCES


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