A PROBLEM CONCERNING MAPPINGS WITH CONSTANT DISPLACEMENT (Nonlinear Analysis and Convex Analysis)

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数理解析研究所講究録 (2002), 1298: 135-140

URL http://hdl.handle.net/2433/42690

Departmental Bulletin Paper

Kyoto University
A PROBLEM CONCERNING MAPPINGS WITH CONSTANT DISPLACEMENT.

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ABSTRACT. We present here an open problem concerning lipschitzian self mappings of closed convex subsets of Banach spaces.

Let $X$ be a Banach space with norm $\| \cdot \|$ and let $C$ be a nonempty, convex closed and bounded subset of $X$. A lot of attention has been focused recently on the behavior of lipschitzian self mappings of such sets $C$. Let us recall that the mapping $T : C \rightarrow C$ is lipschitzian (satisfies Lipschitz condition) if there exists $k \geq 0$ such that

$$\|Tx - Ty\| \leq k \|x - y\|,$$

for all $x, y \in C$. The smallest $k$ for which (1) holds is said to be the Lipschitz constant for $T$ and is denoted by $k(T)$. If (1) holds we also say that $T$ is $k$-lipschitzian or that $T$ is of class $L(k)$, $T \in L(k)$.

If $C$ is compact then due to the Schauder Fixed Point Theorem any continuous (thus also any lipschitzian) mapping $T : C \rightarrow C$ has a point $x$ satisfying $x = Tx$, a fixed point of $T$. If $C$ is not compact, it is no longer true. The strongest known result due to P. K. Lin and Y. Sternfeld [6] states:

- If $C$ is not compact then for any $k > 1$ there exists a mapping $T : C \rightarrow C$ of class $L(k)$ such that,

$$d(T) = \inf \{ \|x - Tx\| : x \in C\} > 0.$$

The number $d(T)$ defined by (2) is called the minimal displacement of $T$ and mappings $T$ which satisfy (2) are called mappings with positive displacement.

Once we have a lipschitzian mapping $T$ with positive displacement $d = d(T) > 0$ we can define a modified mapping $\tilde{T} : C \rightarrow C$ by

$$\tilde{T}x = x + d \frac{Tx - x}{\|Tx - x\|}.$$

It is easy to observe that $\tilde{T}$ is also lipschitzian but the Lipschitz constant $k(\tilde{T})$ is not necessarily the same as $k(T)$.

This modified mapping has constant positive displacement equal $d$, which means that for all $x \in C$ we have

$$\|x - \tilde{T}x\| = d = d(T) > 0.$$

\textit{Mathematics Subject Classification.} Primary 47H10.

\textit{Key words and phrases.} lipschitzian mappings, fixed points, mappings with constant displacement, rotative mappings.

\textit{Date:} November 17, 2002.
Now we can observe that for any $c \in (0, 1]$ the convex combination of the mapping $\tilde{T}$ with the identity mapping $I$,

$$\tilde{T}_c = (1 - c) I + c \tilde{T},$$

is also of positive displacement equal $cd$. Moreover, we have

$$k \left( \tilde{T}_c \right) = k \left( (1 - c) I + c \tilde{T} \right) \leq 1 - c + ck \left( \tilde{T} \right)$$

and consequently, $\lim_{c \to 1} k \left( \tilde{T}_c \right) = 1$.

The above allows us to formulate an equivalent modification of the Lin Sternfeld result.

- **If $C$ is not compact then for any $k > 1$ there exists a mapping $T : C \to C$ of class $L(k)$ with constant positive displacement.**

From now on we shall discuss only mappings with constant positive displacement. Suppose $T : C \to C$ is such a mapping with $d(T) = d > 0$. The iterated mapping $T^2 = T \circ T : C \to C$ is not necessarily of constant displacement. For any $x \in C$ we have an obvious inequality

$$0 \leq \|T^2 x - x\| \leq \|T^2 x - Tx\| + \|Tx - x\| = 2d.$$

If $\|T^2 x - x\| = 2d$ then the line consisting of two linear segments $[x, Tx]$ and $[Tx, T^2 x]$ is isometric to the segment $[x, T^2 x]$ and consequently to the interval $[0, 2d]$. If $\|T^2 x - x\| < 2d$ it means that the vector $T^2 x - Tx$ is in some metric sense “rotated” with respect to the vector $Tx - x$. For this reason it is natural to introduce two coefficients

$$a_- (T) = \inf \left\{ \frac{1}{d} \|T^2 x - x\| : x \in C \right\}$$

and

$$a_+ (T) = \sup \left\{ \frac{1}{d} \|T^2 x - x\| : x \in C \right\}.$$

Intuitively, they represent the minimal metric rotation and global metric rotation. It is understood in the sense that if $a_- (T) < 2$ then for any $\varepsilon > 0$ there are some $x \in C$ for which the vector $T^2 x - Tx$ of length $d$ is rotated with respect to the vector $Tx - x$ of the same length in such a way that

$$(3) \quad \| (T^2 x - Tx) + (Tx - x) \| < (a_- (T) + \varepsilon) d.$$  

If $a_+ (T) < 2$, then (3) holds for all $x \in C$. Especially if $a_+ (T) = 0$ then $T$ is an involution, $T^2 = I$ on $C$.

There are several open problems and questions concerning mutual relations between constants $k(T)$, $a_- (T)$ and $a_+ (T)$. Here is the first observation.
Let $T : C \to C$ be a mapping of class $\mathcal{L}(k)$ with constant displacement $d(T) = d$. Take any point $x \in C$ and put $u = \frac{1}{2} (Tx + T^2x), v = \frac{1}{2} (x + Tx)$. Then we have

$$d = \|u - Tu\| = \left\|\frac{1}{2} (Tx + T^2x) - Tu\right\| \leq$$

$$\leq \frac{1}{2} \|Tx - Tu\| + \frac{1}{2} \|T^2x - Tu\| \leq$$

$$\leq \frac{k}{2} \|x - u\| + \frac{k}{2} \|Tx - u\| \leq$$

$$\leq \frac{k}{2} \|x - v\| + \frac{k}{2} \|v - u\| + \frac{k}{2} \|Tx - u\| =$$

$$= \frac{k}{4} d + \frac{k}{4} \|x - T^2x\| + \frac{k}{4} d =$$

$$= \frac{k}{2} d + \frac{k}{4} \|x - T^2x\| .$$

The conclusion of it can be written in the form

$$\frac{\|x - T^2x\|}{d} \geq 2 \left( \frac{2}{k} - 1 \right)$$

and this shows that the rotation constants and Lipschitz constant of $T$ must satisfy

$$a_+ (T) \geq a_- (T) \geq 2 \left( \frac{2}{k(T)} - 1 \right) .$$

In other words we have

- If $T : C \to C$ is a lipschitzian mapping with constant positive displacement then

$$k(T) \geq \frac{4}{a_- (T) + 2} .$$

The above evaluation is probably not sharp. The main open problem connected with mappings of constant displacement can be described as follows.

**Problem 1.** For any $a \in [0, 2]$ find the value

$$\kappa(a) = \inf \{ k : \text{there exists a mapping } T : C \to C \text{ with } k(T) = k \text{ and } a_- (T) = a \} .$$

The evaluation (5) shows that

$$\kappa(a) \geq \frac{4}{a + 2} .$$

The above has been shown without taking into account any geometrical properties of the set $C$. One can restrict himself to some particular situation of a given set $C$ and define relative function $\kappa_C(a)$. We shall stay here with the general case. However to estimate $\kappa(a)$ from above we have to discuss a concrete construction.

Let $X = C[0, 1]$ with the usual uniform norm and let the set $K$ be defined by

$$K = \{ x \in C [0, 1] : 0 = x(0) \leq x(t) \leq x(1) = 1 \} .$$

Let $e$ be the identity function on $[0, 1], e(t) \equiv t$. Any function $\alpha \in K$ generates a mapping $T_{\alpha} : K \to K$ defined for $x \in K$ by

$$(T_{\alpha}x) (t) = (\alpha \circ x) (t) = \alpha (x (t)) .$$
If \( \alpha \) is lipschitzian, so is \( T_{\alpha} \) and we have
\[
k(T_{\alpha}) = k(\alpha) = \sup \left\{ \frac{|\alpha(t) - \alpha(s)|}{|t - s|} : t, s \in [0,1], t \neq s \right\}.
\]
Moreover, since any \( x \in K \) takes all the values between 0 and 1, we have
\[
||x - T_{\alpha}x|| = \max_{t \in [0,1]} |x(t) - \alpha(x(t))| = \max_{s \in [0,1]} |s - \alpha(s)| = ||e - \alpha||.
\]
Thus for \( \alpha \neq e \), \( T_{\alpha} \) has constant positive displacement \( d(T_{\alpha}) = ||e - \alpha|| > 0 \).
The iterated mapping \( T_{\alpha}^{2} = T_{\alpha} \circ T_{\alpha} = T_{\alpha\alpha} \) is of the same type with \( k(T_{\alpha}^{2}) = k(\alpha \circ \alpha) \leq k(\alpha)^{2} \) and \( d(T_{\alpha}^{2}) = ||e - \alpha \circ \alpha|| > 0 \). In this case
\[
a_{+}(T_{\alpha}) = a_{-}(T_{\alpha}) = \frac{k(T_{\alpha}^{2})}{k(T_{\alpha})} = \frac{||e - \alpha \circ \alpha||}{||e - \alpha||}.
\]
The relation between Lipschitz and rotation constants in this case can be evaluated as follows. There exists at least one point \( t \in [0,1] \) such that
\[
|\alpha(\alpha(t)) - \alpha(t)| = ||e - \alpha|| = d(T_{\alpha}) > 0.
\]
Let us assume that at this point \( \alpha(\alpha(t)) > \alpha(t) \). The case with converse inequality can be treated the same way. Let \( t_{0} \) be the minimal point for which the above holds. It means that
\[
t_{0} = \min \{ t : \alpha(\alpha(t)) - \alpha(t) = ||e - \alpha|| = d(T_{\alpha}) \}.
\]
Obviously
\[
\alpha(\alpha(t_{0})) - \alpha(t_{0}) = ||e - \alpha||.
\]
Observe that at \( t_{0} \) we have \( \alpha(t_{0}) \geq t_{0} \). Indeed, since \( \alpha(0) = 0 \), if \( t_{1} = \alpha(t_{0}) < t_{0} \) then \( \alpha(t_{1}) = \alpha(\alpha(t_{0})) > \alpha(t_{0}) \) implies existence of a point \( t_{2} < t_{0} \) for which \( \alpha(t_{2}) = \alpha(t_{0}) \). For this point we would have \( \alpha(\alpha(t_{2})) - \alpha(t_{2}) = ||e - \alpha|| \) which contradicts (8).

Now, assume that \( \alpha \) is lipschitzian with \( k(\alpha) = k \). Then, we have
\[
\alpha(\alpha(t_{0})) - \alpha(t_{0}) \leq ||\alpha \circ \alpha - \alpha|| = ||T_{\alpha} \alpha - T_{\alpha} e|| \leq k ||\alpha - e|| = k(\alpha(t_{0}) - t_{0}).
\]
Consequently
\[
||\alpha \circ \alpha - e|| \geq \alpha(\alpha(t_{0})) - t_{0} = [\alpha(\alpha(t_{0})) - \alpha(t_{0})] + [\alpha(t_{0}) - t_{0}] \geq \left( 1 + \frac{1}{k} \right) |\alpha(\alpha(t_{0})) - \alpha(t_{0})| = \left( 1 + \frac{1}{k} \right) ||\alpha - e||
\]
and finally, in view of (7)
\[
2 \geq a_{+}(T_{\alpha}) = a_{-}(T_{\alpha}) \geq 1 + \frac{1}{k}.
\]
Both inequalities in (9) are sharp. The case \( a_{+}(T_{\alpha}) = 2 \) occurs for any function \( \alpha \) of the form
\[
\alpha(t) = \begin{cases} 
(1 + \frac{b}{2})t & \text{for } 0 \leq t \leq b < 1 \\
t + \varepsilon & \text{for } b < t \leq 1 - \varepsilon \\
1 & \text{for } 1 - \varepsilon < t \leq 1
\end{cases}
\]
where \( b \in (0,1) \) is arbitrary and \( \varepsilon \) is sufficiently small.
The equalities $k(\alpha) = k > 1$ and $a_- (T_\alpha) = 1 + \frac{1}{k}$ are satisfied for specially chosen family of functions

$$\alpha_k(t) = \begin{cases} kt & \text{for } 0 \leq t \leq \frac{1}{k} \\ 1 & \text{for } \frac{1}{k} < t \leq 1 \end{cases}.$$ 

In this setting the family of mappings $T_k = T_{\alpha_k}, k \geq 1$ fulfils the conditions

$$k(T_k) = k, \text{ with } d(T_k) = 1 - \frac{1}{k},$$

$$k(T_k^2) = k, \text{ with } d(T_k) = 1 - \frac{1}{k^2},$$

(10) $$a_+ (T_\alpha) = a_- (T_\alpha) = 1 + \frac{1}{k}.$$ 

Comparing (10) with the definition of the function $z(a)$ (see 1) and substituting $a = 1 + \frac{1}{k}$ we get

(11) $$z(a) \leq \frac{1}{a-1}$$

for all $a \in (1,2]$.

Summing up the estimates (6) and (11) we conclude with

(12) $$\frac{4}{a+2} \leq z(a) \leq \begin{cases} +\infty & \text{if } a \in [0,1] \\ \frac{1}{a-1} & \text{if } a \in (1,2] \end{cases}.$$ 

The gap between the lower and upper bound given by (12) is large. Both inequalities are probably not sharp. The main problem of finding exact formula for $z(a)$ (see Problem 1) leads to some more specific, seemingly simpler, but still remaining without answer partial questions.

**Problem 2.** Find better then (12) estimate for $z(a)$.

**Problem 3.** Is $z(a) < \infty$ on $[0,1]$? If not, then for which $a \in [0,1], z(a) < \infty$?

In other words. Can one construct an example of a bounded closed convex set $C$ and a lipschitzian mapping with constant positive displacement $T : C \rightarrow C$ such that $a_- (T) \leq 1$? The same with $a_+ (T) \leq 1$?

**Problem 4.** Is $z(0) < \infty$?

More specifically, does there exist a bounded closed and convex set $C$ and a lipschitzian mapping $T : C \rightarrow C$ of constant minimal displacement and such that $a_- (T) = 0$? Replacing $a_- (T)$ in the last question to $a_+ (T)$ we obtain the last „exotic” question.

**Problem 5.** Does there exists a bounded, closed and convex set $C$ for which there is a lipschitzian involution $(T : C \rightarrow C, T^2 = I)$ having constant positive displacement?

The notion of rotation constant and rotative mappings has been introduced by K. Goebel and M. Koter in [2] and [3]. More informations about these notions can be found in an expository article [4] and books [5] and [1].
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