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Some Types of Existence Theorems for Cone Saddle Points

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Abstract

In this paper, we research existence theorems of saddle points for vector valued function and broadly classfy into two categories. One of those classes has been investigated from the beginning of studying about this field and is besed on some fixed point theorems or scalar minimax theorems and are researched by Nieuwenhuis, Ferro, Tanaka and so on. Another type of these theorems have been based on Fan-KKM theorem. This type of theorems have been researched since 2000 by Kazmi, Khan, Kimura and Tanaka and so on. We compare these two types of theorems and consider about the distinction between them.

Keywords: cone saddle point, cone convexity, cone invexity

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1 Introduction

Studies on vector-valued minimax theorems or vector saddle point problems have been extended widely; see [1, 3, 5, 6, 7, 9, 10] and references cited therein. Existence results for cone saddle points can be divided roughly into two categories. First type is based on some fixed point theorems or scalar minimax theorems; see [10, 12]. This type has been started by Nieuwenhuis [5]. Afterwards existence theorems for cone saddle points have investigated moreover by Ferro, Nieuwenhuis, Tanaka and so on. Second type is based on Fan-KKM Thoerem by regarding the problem as a kind of valiational inequality problem. This type was treated by Kazmi and Khan [3] and it has been researched by Kimura and Tanaka [4]. The aim of this paper is introduction of some types of exisetance theorems for cone saddle points and our resent results. Moreover we compare those theorems.

2 Preliminary and terminology

In order to consider saddle points of vector-valued functions, we give some abstract settings for mathematics on vector optimization. Thorughout this section, let Z be an ordered real topological vector space with an ordering $\leq$ on Z defined by a pointed convex cone $C \subset Z$, where 'pointed' means $C \cap (-C) = \{0\}$. If C is solid, i.e., its topological interior $\operatorname{int} C$ is nonempty, then we can consider another ordering cone $C^0 := (\operatorname{int} C) \cup \{0\}$. Now, we can define minimal and maximal elements of a subset A of Z. An element z_0 of a subset A of Z is said to be a C-minimal point of A if $\{ z \in A \mid z_0 - z \in C,\ z \neq z_0 \} = \phi$, and a C-maximal point of A if $\{ z \in A \mid z - z_0 \in C,\ z \neq z_0 \} = \phi$. We denote the set of such all C-minimal [resp., C-maximal] points of A by $\mathrm{Min}A$ [resp., $\mathrm{Max}A$]. If C is R^p_+ then $\mathrm{Min}A$ is the set of pareto solutions, where R^p_+ denotes the nonnegative orthant of R^p and if $p = 1$ then R^p_+ is writen by R_+. Also, C^0-minimal and C^0-maximal points of A are defined similarly, and denoted by $\mathrm{Min_w}A$ and $\mathrm{Max_w}A$, respectively.

Definition 2.1 *A point (x_0, y_0) is said to be a C-saddle point of f with respect to $X \times Y$, if $f(x_0, y_0) \in \mathrm{Max}f(x_0, Y) \cap \mathrm{Min}f(X, y_0)$, where $f(X, y)$ [resp., $f(x, Y)$] denotes $\bigcup_{x \in X} f(x, y)$ [resp., $\bigcup_{y \in Y} f(x, y)$].*

Definition 2.2 *A point (x_0, y_0) is said to be a weak C-saddle point of f with respect to $X \times Y$, if $f(x_0, y_0) \in \mathrm{Max_w}f(x_0, Y) \cap \mathrm{Min_w}f(X, y_0)$.*

3 First type existence results for cone saddle points

In this section, we introduce some existence theorems of cone saddle points for the first type.

Theorem 3.1 *(See Theorem 3.1 in [5].) Let $X \subset R^n$ and $Y \subset R^m$ be nonempty convex compact sets. Let $f : X \times Y \to R^p$ be jointly continuous in (x, y), convex in x for every $y \in Y$ and concave in y for every $x \in X$. Then, f has at least one R^p_+-saddle points.*

Definition 3.1 *Let X be a topological space and Z an ordered topological vector space with an ordering defined by a pointed convex cone C. A vector-valued function $f : X \to Z$ is said to be lower level-closed if $f^{-1}(z - \mathrm{cl}C)$ is closed in X for each $z \in Z$, where $\mathrm{cl}\,A$ stands for closure of a set A.*

Theorem 3.2 *(See Theorems 3.1 and 3.2 in [11] and Theorem 4.1 in [12].) Let X and Y be nonempty compact sets in two topological spaces, respectively, and Z an ordered topological vector space with an ordering defined by a solid pointed convex cone C in Z. A vector-valued function $f : X \times Y \to Z$ has at least weak C-saddle point if one of the following conditions holds:*

(i) f is of the type $f(x, y) = u(x) + v(y)$ where u and $-v$ are lower level-closed;

(ii) f is of the type $f(x, y) = u(x) + \beta(x)v(y)$ where u is continuous, $-v$ is lower level-closed, and $\beta : X \to R_+$ is continuous.

If, in addition, C satisfies $clC + (C\backslash\{0\}) \subset C$, then f has at least one C-saddle point.

Definition 3.2 *Let X be a topological space and Z an ordered topological vector space with an ordering defined by a solid pointed convex cone C. A vector-valued function $f : X \to Z$ is said to be C-lower semicontinuous on X if for each $x_0 \in X$ and any open neighborhood V of $f(x_0)$, there exists an open neighborhood U of x_0 such that $f(x) \in V + C$ for all $x \in U$. If $-f$ is C-lower semicontinuous then f is said to be C-upper semicontinuous.*

Definition 3.3 *Let X be a topological space and let Z be a topological vector space. A vector-valued function $f : X \to Z$ is said to be demicontinuous on X if*

$$f^{-1}(M) := \{\, x \in X \mid f(x) \in M \,\}$$

is closed in X for each closed half-space $M \subset Z$.

Definition 3.4 *Let X be a convex set in a real vector space. A vector-valued function $f : X \to Z$ is said to be C-naturally quasiconvex if*

$$f(\lambda x_1 + (1 - \lambda)x_2) \in \mathrm{co}\{f(x_1), f(x_2)\} - C$$

for every $x_1, x_2 \in X$ and $\lambda \in [0, 1]$, where coA denotes the convex hull of the set A. Also, a vector valued function f is said to be C-naturally quasiconcave on X if $-f$ is C-naturally quasiconvex on X.

Theorem 3.3 *(See Theorem 3.1 in [9] and Theorem 3.3 in [10].) Let X and Y be nonempty compact convex sets in two topological vector spaces, respectively, and Z an ordered topological vector space with an ordering defined by a solid pointed convex cone C in Z. If a vector-valued function $f : X \times Y \to Z$ satisfies* *(i)* $x \mapsto f(x,y)$ *is either C-lower semicontinuous or demicontinuous, and C-naturally quasiconvex on X for every $y \in Y$;*

(ii) $y \mapsto f(x,y)$ *is either C-upper semicontinuous or demicontinuous, and C-naturally quasiconcave on Y for every $x \in X$,*

then the vector-valued function f has at least one weak C-saddle point.

Theorem 3.4 *(See Theorem 4.1 in [8] and Theorem 3.1 in [8].) Let X and Y be nonempty compact convex sets in two locally convex spaces, respectively, and Z an ordered topological vector space with an ordering defined by a solid pointed convex cone C in Z. If a vector-valued function $f : X \times Y \to Z$ is continuous and if the following sets*

$$T(y) := \{\, x \in X \mid f(x,y) \in \mathrm{Min_w} f(X,y) \,\},$$
$$U(x) := \{\, y \in Y \mid f(x,y) \in \mathrm{Max_w} f(x,Y) \,\}$$

are convex for every $y \in Y$ and $x \in X$, respectively, then the vector-valued function f has at least one weak C-saddle point.

4 Second type existence results for cone saddle points

In this section, we deal with the second type of existence thereoms.

Definition 4.1 *Let X be a convex set in a real vector space. A vector-valued function $f : X \to Z$ is said to be C-convex if for each $x, y \in X$ and $\lambda \in [0,1]$,*

$$\lambda f(x) + (1-\lambda) f(y) - f(\lambda x + (1-\lambda) y) \in C.$$

Lemma 4.1 *Let X be a convex set in a real vector space. If a vector-valued function f is C-convex [resp., C-concave] then f is also C-naturally quasiconvex [resp., C-naturally quasiconcave].*

Theorem 4.1 *(See Theorem 2.3 in [3].) Let $X \subset R^n$ and $Y \subset R^m$ be a nonempty closed convex set and a nonempty compact set, respectively. Assume that $f : X \times Y \to R^p$ is continuously Fréchet differentiable and R^p_+-convex in the first argument; moreover assume that a multifunction $T : X \to 2^Y$ is defined by $T(x) := \mathrm{Max_w} f(x,Y)$. Suppose that, for each fixed $(x,y) \in X \times Y$, the function $\langle f'(x,y), u - x \rangle$ is a R^p_+-naturally quasiconvex function in $u \in R^p$, where $f'(x,y)$ stands for Fréchet derivartive of f with respect to first variable at (x,y). If there exist a nonempty compact subset B of R^n and $x_0 \in (B \cap X)$ such that for any $x \in (X \backslash B)$, there exists $y \in T(x)$ such that*

$$\langle f'(x,y), x_0 - x \rangle \in -\mathrm{int}\, R^p_+,$$

then the vector-valued function f has at least one weak R^p_+-saddle point.

Theorem 4.2 *(See Theorem 2.3 in [4].) Let X and Y be a nonempty closed convex subset of a normed space E and a nonempty compact subset of a topological vector space F, respectively, and Z an ordered normed space with ordering defined by a solid pointed closed convex cone C in Z. Assume that the vector-velued function $f : X \times Y \to Z$ is continuously Fréchet differentiable and C-convex in the first argument and f' is continuous in both x and y, and let $T : X \to 2^Y$ be the multifunction defined by $T(x) := \mathrm{Max_w} f(x, Y)$. If there exist a nonempty compact subset B of X and $x_0 \in (B \cap X)$ such that for any $x \in (E \backslash (X \cap B))$ and $y \in T(x)$,*

$$\langle f'(x, y), x_0 - x \rangle \in -\operatorname{int} C$$

then the vector-valued function f has at least one weak C-saddle point.

Definition 4.2 *Let X be a convex subset of a normed space and Z an ordered normed space; let a vector-valued function $\eta : X \times X \to E$. Suppose that a vector-valued function $f : X \to Z$ is Fréchet differentiable on X. A vector-valued function f is said to be C-invex with respect to η if*

$$f(x) - f(y) - \langle f'(y), \eta(x, y) \rangle \in C$$

for every $x, y \in X$.

Lemma 4.2 *Let X and Y be a nonempty closed convex subset of a normed space E and a nonempty compact subset of a topological vector space F, respectively. Assume that the vector-valued function f is Fréchet differentiable and C-invex with respect to η in the first argument, where $\eta : X \times X \to E$ satisfies the following three conditions: for all $x \in X$,*

(i) $\eta(\cdot, x)$ *is affine,*

(ii) $\eta(x, \cdot)$ *is continuous, and*

(iii) $\eta(x, x) = 0$.

Moreover assume that Fréchet derivative f' is continuous in both x and y. If there exist a nonempty compact subset B of E and $x_0 \in (B \cap X)$ such that for any $x \in (X \backslash B)$ and $y \in T(x)$,

$$\langle f'(x, y), \eta(x_0, x) \rangle \in -\operatorname{int} C,$$

then the vector-valued function f has at least one weak C-saddle point.

In order to prove Theorem 4.5, we need the following two theorems.

Theorem 4.3 *Suppose that $X \subset R^n$ is nonempty convex, $Y \subset R^m$ is nonempty and $f : X \times Y \to R^p_+$ is subdifferentiable with respect to η in the first argument. Moreover assume that a multifunction $T : X \to 2^Y$ is defined by $T(x) := \mathrm{Max_w} f(x, Y)$. Then*

$$\{ (x_0, y_0) \in X \times Y \mid \langle A, \eta(x_0, x) \rangle \notin \operatorname{int} R^p_+, y_0 \in T(x_0) \text{ and } A \in \partial f(x_0, y_0) \} \\ \subset \{ (x_0, y_0) \in X \times Y \mid \mathrm{Max_w} f(x_0, Y) \cap \mathrm{Min_w} f(X, y_0) \}$$

Theorem 4.4 *(See [2]) Let Y be a subset of the topological vector space X. For each $x \in Y$, let a nonempty closed set $F(x)$ in X be given such that $F(x)$ is compact for at least one $x \in Y$. If the convex hull of every finite subset $\{x_1, \ldots, x_n\}$ of Y is contained in the corresponding union $\bigcup_{i=1}^n F(x_i)$, then $\bigcap_{x \in Y} F(x) \neq \phi$.*

The mapping $F : Y \to 2^Y$ is called the KKM-map if $\mathrm{conv}\{x_1, \ldots, x_n\} \subset \bigcup_{i=1}^n F(x_i)$ for every finite subset $\{x_1, \ldots, x_n\}$ of Y, where $\mathrm{conv}\, D$ denotes the convex hull of the set D.

Definition 4.3 *Let X be a convex subset of R^n and a vector-valued function $\eta : X \times X \to R^n$. Assume that a multifunction $\partial f : X \to \mathcal{L}(R^n, R^p)$ is defined by*

$$\partial f(a) := \{ A \in \mathcal{L}(R^n, R^p) \mid f(x) - f(a) - \langle A, \eta(x, a) \rangle \in R^p_+ \text{ for all } x \in X \},$$

where $\mathcal{L}(R^n, R^p)$ denotes the set of bounded linear operater from $R^n \to R^p$. A vector-valued function $f : R^n \to R^p$ is said to be subdifferentiable on X with respect to η if for every $x \in X$, $\partial f(x) \neq \phi$.

Theorem 4.5 *Let X and Y be a nonempty closed convex subset and a nonempty compact subset of R^n and R^m, respectively. Assume that the vector-valued function $f : X \times Y \to R^p$ is subdifferentiable with respect to η in the first argument, where $\eta : X \times X \to R^n$ satisfies the following three conditions: for all $x \in X$,*

(i) $\eta(\cdot, x)$ is affine,

(ii) $\eta(x, \cdot)$ is continuous, and

(iii) $\eta(x, x) = 0$.

Moreover assume that a multifunction $T : X \to 2^Y$ is defined by $T(x) := \mathrm{Max_w} f(x, Y)$. If there exist a nonempty compact subset B of R^p and $x_0 \in (B \cap X)$ such that for any $x \in (X \backslash B)$, $y \in T(x)$, $A \in \partial f(x, y)$

$$\langle A, \eta(x_0, x) \rangle \in -\operatorname{int} R^p_+,$$

then the vector-valued function f has at least one weak C-saddle point.

Proof. Define a multifunction $F : X \to 2^X$ by

$$\begin{aligned} F(u) := \ & \{ x \in X \mid \langle A, \eta(u,x)\rangle \notin -\mathrm{int}\, R^p_+, \\ & \qquad \text{for some } y \in T(x) \text{ and } A \in \partial L(x,y) \}, \quad u \in X. \end{aligned}$$

In order to prove the theorem, it is sufficient to show that the set $\{ (x_0, y_0) \in X \times Y \mid \langle A, \eta(x_0, x)\rangle \notin \mathrm{int}\, R^p_+$, for some $y_0 \in T(x_0)$ and $A \in \partial f(x_0, y_0) \} \neq \phi$ by Theorem 4.3. So we should show, by Theorem 4.4, the following three points:

(a) F is a KKM-map;

(b) $F(x)$ is closed for each $x \in X$; and

(c) there exists $\hat{x} \in X$ such that $F(\hat{x})$ is compact.

First, we prove the condition (a). Suppose to the contrary that there exist $x_1, x_2, \ldots, x_m$ and $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that

$$\hat{x} := \sum_{i=1}^{m} \alpha_i x_i \notin \bigcup_{i=1}^{m} F(x_i), \quad \sum_{i=1}^{m} \alpha_i = 1.$$

Then, $\hat{x} \notin F(x_i)$ for all $i = 1, \ldots, m$, and hence for any $y \in T(\hat{x})$, $A \in \partial L(\hat{x}, y)$,

$$\langle A, \eta(x_i, \hat{x})\rangle \in -\mathrm{int}\, R^p_+,$$

for all $i = 1, \ldots, m$. Since $\mathrm{int}\, R^p_+$ is convex, we have

$$\sum_{i=1}^{m} \alpha_i \langle A, \eta(x_i, \hat{x})\rangle \in -\mathrm{int} R^p_+.$$

Since A is a linear operater and η is an affine operater, we have

$$\left\langle A, \eta\left(\sum_{i=1}^{m} \alpha_i x_i, \sum_{i=1}^{m} \alpha_i \hat{x}\right)\right\rangle \in -\mathrm{int}\, R^p_+.$$

Therefore

$$\langle A, \eta(\hat{x}, \hat{x})\rangle = 0 \in -\mathrm{int}\, R^p_+,$$

which is inconsistent. Thus, we deduce that

$$\mathrm{conv}\{x_1, x_2, \ldots, x_m\} \subset \bigcup_{i=1}^{m} F(x_i).$$

Next, we show that the condition (b) holds. For each $u \in X$, let $\{x_n\} \subset F(u)$ such that $x_n \to x \in X$. Since $x_n \in F(u)$ for all n, there exist $y_n \in T(x_n)$ and $A_n \in \partial L(x_n, y_n)$ such that

$$\langle A_n, \eta(u, x_n)\rangle \in W,$$

where $W := R^p \backslash (-\mathrm{int}\, R^p_+)$. As $\{y_n\} \subset Y$, without loss of generality, we can assume that there exists $y \in Y$ such that $y_n \to y$. Now T is closed, so $y \in T(x)$. Because of the closedness of W, the upper semicontinuity of ∂L and $\langle A_n, \eta(u, x_n)\rangle \in (R^p \backslash - \mathrm{int}\, R^p_+)$ for all n, there exists $A \in \partial L(x, y)$

$$\langle A, \eta(u, x)\rangle \in (R^p \backslash - \mathrm{int}\, R^p_+).$$

Hence $x \in F(u)$. As a result the condition (b) holds.

Finally we prove the condition (c). Since $F(\bar{x})$ is closed and B is compact, it is sufficient to show that $F(\bar{x}) \subset B$. Suppose to the contrary that there exists $\hat{x} \in F(\bar{x})$ such that $\hat{x} \notin B$. Since $\hat{x} \in F(\bar{x})$, there exist $\hat{y} \in T(\hat{x})$ and $\hat{A} \in \partial L(\hat{x}, \hat{y})$ such that

$$\langle \hat{A}, \eta(\bar{x}, \hat{x})\rangle \notin -\mathrm{int}\, R^p_+. \tag{1}$$

Since $\hat{x} \notin B$, by the hypothesis, for any $y \in T(\hat{x})$ and $A \in \partial L(\hat{x}, y)$,

$$\langle A, \eta(\bar{x}, \hat{x})\rangle \in -\mathrm{int}\, R^p_+,$$

which contradicts condition (1). Hence $F(\bar{x}) \subset B$. Since B is compact and $F(\bar{x})$ is also closed, $F(\bar{x})$ is compact, i.e., the condition (c) holds. Consequently by Fan-KKM Theorem, it follows that $\bigcap_{x \in X} F(x) \neq \phi$. Thus, there exists $x_0 \in X$ and $y_0 \in T(y_0)$ such that

$$\langle A, \eta(x, x_0)\rangle \notin -\mathrm{int}\, R^p_+,$$

for all $x \in X$. As a result the vector-valued function f has at least one weak C-saddle point. ∎

Definition 4.4 *Let $f : X \to R$ be a lower semi-continuous function, where X is a nonempty convex set in R^n. Then the convex envelope of $f(x)$ taken over X is a function $F(x)$ such that*

(i) $F(x)$ is convex on X;

(ii) $F(x) \leq f(x)$ for all $x \in X$;

(iii) If $h(x)$ is any convex function defined on X such that $h(x) \leq f(x)$ for all $x \in X$, then $h(x) \leq F(x)$ for all $x \in X$.

Geometrically, $F(x)$ is precisely the function whose epigraph coincides with the convex hull of the epigraph of f.

Definition 4.5 *Suppose that vector-valued functions f and h consist of p real-valued functions $f_1, \ldots, f_p$ and $h_1, \ldots, h_p$ on $X \times Y$, respectively. If each of components of h are the convex envelope of $f_1, \ldots, f_p$, respectively, then h is called the vector convex envelope of f.*

Assumption A. For $f : X \to R^p$ and its vector convex envelope h, the following condition holds:

$$\{ x \in X \mid h(x) - h(y) \notin \mathrm{int} R^p_+ \ \forall y \in X \} \\ \subset \{ x \in X \mid f(x) - f(y) \notin \mathrm{int} R^p_+ \ \forall y \in X \}.$$

Corollary 4.1 *Let X and Y be a nonempty closed convex subset and a nonempty compact subset of R^n and R^m, respectively. Suppose that a vector-valued function $H : X \times Y \to R^p$ is the convex envelope of $L : X \times Y \to R^p$ in the first argument and that H satisfies the conditions on L in Theorem 4.5. If $h(x) := H(x, y)$ and $f(x) := L(x, y)$ satisfy Assumption A for each $y \in Y$, then L has at least one solution.*

Proof. Since H satisfies the conditions on L in Theorem 4.5, H has at least one weak R^p_+-saddle point by Theorem 4.5. Since H satisfies Assumption A, then L has at least one solution. ∎

5 Conclusions

We have seen existence theorems which are classified roughly into two types. In the first type of theorems, each payoff function is a saddle function, which has some dualities, e.g., convexity of $f(\cdot, y)$ for every $y \in Y$ and concavity of $f(x, \cdot)$ for every $x \in X$, lower-semicontinuity of $f(\cdot, y)$ for every $y \in Y$ and upper-semicontinuity of $f(x, \cdot)$ for every $x \in X$ and so on. Those theorems seem to be much polished. For the second type theorems, though those required conditions are anti-duality and there are some stronger conditions than the first type of theorems, there seems to be a room for evolution.

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