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On A Dynamic Fractional Game

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In Memory of Professor Kensuke Tanaka [2]

Abstract Consider a two-person zero-sum game constructed by a dynamic fractional form. We establish the upper value as well as the lower value of a dynamic fractional game, and prove that the dual gap is equal to zero under certain conditions. It is also established that the saddle point function exists in the fractional game system under certain conditions so that the equilibrium point exists in this game system.

1. Introduction

In 1953 Fun [5] proved minimax theorems for a function f defined on the product set $X \times Y$ of two arbitrary sets X, Y (not necessary topologized and need not be linear). That is the equality

$$\min_{x\in X}\max_{y\in Y} f(x,y) = \max_{y\in Y}\min_{x\in X} f(x,y)$$

holds under certain conditions. Fun's results were widely applied to many directions. By the above idea on minimax identity, we will constitute a two-person zero-sum dynamic game for fractional type :

$$\phi(x,y) = \frac{f(x,y)}{g(x,y)}, \quad (x,y) \in X \times Y.$$

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⋆ The whole paper will be appeared elsewhere.

Several types of game systems have been discussed and investigated by Lai and Tanaka in [12-19] and [24]. See also the related work in [7], [20, 21, 25]. These games involved n-person noncooperative dynamic systems in various spaces (cf. [12-17], [19] and [25]) and two-person zero-sum games (see [7], [20-21], [24]). Recently, many authors investigated fractional programming; see for example [3, 4, 6, 9, 10, 11], [22] and [23]. Lai et al. investigated minimax fractional programming in [9, 10, 11] and propose that a minimax theory for fractional objective $f(x,y)/g(x,y)$ could be applied to two-person zero-sum game theory.

Following this approach, we consider a two-person zero-sum dynamic fractional game in this paper, and investigate an existence theorem for the saddle value function in a fractional game system.

2. Preliminaries

In a two-person zero-sum game, we will investigate whether two persons will attain a saddle point in the game system, that is we want to find a value function such that the two persons can obtain an equilibrium point.

A two-person zero-sum dynamic game with a parameter θ at a discrete time $n \in N$, denoted briefly by the game (DGP$_\theta$), includes the following seven elements:

$$(S_n, A_n, B_n, t_{n+1}, u_n, v_n, \theta)$$

where each element is defined as follows, and for convenience of the mathematical analysis, the assumptions below are made.

1) S_n is the **state space** at time $n \in N$, which is assumed to be a separable complete metrizable Borel space, so that the Borel functions defined on S_n are integrable over such a space.

2) A_n and B_n are, respectively, the **action spaces** at time $n \in N$ for players I and II in which each player chooses his (or her) actions in the game system. Here A_n and B_n are always assumed to be Borel spaces.

3) $\{t_{n+1}\}$ is a sequence of **transition probabilities** from time n to time $n+1$ in the law of motion for the game system. When the two players have finished their actions at time n, denoted by $H_nA_nB_n$, then the system is moved to state S_{n+1}. Here H_n stands for the histories up to time n, thus $H_1 = S_1$, $H_n = S_1A_1B_1S_2A_2B_2\cdots S_{n-1}A_{n-1}B_{n-1}S_n$, $n = 2, 3, \cdots$ and H_∞ stands for the set of infinite histories of the game system.

4) $u_n : H_nA_nB_n \to R$ and $v_n : H_nA_nB_n \to R^+$ are bounded Borel measurable functions, and as the time n goes to infinity, they have the limits as follows

$$\lim_{n\to\infty} u_n = u \in R, \qquad \lim_{n\to\infty} v_n = v \in R^+.$$

Of course u_n and v_n are also regarded as functions on H_∞.

5) $\theta : S_1 \to R$ is a given parameter function on which the **loss** function of player I at time $n \in N$ is given by

$$T_\theta^n = u_n - \theta v_n$$

and the **gain** (loss) function of player II at time $n \in N$ is given by

$$-T_\theta^n.$$

Then the sum of the two values is always zero.

We denote by F_n(resp. G_n) the set of all universal measurable transition probabilities from history H_n to A_n(resp. B_n), and consider the sequence $f = \{f_n\}$ (resp. $g = \{g_n\}$) with $f_n \in F_n$ (resp. $g_n \in G_n$) for each time $n \in N$.

Let E_{f_n}, E_{g_n}, $E_{t_{n+1}}$ denote the conditional expectation operators with respect to $f_n \in F_n$, $g_n \in G_n$ and t_{n+1}, respectively. Then each pair of strategies $f = \{f_n\}$ and $g = \{g_n\}$, together with the law of motion $\{t_{n+1}\}$, defines a unique universally measurable transition probability by

$P_{fg}(\,\cdot \mid \cdot\,)$ from $S_1 \to A_1B_1S_2A_2B_2S_3\cdots$

such that, for two bounded Borel measurable functions u_n, v_n defined on $H_nA_nB_n$ ($n \in N$), and for $s_1 \in S_1$ and $h \in H_\infty$, we have

$$\begin{aligned} E(u_n, f, g)(s_1) &= \int_{H_\infty} u_n(h)P_{fg}(dh|s_1) \\ &= E_{f_1}E_{g_1}E_{t_2}\cdots E_{f_{n-1}}E_{g_{n-1}}E_{t_n}E_{f_n}E_{g_n}u_n(s_1) \end{aligned}$$

and

$$\begin{aligned} E(v_n, f, g)(s_1) &= \int_{H_\infty} v_n(h)P_{fg}(dh|s_1) \\ &= E_{f_1}E_{g_1}E_{t_2}\cdots E_{f_{n-1}}E_{g_{n-1}}E_{t_n}E_{f_n}E_{g_n}v_n(s_1). \end{aligned}$$

Under our assumptions, by the dominated convergence theorem and the Fubini theorem, we infer that, for each $s_1 \in S_1$, $f = \{f_n\} \in F$ and $g = \{g_n\} \in G$, it would have

$$\begin{aligned} U(f,g)(s_1) &= \lim_{n\to\infty} E(u_n, f, g)(s_1) \\ &= \lim_{n\to\infty} E_{f_1}E_{g_1}E_{t_2}\cdots E_{f_{n-1}}E_{g_{n-1}}E_{t_n}E_{f_n}E_{g_n}u_n(s_1) \\ &= \lim_{n\to\infty} E_{g_1}E_{f_1}E_{t_2}\cdots E_{g_{n-1}}E_{f_{n-1}}E_{t_n}E_{g_n}E_{f_n}u_n(s_1) \end{aligned}$$

and

$$\begin{aligned} V(f,g)(s_1) &= \lim_{n\to\infty} E(v_n,f,g)(s_1) \\ &= \lim_{n\to\infty} E_{f_1}E_{g_1}E_{t_2}\cdots E_{f_{n-1}}E_{g_{n-1}}E_{t_n}E_{f_n}E_{g_n}v_n(s_1) \\ &= \lim_{n\to\infty} E_{g_1}E_{f_1}E_{t_2}\cdots E_{g_{n-1}}E_{f_{n-1}}E_{t_n}E_{g_n}E_{f_n}v_n(s_1). \end{aligned}$$

Then, for given $s_1 \in S_1$, $f = \{f_n\} \in F$ and $g = \{g_n\} \in G$, we can evaluate the total loss function

$$\begin{aligned} T_\theta(f,g)(s_1) &= \lim_{n\to\infty} E_{fg}T_\theta^n(f,g)(s_1) \\ &= U(f,g)(s_1) - \theta(s_1)V(f,g)(s_1), \end{aligned}$$

together with the **upper value** function of the game

$$\bar{T}_\theta(s_1) = \inf_{f\in F}\sup_{g\in G} T_\theta(f,g)(s_1),$$

and the **lower value** function of the game

$$\underline{T}_\theta(s_1) = \sup_{g\in G}\inf_{f\in F} T_\theta(f,g)(s_1).$$

We call the interval $[\underline{T}_\theta(s_1), \bar{T}_\theta(s_1)]$ the **dual gap** of (DGP$_\theta$), and say that the game system has a **saddle value function**, (or shortly, a **value function**) if

$$\bar{T}_\theta(s_1) = \underline{T}_\theta(s_1) = T_\theta^*(s_1), \quad \text{for } \; s_1 \in S_1.$$

In this paper, we will consider the fractional dynamic game of the form

$$W(f,g)(s_1) = \frac{U(f,g)(s_1)}{V(f,g)(s_1)}, \qquad (f,g) \in F \times G$$

and investigate the **upper value function**

$$\bar{\theta}(s_1) = \inf_{f\in F}\sup_{g\in G} W(f,g)(s_1)$$

and the **lower value function**

$$\underline{\theta}(s_1) = \sup_{g\in G}\inf_{f\in F} W(f,g)(s_1).$$

Furthermore, it is natural to ask whether a *zero duality gap* exists in the game system. That is, *under what conditions* one can get a common value function for upper value function and lower value function, that is,

$$\bar{\theta}(s_1) = \underline{\theta}(s_1) = \theta^*(s_1), \quad \text{for } \; s_1 \in S_1.$$

3. A Two-Person Zero-Sum Dynamic Fractional Game

According to the arguments given in Section 2, we define a two-person zero-sum dynamic fractional game (DFG) as the following set of elements:

$$(S_n, A_n, B_n, t_{n+1}, u_n, v_n, \bar{\theta}, \underline{\theta}), \quad n \in N.$$

In this game system, all notation and symbols are used as introduced in section 2. Recall the state space S_n, a separable complete metrizable Borel space; A_n and B_n, the action spaces of players I and II respectively, at time $n \in N$; $\{t_{n+1}\}$ the sequence of transition probability regarded as the law of motion in the game system; the functions

$$u_n : H_n A_n B_n \to R \quad \text{and} \quad v_n : H_n A_n B_n \to R^+ = (0, \infty)$$

which are bounded Borel measurable, respectively, and letting time n goes to infinity, they converge to

$$\lim_{n\to\infty} u_n = u \in R \quad \text{and} \quad \lim_{n\to\infty} v_n = v \in R^+;$$

For each $s_1 \in S_1$, $f = \{f_n\} \in F$ and $g = \{g_n\} \in G$, we assume that the limits of expectations

$$U(f,g)(s_1) = \lim_{n\to\infty} E(u_n, f, g)(s_1)$$

and

$$V(f,g)(s_1) = \lim_{n\to\infty} E(v_n, f, g)(s_1) > 0$$

exist, so that the fraction

$$W(f,g)(s_1) = \frac{U(f,g)(s_1)}{V(f,g)(s_1)}$$

is well defined. For an initial state $s_1 \in S_1$, we define, the **upper** and **lower** value functions of the game (DFG) by

$$\bar{\theta}(s_1) = \inf_{f\in F} \sup_{g\in G} W(f,g)(s_1)$$

and

$$\underline{\theta}(s_1) = \sup_{g\in G} \inf_{f\in F} W(f,g)(s_1).$$

respectively.
Of course $\bar{\theta}(s_1) \geq \underline{\theta}(s_1)$ for all $s_1 \in S_1$, and call the interval $[\underline{\theta}(s_1), \bar{\theta}(s_1)]$ as the **duality gap** of the game (DFG).

<u>**Definition 3.1**</u> The game (DFG) is said to have a **value function** if the duality gap is equal to zero, and we call the common value function the value function

$$\bar{\theta}(s_1) = \underline{\theta}(s_1) = \theta^*(s_1).$$

Furthermore, if there exists $g^* \in G$ such that

$$\bar{\theta}(s_1) = \inf_{f\in F} \sup_{g\in G} W(f,g)(s_1) = \inf W(f,g^*)(s_1),$$

then we call g^* a maximizer [of $W(f,g)(s_1)$] over $g \in G$ for each $f \in F$ in the game (DFG).

Similarly, if there exists $f^* \in F$ such that

$$\underline{\theta}(s_1) = \sup_{g\in G} \inf_{f\in F} W(f,g)(s_1) = \sup_{g\in G} W(f^*,g)(s_1),$$

then $f^* \in F$ is called a minimizer [of $W(f,g)(s_1)$] over $f \in F$ for each $g \in G$ in the game (DFG).

Next, we analyze some relationships between the upper as well as the lower value functions of (DGP$_\theta$) and (DFG). Then we can state some properties for $\bar{T}_\theta(s_1)$ as well as $\underline{T}_\theta(s_1)$ in the following propositions.

Proposition 3.1.

(1) For two parameter functions $\theta_1(s_1)$ and $\theta_2(s_1)$, if $\theta_1(s_1) > \theta_2(s_1) \geq 0$, then $\bar{T}_{\theta_1}(s_1) \leq \bar{T}_{\theta_2}(s_1)$.

(2) If $\bar{T}_\theta(s_1) < 0$, then $\theta(s_1) \geq \bar{\theta}(s_1)$.

(3) If $\bar{T}_\theta(s_1) > 0$, then $\theta(s_1) \leq \bar{\theta}(s_1)$.

(4) If $\theta(s_1) > \bar{\theta}(s_1)$, then $\bar{T}_\theta(s_1) \leq 0$.

(5) If $\theta(s_1) < \bar{\theta}(s_1)$, then $\bar{T}_\theta(s_1) \geq 0$.

Proof. A brief proof is given here.

(1) If $\theta_1(s_1) > \theta_2(s_1) \geq 0$, then for all (f,g)

$$\theta_1(s_1)V(f,g)(s_1) > \theta_2(s_1)V(f,g)(s_1),$$

$$U(f,g)(s_1) - \theta_1(s_1)V(f,g)(s_1) < U(f,g)(s_1) - \theta_2(s_1)V(f,g)(s_1),$$

or

$$T_{\theta_1}(f,g)(s_1) < T_{\theta_2}(f,g)(s_1),$$

so $\bar{T}_{\theta_1}(s_1) = \inf_{f\in F} \sup_{g\in G} T_{\theta_1}(f,g)(s_1) \leq \inf_{f\in F} \sup_{g\in G} T_{\theta_2}(f,g)(s_1) = \bar{T}_{\theta_2}(s_1)$.

(2) If $\bar{T}_\theta(s_1) < 0$, then there exists $\bar{f} \in F$ such that $\sup_{g\in G} T_\theta(\bar{f},g)(s_1) < 0$. It follows that

$$T_\theta(\bar{f},g)(s_1) = U(\bar{f},g)(s_1) - \theta(s_1)V(\bar{f},g)(s_1) < 0.$$

and

$$W(\bar{f},g)(s_1) = \frac{U(\bar{f},g)(s_1)}{V(\bar{f},g)(s_1)} < \theta(s_1),$$

Therefore $\bar{\theta}(s_1) = \inf_{f\in F}\sup_{g\in G} W(f,g)(s_1) < \theta(s_1)$.

(3) If $\bar{T}_\theta(s_1) > 0$, it can be proved as the same line in the proof of (2) to get

$$\bar{\theta}(s_1) \geq \theta(s_1).$$

(4) If $\theta(s_1) > \bar{\theta}(s_1)$, then there exists $\bar{f} \in F$ such that

$$\theta(s_1) > \sup_{g\in G} W(\bar{f},g)(s_1).$$

So $\theta(s_1) > W(\bar{f},g)(s_1)$ for all $g \in G$.

$$T_\theta(\bar{f},g)(s_1) = U(\bar{f},g)(s_1) - \theta(s_1)V(\bar{f},g)(s_1) < 0.$$

Hence $0 \geq \sup_{g\in G} T_\theta(\bar{f},g)(s_1) \geq \inf_{f\in F}\sup_{g\in G} T_\theta(f,g)(s_1) = \bar{T}_\theta(s_1)$.

(5) If $\bar{\theta}(s_1) > \theta(s_1)$, the proof is similar to the case of (4), and get

$$\bar{T}_\theta(s_1) = \inf_{f\in F}\sup_{g\in G} T_\theta(f,g)(s_1) \geq 0.$$

□

Next we can state some properties for $\underline{T}_\theta(s_1)$ by following similar arguments as for $\bar{T}_\theta(s_1)$ in the above proposition.

Proposition 3.2.

(1) If $\theta_1(s_1) > \theta_2(s_1) \geq 0$, then $\underline{T}_{\theta_1}(s_1) \leq \underline{T}_{\theta_2}(s_1)$.

(2) If $\underline{T}_\theta(s_1) < 0$, then $\theta(s_1) \geq \underline{\theta}(s_1)$.

(3) If $\underline{T}_\theta(s_1) > 0$, then $\theta(s_1) \leq \underline{\theta}(s_1)$.

(4) If $\theta(s_1) > \underline{\theta}(s_1)$, then $\underline{T}_\theta(s_1) \leq 0$.

(5) If $\theta(s_1) < \underline{\theta}(s_1)$, then $\underline{T}_\theta(s_1) \geq 0$.

Proof. Using $\underline{T}_\theta(s_1)$ and $\underline{\theta}(s_1)$ instead of $\bar{T}_\theta(s_1)$ and $\bar{\theta}(s_1)$ respectively, we can prove this proposition by similar arguments as in the previous proof. □

4. The Saddle Value Function of the Game (DFG)

Now we can prove the existence theorem for saddle value function in the game (DFG), and the relationship between the games (DFG) and (DGP$_\theta$).

Theorem 4.1. *Suppose that $g^* \in G$ is a maximizer in the game (DFG). Then*

(1) $\bar{\theta}(s_1) = \underline{\theta}(s_1) = \theta^*(s_1)$, *and*

(2) if $\bar{T}_{\theta^*}(s_1) \leq 0$, *then* g^* *is a maximizer in the game (DGP*$_{\theta^*}$*).*

From Theorem 4.1, we can easily derive the following result.
Corollary 4.2. *Suppose that* $(f^*, g^*) \in F \times G$ *is a saddle point of (DFG). Then*

(1) $T_{\theta^*}(f^*, g^*)(s_1) = 0$, *and*

(2) (f^*, g^*) *is a saddle point of the game (DGP*$_{\theta^*}$*).*

A theorem similar to Theorem 4.1 is given as follows.
Theorem 4.3. *Suppose that* $f^* \in F$ *is a minimizer in (DFG). Then,*

(1) $\bar{\theta}(s_1) = \underline{\theta}(s_1) = \theta^*(s_1)$, *and*

(2) if $\underline{T}_{\theta^*}(s_1) \geq 0$, *then* f^* *is also a minimizer in the game (DGP*$_{\theta^*}$*).*

Combine Theorems 4.1~4.3, we can state the main results as follows.

Theorem 4.4. *Suppose that* $\bar{\theta}(s_1) = \underline{\theta}(s_1) = \theta^*(s_1)$.
(1) If $g^* \in G$ *is a maximizer for* $W(f, g)$ *in the game (DGP*$_{\theta^*}$*) with*

$$\inf_{f \in F} T_{\theta^*}(f, g^*)(s_1) = \bar{T}_{\theta^*}(s_1) \geq 0,$$

then, g^* *is also a maximizer for* $W(f, g)$ *in the game (DFG).*
(2) If $f^* \in F$ *is a minimizer for* $W(f, g)$ *in (DGP*$_{\theta^*}$*) with*

$$\sup_{g \in G} T_{\theta^*}(f^*, g)(s_1) = \underline{T}_{\theta^*}(s_1) \leq 0$$

then f^* *is also a minmizer of* $W(f, g)$ *in the game (DFG).*

Corollary 4.5. *Suppose that* $\bar{\theta}(s_1) = \underline{\theta}(s_1) = \theta^*(s_1)$, *and that* $(f^*, g^*) \in F \times G$ *is a saddle point of the game (DGP*$_{\theta^*}$*) with* $T_{\theta^*}(f^*, g^*)(s_1) = 0$. *Then* (f^*, g^*) *is a saddle point of the game (DFG).*

The proof of the corollary is easily obtained from Theorem 4.4.

5. A Remark for further Development

The objective function of a fractional dynamic game is of the form

$$(\star) \quad W(x, y) = \frac{U(x, y)}{V(x, y)}, \quad x \in X, y \in Y.$$

Our problem is to show that $(\star)$ has zero gap under certain conditions. That is,

$$\inf_{x\in X}\sup_{y\in Y} W(x,y) = \sup_{y\in Y}\inf_{x\in X} W(x,y),$$

where X and Y denote the universal strategy spaces in the sense of measurable transition probabilities. If X and Y are assumed to be non-discrete compact strategy spaces for players I and II, then a question arises in the mathematical analysis for deterministic situations that under what conditions on the denominator and the numerator functions, $V(x,y)$ and $U(x,y)$, the fractional function $W(x,y)$ will have a saddle point? There are many authors who investigated this problem deriving minimax theorems with respect to a two-variable function in x and y; see for example [5], [9], [10] and [11]. One can see that these papers in minimax fractional programming are taking x to be discrete as counting functions of y. Further problems are implicit in the fractional function $W(x,y)$.

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