Sperner Matroid and Sperner Map

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Abstract. A reconstruction of a Sperner map from a Sperner matroid is illustrated. As an application of this reconstruction, we give a new proof of a completion theorem of Lovász's matroid version of Sperner's lemma.

The purpose of this note is to give a simple procedure which allows us to retrieve a Sperner map from a Sperner matroid.

1. Reconstruction of Sperner map from Sperner matroid

Let $K$ be a triangulation of a $d$-simplex $a_0a_1 \ldots a_d$ in a Euclidean space, and $V(K)$ the vertex set of $K$. A map $\varphi : V(K) \to \{0, 1, \ldots, d\}$ is said to be a Sperner map if for each $i_0, i_1, \ldots, i_k$ with $0 \leq i_0 < i_1 < \ldots < i_k \leq d$ and for each $v \in V(K) \cap a_0a_1 \ldots a_i$, $\varphi(v) \in \{i_0, i_1, \ldots, i_k\}$. A matroid $M$ on $V(K)$ is called a Sperner matroid over $K$ if for each $S \subset \{a_0, a_1, \ldots, a_d\}$ and for each $v \in V(K) \cap \text{conv}(S)$, $v \in \text{cl}_M(S)$, where $\text{conv}(S)$ stands for the convex hull of $S$ and $\text{cl}_M(S)$ denotes the closure of $S$ in $M$. Let $M$ be a Sperner matroid over $K$ such that $\{a_0, a_1, \ldots, a_d\}$ forms a basis of $M$. Put

$$F_j \equiv \text{cl}_M(\{a_0, a_1, \ldots, a_j\}) \quad (j = 0, 1, \ldots, d).$$

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Let us define \( \varphi : V(K) \rightarrow \{0,1,\ldots,d\} \) by setting

\[
\varphi(v) = 0 \quad \text{if} \quad v \in F_0,
\]

\[
\varphi(v) = j \quad \text{if} \quad v \in F_j \setminus F_{j-1} \quad (j = 1,2,\ldots,d).
\]

The problem that we consider in this paper is the following: *Under what condition is \( \varphi \) a Sperner map?* It is clearly necessary that the \((d-1)\)-face \( a_1a_2\ldots a_d \) contains no loops. To see this, if \( a_1a_2\ldots a_d \) contains a loop \( v \) of \( M \), then \( \varphi(v) = 0 \) and \( v \in V(K) \cap a_1a_2\ldots a_d \). As \( \varphi \) is a Sperner map, we have \( \varphi(v) \in \{1,2,\ldots,d\} \), in contradiction. What is perhaps surprising is that this condition is also sufficient. Indeed, we have

**Theorem 1.** Let \( M \) be a Sperner matroid over a triangulation \( K \) of a \( d \)-simplex \( a_0a_1\ldots a_d \) such that \( \{a_0,a_1,\ldots,a_d\} \) forms a basis. Put \( F_j \equiv c_M(\{a_0,a_1,\ldots,a_j\}) \) \((j = 0,1,\ldots,d)\), and let \( \varphi : V(K) \rightarrow \{0,1,\ldots,d\} \) be defined by

\[
\varphi(v) = 0 \quad \text{if} \quad v \in F_0, \quad \varphi(v) = j \quad \text{if} \quad v \in F_j \setminus F_{j-1} \quad (j = 1,2,\ldots,d).
\]

Then \( \varphi \) is a Sperner map if and only if the \((d-1)\)-face \( a_1a_2\ldots a_d \) contains no loops of \( M \).

The proof of Theorem 1 is based on the following:

**Lemma.** Let \( B \) be a basis of a matroid \( M \). Suppose

(a) \( S \subset T \subset B \),

(b) \( X \subset B \), \( X \cap T = \emptyset \),

(c) \( y \in c_M(T) \setminus c_M(S) \).

Then \( y \in c_M(T \cup X) \setminus c_M(S \cup X) \).

2. A sign function

Let \( K \) be a triangulation of a \( d \)-simplex \( a_0a_1\ldots a_d \) in a Euclidean space, \( M \) a Sperner matroid over \( K \), \( B \equiv (a_0,a_1,\ldots,a_d) \) an ordered basis of \( M \). Let \( \Lambda_B : K \rightarrow \{-1,0,1\} \). We define \( \Lambda_B(v_0v_1\ldots v_d) = 1 \) (resp. \(-1\)) if \( v_0 \in F_0 \) and \( v_j \in F_j \setminus F_{j-1} \) \((j = 1,2,\ldots,d)\), where \( F_j \equiv c_M(\{v_0,v_1,\ldots,v_j\}) \) \((j = 0,1,\ldots,d)\), and \( \det(\alpha_{ij}) > 0 \) (resp. \( \det(\alpha_{ij}) < 0 \)), where

\[
\begin{pmatrix}
v_0 \\
\vdots \\
v_d
\end{pmatrix} =
\begin{pmatrix}
\alpha_{00} & \ldots & \alpha_{0d} \\
\vdots & \ddots & \vdots \\
\alpha_{d0} & \ldots & \alpha_{dd}
\end{pmatrix}
\begin{pmatrix}
a_0 \\
\vdots \\
ad
\end{pmatrix},
\]

\[
d \sum_{j=0}^{d} \alpha_{ij} = 1 \quad (0 \leq i \leq d). \quad (*)
\]

We define \( \Lambda_B(v_0v_1\ldots v_d) = 0 \) otherwise. Now, let \( \varphi : V(K) \rightarrow \{0,1,\ldots,d\} \). We call a \( d \)-simplex \( v_0v_1\ldots v_d \in K \) positively (resp. negatively) completely labelled if \( \varphi(v_j) = j \) \((j = 0,1,\ldots,d)\), and \( \det(\alpha_{ij}) > 0 \) (resp.
$\det(\alpha_{ij}) < 0$, where the matrix $(\alpha_{ij})$ is given in $(*)$. A $d$-simplex of $K$ is completely labelled if it is positively or negatively completely labelled. It is obvious that $v_0v_1 \ldots v_d \in K$ is completely labelled if and only if 
\{\varphi(v_0), \varphi(v_1), \ldots, \varphi(v_d)\} = \{0, 1, \ldots, d\}. The celebrated Sperner lemma asserts that if $\varphi$ is a Sperner map then $\sharp\{\sigma \in K ; \ \sigma \text{ is completely labelled }\} \equiv 1 \pmod{2}$. The oriented Sperner lemma states that

$$\sharp\{\sigma \in K ; \ \sigma \text{ is positively completely labelled }\} - \sharp\{\sigma \in K ; \ \sigma \text{ is negatively completely labelled }\} = 1. $$

By Theorem 1 and the oriented Sperner lemma, we have

**Theorem 2.** Let $K$ be a triangulation of a $d$-simplex $a_0a_1 \ldots a_d$ in a Euclidean space, and $M$ a Sperner matroid over $K$ such that the $(d-1)$-face $a_1a_2 \ldots a_d$ contains no loops of $M$. If $B \equiv (a_0,a_1,\ldots,a_d)$ is an ordered basis of $M$, then

$$\sum_{\sigma \in K} \Lambda_B(\sigma) = 1. $$

Theorem 2 was recently obtained by the authors in [4] with a completely different proof. An example given in [4] shows that the condition “the $(d-1)$-face $a_1a_2 \ldots a_d$ contains no loops” cannot be dispensed with.

By disregarding orientation in Theorem 2, we have

**Theorem 3.** Under the assumptions of Theorem 2,

$$\sum_{\sigma \in K} |\Lambda_B(\sigma)| \equiv 1 \pmod{2}. $$

Theorem 3 is an extension of Lovász's theorem. It complements Lovász's theorem in two aspects: one concerns an arbitrary matroid while the other is the assertion of oddness. Indeed, Lovász[2] proved the following:

**Theorem 4** (L. Lovász). Let $K$ be a triangulation of a $d$-simplex $a_0a_1 \ldots a_d$ in a Euclidean space, and $M$ a Sperner matroid over $K$ such that $M$ contains no loops. If \{a_0,a_1,\ldots,a_d\} is a basis of $M$, then $K$ has a $d$-simplex $v_0v_1 \ldots v_d$ such that \{v_0, v_1, \ldots, v_d\} is also a basis of $M$.

Finally, let us mention that it is perhaps worth developing a general matroid version which contains multiple balanced Sperner lemma[6], combinatorial Lefschetz fixed-point formula[5], and multiple combinatorial Stokes' theorem[3].
References


