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DECOMPOSABILITY OF NONSATURATED FRACTAL GEOMETRIC DYNAMICAL SYSTEMS

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1. Introduction

In this note, we show the method of visualizing the set of all orbits of fractal geometric dynamical systems. It is well known that a fractal geometric dynamical system defined on $\mathbb{R}^2$ can be exactly characterized by a system of contractive affine transformations on $\mathbb{R}^2$ and by a production rule which represents the sequential evolution of orbits. It is quite natural that if we exchange another production rule for the previously given rule without changing the previously given system of contractive affine transformations, we can construct another fractal geometric dynamical system.

In general, for a given fractal geometric dynamical system, a certain fractal geometric dynamical system is called the enveloping dynamical system, if the set of all orbits corresponding to this system, on condition that the contractive affine transformation system is the same one as previously given, and the previously given system can be topologically embedded into this system. Especially, if the enveloping dynamical system is exactly equal to the previously given system, then this system is said to be saturated. Moreover, if a fractal geometric dynamical system can be represented as a disjoint union of several saturated fractal geometric subdynamical systems, then this system is said to be decomposable and these saturated subdynamical systems are called the components of this system. It is easy to prove that decomposable fractal geometric dynamical systems are nonsaturated, but as we see later, not all nonsaturated fractal geometric dynamical systems are decomposable. It is well known that the topological structure of saturated systems are more simple than that of nonsaturated ones.

Let k be a positive integer and let $\mathcal{M}_k$ be the set of all positive integers which are less than $k+1$. $f_1, \dots, f_k$ denote contraction mappings whose contraction coefficients are $r_1, \dots, r_k$, respectively. Then, it is known that there uniquely exists a compact subset K of X satisfying

$$K = f_1(K) \cup \cdots \cup f_k(K).$$

Here, we assume that the intersection of $f_i(K)$ and $f_j(K)$ is empty if $i \neq j$ holds. Let ϕ_0 be the mapping on K with values in $\mathcal{M}_k$ defined as

$$x \in f_{\phi_0(x)}(K), \quad x \in K.$$

For any positive integer n, let ϕ_n be the mapping on K with values in $\mathcal{M}_k$ defined as

$$x \in f_{\phi_n(x)}\left(f_{\phi_{n-1}(x)}\left(\cdots f_{\phi_0(x)}(K)\ldots\right)\right), \quad x \in K,$$

inductively. Then, the address mapping Φ on K with values in $\mathcal{M}_k^{\mathbb{N}}$ is defined as

$$\Phi(x) = \{\phi_n(x)\}_{n=0}^{\infty}, \quad x \in K.$$

Here, $\Phi(x)$ is called the address of x and it is clear that Φ is injective. Since K is a subset of X, K can be equipped with the relative topology induced by (X,d). It is clear that the relative topology over K is equal to Tychonoff's product topology over $\mathcal{M}_k^{\mathbb{N}}$. Let D be a compact subset of K and let T be a mapping on D with values in D. Then (D,T) is called a fractal geometric dynamical system if T satisfies the following condition:

$$\phi_{n+1}(x) = \phi_n(Tx), \quad x \in D,\, n \in \mathbb{N}.$$

2. Examples

In this section, we present two types of examples showing graphical relations between a fractal geometric dynamical system and the corresponding enveloping dynamical system. More exactly speaking, the first example shows a nonsaturated fractal geometric dynamical system which is not decomposable. The second example shows a decomposable fractal geometric dynamical system satisfying that the total number of all corresponding components is equal to two.

Example 1. Let f_1, f_2, f_3, f_4 and f_5 be the five affine transformations on $\mathbb{R}^2$ with values in $\mathbb{R}^2$ which are defined as

$$f_1 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto 0.3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.35 \\ 0.35 \end{bmatrix},$$
$$f_2 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto 0.3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -0.35 \\ 0.35 \end{bmatrix},$$
$$f_3 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto 0.3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -0.35 \\ -0.35 \end{bmatrix},$$
$$f_4 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto 0.3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.35 \\ -0.35 \end{bmatrix},$$
$$f_5 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto 0.3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

respectively, and let D_{125}, D_{235}, D_{345} and D_{415} be the compact subsets which are defined as

$$\begin{aligned} D_{125} &= f_1(D_{125}) \cup f_2(D_{125}) \cup f_5(D_{125}), \\ D_{235} &= f_2(D_{235}) \cup f_3(D_{235}) \cup f_5(D_{235}), \\ D_{345} &= f_3(D_{345}) \cup f_4(D_{345}) \cup f_5(D_{345}), \\ D_{415} &= f_4(D_{415}) \cup f_1(D_{415}) \cup f_5(D_{415}), \end{aligned}$$

respectively. Then, the set of all orbits of the fractal geometric dynamical system constructed from the above production rule, which is denoted by $D_{125} \cup D_{235} \cup D_{345} \cup D_{415}$, can be illustrated with FIGURE 1.

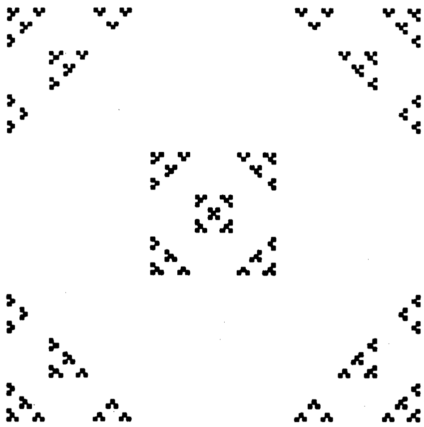

FIGURE 1

Let K_{12345} be the compact subsets which is defined as

$$K_{12345} = f_1(K_{12345}) \cup f_2(K_{12345}) \cup f_3(K_{12345}) \cup f_4(K_{12345}) \cup f_5(K_{12345}).$$

Then the set of all orbits of the fractal geometric dynamical system constructed from the above production rule, which is equal to K_{12345}, can be illustrated with FIGURE 2.

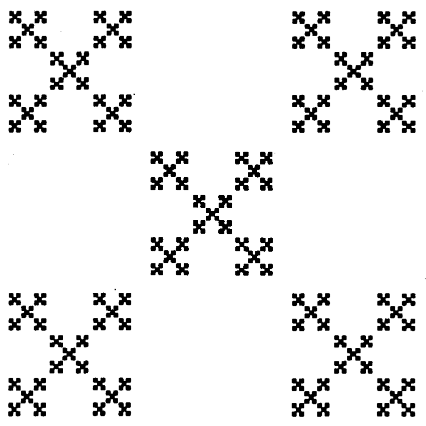

FIGURE 2

Here K_{12345} is equal to the enveloping dynamical system of $D_{125} \cup D_{235} \cup D_{345} \cup D_{415}$. The total number of all components of K_{12345} is equal to one.

Example 2. Let g_1, g_2, g_3, g_4, g_5 and g_6 be the six affine transformations on $\mathbb{R}^2$ with values in $\mathbb{R}^2$ which are defined as

$$g_1 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto 0.45 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$g_2 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto 0.45 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.55 \end{bmatrix},$$
$$g_3 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto 0.45 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.55 \\ 0 \end{bmatrix},$$
$$g_4 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto 0.45 \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix},$$
$$g_5 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto 0.45 \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.5 \\ -0.05 \end{bmatrix},$$
$$g_6 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \longmapsto 0.45 \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -0.05 \\ 0.5 \end{bmatrix},$$

respectively, and let E_{123} and E_{456} be the compact subsets which are defined as

$$\begin{aligned} E_{123} &= g_1(E_{123}) \cup g_2(E_{123}) \cup g_3(E_{123}), \\ E_{456} &= g_4(E_{456}) \cup g_5(E_{456}) \cup g_6(E_{456}), \end{aligned}$$

respectively. Then the sets of all orbits of the fractal geometric dynamical system constructed from the above production rule, which is denoted by $E_{123} \cup E_{456}$, can be illustrated with FIGURE 3. Since E_{123} and E_{456} are saturated and $E_{123} \cap E_{456}$ is empty, $E_{123} \cup E_{456}$ is decomposable.

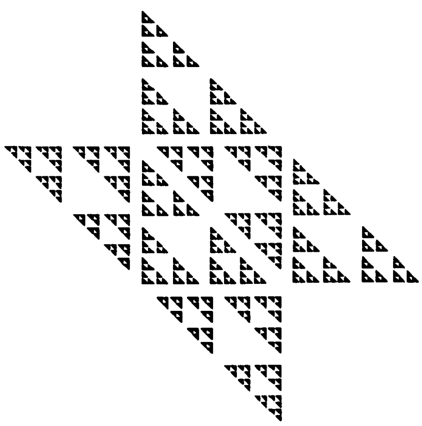

FIGURE 3

Let h_1, h_2, h_3, h_4, h_5 and h_6 be the six affine transformations on $\mathbb{R}^2$ with values in $\mathbb{R}^2$ which are defined as

$$
\begin{aligned}
h_1 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &\longmapsto 0.2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.3 \\ 0 \end{bmatrix}, \\
h_2 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &\longmapsto 0.2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.3 \\ 0.5 \end{bmatrix}, \\
h_3 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &\longmapsto 0.2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0.8 \\ 0 \end{bmatrix}, \\
h_4 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &\longmapsto 0.2\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -0.3 \\ 0 \end{bmatrix}, \\
h_5 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &\longmapsto 0.2\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -0.3 \\ -0.5 \end{bmatrix}, \\
h_6 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &\longmapsto 0.2\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -0.8 \\ 0 \end{bmatrix},
\end{aligned}
$$

respectively, and let F_{123} and F_{456} be the compact subsets which are defined as

$$
\begin{aligned}
F_{123} &= h_1(F_{123}) \cup h_2(F_{123}) \cup h_3(F_{123}), \\
F_{456} &= h_4(F_{456}) \cup h_5(F_{456}) \cup h_6(F_{456}),
\end{aligned}
$$

respectively. Then the sets of all orbits of the fractal geometric dynamical system constructed from the above production rule, which is denoted by $F_{123} \cup F_{456}$, can be illustrated with FIGURE 4. Since $F_{123} \cup F_{456}$ is homeomorphic to $E_{123} \cup E_{456}$, $F_{123} \cup F_{456}$ is also decomposable.

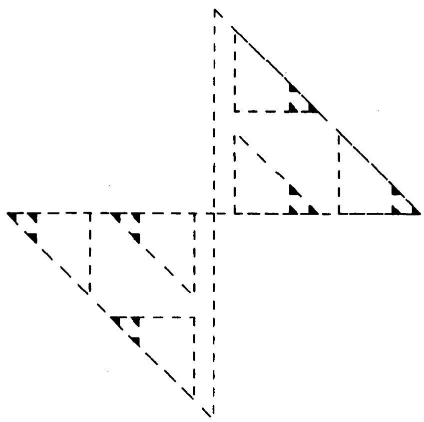

FIGURE 4

Let L_{123456} be the compact subsets which is defined as

$$
L_{123456} = \bigcup_{i=1}^{6} h_i(L_{123456}).
$$

Then, the set of all orbits of the fractal geometric dynamical system constructed from the above production rule, which is equal to L_{123456}, can be illustrated with FIGURE 5.

Here L_{123456} is equal to the enveloping dynamical system of $F_{123} \cup F_{456}$. The total number of all components of $F_{123} \cup F_{456}$ is equal to two.

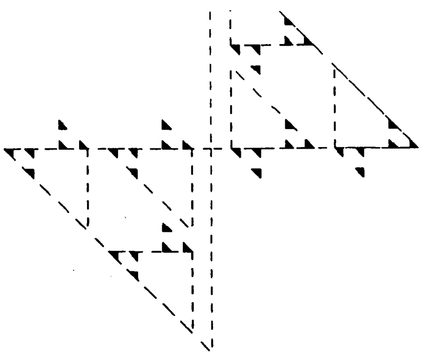

FIGURE 5

REFERENCES

[1] R. L. Adler, A. G. Konheim and M. H. McAndrew: Topological entropy, *Trans. Amer. Math. Soc.*, **114**, 309–319(1965).

[2] S. Akashi: Embedding of expansive dynamical systems into symbolic dynamical systems, *Rep. Math. Phys.*, **46**, 11–14(2000).

[3] M. F. Barnsley: *Fractal Everywhere, 2nd Edition*, AP Professional, Boston, 1993.

[4] R. S. Ingarden, A. Kossakowski and M. Ohya: *Infomation Dynamics and Open Systems, -classicl and Quantum Approach-*, Fundamental Theories of Physcis, vol.86, Kluwer Academic Publishers, Dordrecht, 1997.