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# Discrete Mittag-Leffler function and its applications

# 離散ミッタークレフラー関数とその応用

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**Abstract**

Discrete and $q$-discrete analogues of Mittag-Leffler function are presented. Their relations to fractional difference are also investigated. Applications of these functions to numerical analysis and integrable systems are also made.

## 1 Fractional derivative

Fractional derivative goes back to the Leipniz's note in his list to L'Hospital in 1695 and we now have many definitions of fractional derivatives [11, 13]. In the last few decades, many authors pointed out that derivatives and integrals of fractional order, especially 1/2-derivative, are very suitable for the description of physical phenomena (See ref. [14] for example.).

We first define a fractional integral operator $I^a$ as follows.

**Definition 1** *Let $a$ be a nonnegative real number and $u(t)(0 < t)$ be piecewise continuous on $(0, \infty)$ and integrable on any subinterval $[0, \infty)$. Then for $t > 0$, we call*

$$I^a u(t) = \int_0^t K(a; t-s) u(s) \mathrm{d}s \tag{1}$$

$$I^0 u(t) = u(t) \tag{2}$$

*the fractional integral of $u$ of order $a$. $K(a;t)$ is a monomial given by*

$$K(a;t) \equiv \frac{t^{a-1}}{\Gamma(a)} \quad (t > 0,\ a > 0). \tag{3}$$

Fractional derivatives of order $a > 0$ are defined by a combination of normal derivative and fractional integral in the following two manners.

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**Definition 2** *Let $m$ be a positive integer such as $m-1 < a \leq m$ and $u(t)$ be a given function which satisfies the conditions in the previous Definition 1 and is $m$ times continuously differentiable. Then, its fractional derivative of order $a$ is defined by*

$$D^a u(t) \equiv (I^{m-a} D^m u)(t) = \int_0^t K(m-a; t-s) u^{(m)}(s) \mathrm{d}s \tag{4}$$

**Definition 3** *For the same $a, m, u(t)$ in the previous Definition 2, its derivative of order $a$ is defined by*

$$D^a u(t) \equiv (D^m I^{m-a} u)(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^m \int_0^t K(m-a; t-s) u(s) \mathrm{d}s. \tag{5}$$

These two definitions are called Caputo and Riemann-Liouville fractional derivatives, respectively. We here adopt Caputo's definition 3.

The Mittag-Leffler function,

$$E_a(z) = \sum_{j=0}^{\infty} \frac{z^j}{\Gamma(aj+1)} \quad (a > 0, z \in \mathbb{C}) \tag{6}$$

was proposed by Mittag-Leffler [9] in 1903 as an entire function whose order can be calculated exactly. Afterwards, it was clarified that the Mittag-Leffler function also plays an important role in fractional calculus (See refs. [8, 10] for example).

The main purpose of this paper is to discretize the Mittag-Leffler function and to investigate its relation to fractional difference proposed by Hirota in 1991 [6]. In section 2, we elaborate on the property of the Mittag-Leffler function concerning its relation to fractional derivative. Especially, we review Kametaka's result [7] in which a solution to a certain fractional differential equation is given by means of the Mittag-Leffler function. Section 3 and 4 is devoted to discretization and $q$-discretizations of the Mittag-Leffer function. It is also shown that they are eigen functions of a certain fractional difference and $q$-difference operators, respectively. Two applications are given in the final 2 sections. First is an application to numerical computation of fractional differential equations and the second is to integrable systems, in which a new type of discrete nonlinear integrable equation equipped with fractional difference is proposed.

## 2 Half-order differential equation and Mittag-Leffle function

We start with a fractional differential equation,

$$\begin{cases}(D + aD^{1/2} + b)u(t) = \beta K(1/2;t) + f(t) \quad (t > 0),\\ u(0) = \alpha,\end{cases} \tag{7}$$

or equivalently

$$\begin{cases}u'(t) + a\displaystyle\int_0^t K(1/2;t-s)u'(s)\mathrm{d}s + bu(t) = \beta K(1/2;t) + f(t) \quad (t > 0),\\ u(0) = \alpha,\end{cases} \tag{8}$$

where $p, q$ are positive constants and $\{\alpha, \beta, f(t)\}$ are given data. Equation (7) describes a motion of a particle in a fluid and the unknown function $u(t)$ stands for a relative velocity of a particle with respect to its surrounding fluid. It was known from experimental result that $u(t)$ decays to 0 at the order of $O(1/\sqrt{t})$. However its mathematical proof had not been given.

In around 1986, Kametaka [7] gave a mathematical proof on the above fact by considering the following expansion of $u(t), f(t)$.

$$\begin{cases}u(t) = \displaystyle\sum_{j=0}^{\infty} u_j K\left(\frac{j+2}{2};t\right) = u_0 + u_1K(3/2;t) + u_2K(2;t) + \cdots\\ f(t) = \displaystyle\sum_{j=0}^{\infty} f_j K\left(\frac{j+2}{2};t\right) = f_0 + f_1K(3/2;t) + f_2K(2;t) + \cdots\end{cases}$$

Substituting the above expression into eq. (7), we have

$$\begin{aligned}u_1K(1/2;t) + \sum_{j=0}^{\infty}(u_{j+2} + au_{j+1} + bu_j)K\left(\frac{j+2}{2};t\right)\\ = \beta K(1/2;t) + \sum_{j=0}^{\infty} f_jK\left(\frac{j+2}{2};t\right)\end{aligned} \tag{9}$$

and obtain the following linear difference equation.

$$u_{j+2} + au_{j+1} + bu_j = f_j, \ u_0 = \alpha, \ u_1 = \beta \tag{10}$$

Hence, the solution to eq. (7) is given by

$$\begin{aligned}u(t) &= \frac{-\alpha\lambda_+\lambda_-}{\lambda_+-\lambda_-}\left(\lambda_+^{-1}E_{1/2}(\lambda_+t^{1/2})-\lambda_-^{-1}E_{1/2}(\lambda_-t^{1/2})\right)\\ &+\frac{\beta}{\lambda_+-\lambda_-}\left(E_{1/2}(\lambda_+t^{1/2})-E_{1/2}(\lambda_-t^{1/2})\right)\\ &+\frac{1}{\lambda_+-\lambda_-}\left(\lambda_+E_{1/2}(\lambda_+t^{1/2})-\lambda_-E_{1/2}(\lambda_-t^{1/2})\right)*f(t) \qquad (11)\\ \lambda_\pm &= \frac{-a\pm\sqrt{a^2-4b}}{2}\end{aligned}$$

The function $E_{1/2}(\lambda_\pm t^{1/2})$ is the Mittag-Leffler function and is also expressed as follows.

$$E_{1/2}(\lambda_\pm t^{1/2}) = \sum_{j=0}^{\infty}\lambda_\pm^j K(j/2+1;t), \quad (z\in\mathbb{C}) \tag{12}$$

It is a well-known result that if $0<a<2, a\neq 1$ the Mittag-Leffler function possesses the following asymptotic behavior [12].

$$E_a(z)\sim -\sum_{k=1}^{\infty}\frac{1}{\Gamma(1-ak)}z^{-k} \quad \left(|z|\to\infty,\ \frac{a\pi}{2}<|\arg z|\leq\pi\right) \tag{13}$$

Noticing $\mathrm{Re}\lambda_\pm<0$, one can conclude that $u(t)$ decays to 0 at the order of $O(1/\sqrt{t})$.

It can be confirmed through simple calculations that the Mittag-Leffer function,

$$u(t)=\sum_{j=0}^{\infty}\lambda^j K(aj+1;t)=E_a(\lambda t^a) \tag{14}$$

is an eigen function of Caputo's fractional derivative [8],

$$D^a u(t)=\lambda u(t) \quad (t\geq 0). \tag{15}$$

In the next section, we consider discrete analogue of Mittag-Leffler function, which preserves the property (15).

# 3 Fractional difference and discrete Mittag-Leffler function

We here give a definition of fractional difference operator and its eigenfunction. Before going to its definition, let us introduce fundamental functions

$M(a;n)$ defined by

$$M(a;n) = \varepsilon^{a-1}\begin{pmatrix} n+a-2 \\ n-1 \end{pmatrix} = \frac{1}{\Gamma(a)}\varepsilon^{a-1}\frac{\Gamma(n+a-1)}{\Gamma(n)} \quad (a>0, n\in\mathbb{Z}_{\geq 1}), \tag{16}$$

$$M(a;0) = \begin{cases} 1 & (a=1) \\ 0 & (a\neq 1) \end{cases} \tag{17}$$

where $\varepsilon$ is an interval length and $\begin{pmatrix} a \\ n \end{pmatrix}$ $(a\in\mathbb{R}, n\in\mathbb{Z})$ stands for a binomial coefficient defined by

$$\begin{pmatrix} a \\ n \end{pmatrix} = \begin{cases} \dfrac{a(a-1)\cdots(a-n+1)}{n!} = \dfrac{\Gamma(a+1)}{\Gamma(a-n+1)\Gamma(n+1)} & (n>0) \\ 1 & (n=0) \\ 0 & (n<0) \end{cases}$$

This function satisfies the following lemma.

**Lemma 1** *The following relations hold.*

$$\Delta_{-n}M(a+1;n) = \varepsilon^{-1}(M(a+1;n) - M(a+1;n-1)) = M(a;n) \quad (a>0) \tag{18}$$

**Proof of Lemma 1**: This is proved by using the relation $\Gamma(x+1) = x\Gamma(x)$ as follows.

$$\begin{aligned} &\varepsilon^{-1}(M(a+1;n) - M(a+1;n-1)) \\ &= \frac{\varepsilon^{a-1}}{\Gamma(a+1)}\left(\frac{(n+a-1)\Gamma(n+a-1)}{\Gamma(n)} - \frac{(n-1)\Gamma(n+a-1)}{\Gamma(n)}\right) \\ &= \frac{a\varepsilon^{a-1}}{\Gamma(a+1)}\frac{\Gamma(n+a-1)}{\Gamma(n)} = M(a;n). \end{aligned}$$

■

Next we go to the definition of fractional difference. Hirota [6] took the first $n$ terms of Taylor series of $\Delta_{-n}^{\alpha} = \varepsilon^{-\alpha}(1-E^{-1})^{\alpha}$ and gave the following definition.

**Definition 4** *Let $\alpha \in \mathbb{R}$. Then difference operator of order $\alpha$ is defined by*

$$\Delta_{-n}^{\alpha} u_n = \begin{cases} \varepsilon^{-\alpha} \sum_{j=0}^{n-1} \begin{pmatrix} \alpha \\ j \end{pmatrix} (-1)^j u_{n-j} & \alpha \neq 1, 2, \cdots \\ \varepsilon^{-m} \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} (-1)^j u_{n-j} & \alpha = m \in \mathbb{Z}_{>0} \end{cases} \tag{19}$$

It should be noted that Diaz, Osler [4] gave another definition of fractional difference,

$$\Delta_{-n}^{\alpha} u(t) = \varepsilon^{-\alpha} \sum_{j=0}^{\infty} \begin{pmatrix} \alpha \\ j \end{pmatrix} (-1)^j u(t + (\alpha - j)\varepsilon) \tag{20}$$

We here adopt another difference oparator $\Delta_{*,-n}^{\alpha}$ by modifying Hirota's operator.

**Definition 5** *Let $\alpha \in \mathbb{R}$ and $m$ be an integer such that $m - 1 < \alpha \leq m$. We define difference operator of order $\alpha$, $\Delta_{*,-n}^{\alpha}$, by*

$$\Delta_{*,-n}^{\alpha} u_n \equiv \begin{cases} \Delta_{-n}^{\alpha-m} \Delta_{-n}^{m} u_n & \\ \quad = \varepsilon^{m-\alpha} \sum_{j=0}^{n-1} \begin{pmatrix} \alpha - m \\ j \end{pmatrix} (-1)^j \Delta_{-k}^{m} u_k |_{k=n-j} & (\alpha > 0) \\ u_n & (\alpha = 0) \\ \Delta_{-n}^{\alpha} u_n & (\alpha < 0) \end{cases} \tag{21}$$

We define a new function,

$$F_p(\lambda, n) = \sum_{j=0}^{\infty} \lambda^j M(pj + 1; n). = \sum_{j=0}^{\infty} \lambda^j \varepsilon^{pj} \frac{\Gamma(n + pj)}{\Gamma(pj + 1)\Gamma(n)} \tag{22}$$

**Remark 1** *Putting $p = 1$ in the above definition, we have a discrete exponential function.*

$$\begin{aligned} F_1(\lambda, n) &= \sum_{j=0}^{\infty} \lambda^j \varepsilon^j \frac{\Gamma(n + j)}{\Gamma(j + 1)\Gamma(n)} \\ &= \sum_{j=0}^{\infty} (\lambda\varepsilon)^j \begin{pmatrix} n + j - 1 \\ j \end{pmatrix} \\ &= \sum_{j=0}^{\infty} (-\lambda\varepsilon)^j \begin{pmatrix} -n \\ j \end{pmatrix} = (1 - \lambda\varepsilon)^{-n}. \end{aligned} \tag{23}$$

**Remark 2** *In the limit of $\varepsilon \to 0, n \to \infty$ with $t = n\varepsilon$ fixed, the function $F_p(\lambda, n)$ converges to the Mittag-Leffler function.*

$$F_p(\lambda, n) \to \sum_{j=0}^{\infty} \frac{\lambda^j t^{pj}}{\Gamma(pj+1)} = E_p(\lambda t^p). \tag{24}$$

The following theorem states that $F_p(\lambda; n)$ is an eigen-function of fractional difference operator $\Delta^p_{*,-n}$.

**Theorem 1** *If $p > 0$, the function $F_p(\lambda, n)$ satisfies the following relation.*

$$\Delta^p_{*,-n} F_p(\lambda, n) = \lambda F_p(\lambda, n) \tag{25}$$

**Proof of Theorem 1**: Let $m$ be an integer such that $m - 1 < p \leq m$. Then we have

$$\begin{aligned}
\Delta^p_{*,-n} F_p(\lambda, n) &= \Delta^p_{*,-n}\left(1 + \sum_{j=1}^{\infty} \lambda^j M(pj+1; n)\right) \\
&= \Delta^p_{*,-n} \sum_{j=1}^{\infty} \lambda^j M(pj+1; n) \\
&= \Delta^{p-m}_{-n} \sum_{j=1}^{\infty} \lambda^j \Delta^m_{-n} M(pj+1; n) \\
&= \sum_{j=1}^{\infty} \lambda^j \Delta^{p-m}_{-n} M(pj+1-m; n)
\end{aligned} \tag{26}$$

Each summand in the above equation is given by

$$\begin{aligned}
\Delta^{p-m}_{-n} M(pj+1-m; n) &= \sum_{k=0}^{n-1} \binom{p-m}{k} (-1)^k M(pj+1-m; n-k) \\
&= \sum_{k=0}^{n-1} \binom{p-m}{k} (-1)^k \binom{pj-m+n-k-1}{n-k-1} \\
&= \sum_{k=0}^{n-1} \binom{p-m}{k} (-1)^{n-1} \binom{-pj+m-1}{n-k-1} \\
&= (-1)^{n-1} \binom{p-pj-1}{n-1} \\
&= \binom{n+pj-p-1}{n-1} = M(pj-p+1; n),
\end{aligned} \tag{27}$$

where we have employed an upper nagation rule twice and a Vandermonde convolution rule of binomial coefficients. Therefore, substitution of eq. (27) into eq. (26) gives

$$\begin{aligned}\Delta^{p}_{*,-n}F_p(\lambda,n) &= \sum_{j=1}^{\infty}\lambda^j M(pj-p+1;n)\\ &= \sum_{j=0}^{\infty}\lambda^{j+1}M(pj+1;n) = \lambda F_p(\lambda,n)\end{aligned} \tag{28}$$

which completes the proof. ■

**Remark 3** *It should be noted that the Mittag-Leffler function satisfies more abundant properties other that its relation to fractional derivative, as can be observed in ref. [12]. However, it is unknown whether the function $F_p(\lambda;n)$ proposed here also satisfies such properties.*

# 4 Fractional $q$-difference and $q$-Mittag-Leffler function

## 4.1 Fractional $q$-difference

In this section, we present fractional $q$-addition and $q$-difference operators and investigate their properties.

Before getting onto the main subject, we first give definitions of $q$-number, $q$-binomial coefficient and $q$-difference operator, together with their properties, which are required in this paper.† Let $q$ be a given complex number. Throughout this paper, we impose the assumption,

$$|q| > 1. \tag{29}$$

We introduce $q$-number $[a]_q$ defined by

$$[a]_q = \frac{q^a-1}{q-1}, \tag{30}$$

we here rewrite $[a]_q$ as $[a]$ for the sake of simplicity. By making use of the $q$-number, $q$-binomial coefficient is given as follows.

$$\begin{bmatrix} x \\ n \end{bmatrix} = \frac{[x][x-1]\cdots[x-n+1]}{[n]!} = \frac{[x][x-1]\cdots[x-n+1]}{[n][n-1]\cdots[1]} \tag{31}$$

†For details of $q$-analysis, see ref. [2] for example.

We here list some important properties of $q$-number and $q$-binomial coefficient used in future.

$$[-x] = -q^{-x}[x] \tag{32}$$

$$\begin{bmatrix} -x \\ n \end{bmatrix} = (-1)^n q^{-nx-\frac{1}{2}n(n-1)} \begin{bmatrix} x+n-1 \\ n \end{bmatrix} \tag{33}$$

$$\begin{bmatrix} x \\ n \end{bmatrix} - \begin{bmatrix} x-1 \\ n \end{bmatrix} = q^{x-n} \begin{bmatrix} x-1 \\ n-1 \end{bmatrix} \tag{34}$$

$$\begin{bmatrix} x \\ n \end{bmatrix} - \begin{bmatrix} x-1 \\ n-1 \end{bmatrix} = q^{n} \begin{bmatrix} x-1 \\ n \end{bmatrix} \tag{35}$$

$$\sum_{k=0}^{n} \begin{bmatrix} x \\ n-k \end{bmatrix} \begin{bmatrix} y \\ k \end{bmatrix} q^{k^2-nk+kx} = \begin{bmatrix} x+y \\ n \end{bmatrix} \tag{36}$$

We here adopt backward $q$-difference operator $\Delta_q$ defined by

$$\Delta_q f(x) = \frac{f(x) - f(q^{-1}x)}{(1-q^{-1})x} \tag{37}$$

Through dependent and independent variable transformations

$$x = q^n, f(x) = f(q^n) = f_n, \tag{38}$$

the $q$-difference operator in eq. (37) is rewritten equivalently as

$$\Delta_q f_n = \frac{f_n - f_{n-1}}{q^n - q^{n-1}}. \tag{39}$$

We next introduce a fractional $q$-addition operator $I_q^\alpha$ defined as follows.

**Definition 6** *Let $\alpha$ be a non-negative real number and $\{f_n\}$ is a given complex sequence. Then a q-addition operator of fractional order $\alpha$ for $\{f_n\}$ is defined by*

$$\begin{aligned} I_q^\alpha f_n &= q^{(n-1)\alpha}(q-1)^\alpha \sum_{k=0}^{n-1} (-1)^k \begin{bmatrix} -\alpha \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} f_{n-k} \quad (\alpha > 0, n \geq 1) \\ I_q^0 f_n &= f_n \quad (n \geq 1) \end{aligned} \tag{40}$$

Substitution of $\alpha = 1$ into eq. (40) gives

$$
\begin{aligned}
I_q f_n &= q^{n-1}(q-1)\sum_{k=0}^{n-1} \begin{bmatrix} -1 \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} f_{n-k} \\
&= q^{n-1}(q-1)\sum_{k=0}^{n-1}(-1)^k(-1)^k q^{-\frac{1}{2}k(k+1)} q^{\frac{1}{2}k(k-1)} f_{n-k} \\
&= (q-1)\sum_{k=0}^{n-1} q^{n-1-k} f_{n-k} \\
&= (q-1)\sum_{k=1}^{n} q^{k-1} f_k,
\end{aligned}
$$

which is a finite version of Jackson integral. This fractional $q$-addition operator satisfies the following lemma.

**Lemma 2** *Let $\alpha, \beta$ be non-negative real numbers, $a, b$ be complex numbers and $\{f_n\}, \{g_n\}$ be given complex sequences. Then $q$-addition operators satisfy the following linearity and commutation rules.*

$$
I_q^\alpha (a f_n + b g_n) = a(I_q^\alpha f_n) + b(I_q^\alpha g_n) \tag{42}
$$

$$
I_q^\alpha I_q^\beta f_n = I_q^\beta I_q^\alpha f_n = I_q^{\alpha+\beta} f_n \tag{43}
$$

**Proof of Lemma 2.** Equation (42) is obvious. We prove a commutation rule (43) by employing some properties of a $q$-binomial coefficient.

$$
\begin{aligned}
&I_q^\alpha I_q^\beta f_n \\
=&q^{(n-1)\alpha}(q-1)^\alpha \sum_{k=0}^{n-1}(-1)^k \begin{bmatrix} -\alpha \\ k \end{bmatrix} q^{k(k-1)/2} q^{(n-k-1)\beta}(q-1)^\beta \\
&\times \sum_{j=0}^{n-k-1}(-1)^j \begin{bmatrix} -\beta \\ j \end{bmatrix} q^{j(j-1)/2} f_{n-k-j} \\
=&q^{(n-1)(\alpha+\beta)}(q-1)^{\alpha+\beta} \sum_{k=0}^{n-1}(-1)^k \begin{bmatrix} -\alpha \\ k \end{bmatrix} q^{k(k-1)/2} q^{-\beta k} \\
&\times \sum_{j=0}^{n-k-1}(-1)^j \begin{bmatrix} -\beta \\ j \end{bmatrix} q^{j(j-1)/2} f_{n-k-j} \\
=&q^{(n-1)(\alpha+\beta)}(q-1)^{\alpha+\beta} \sum_{k=0}^{n-1}(-1)^k \begin{bmatrix} -\alpha \\ k \end{bmatrix} q^{k(k-1)/2} q^{-\beta k} \\
&\times \sum_{j=0}^{n-k-1}(-1)^{n-k-1-j} \begin{bmatrix} -\beta \\ n-j-1-k \end{bmatrix} q^{(n-k-1-j)(n-k-2-j)/2} f_{j+1}
\end{aligned}
$$

$$
\begin{aligned}
&= q^{(n-1)(\alpha+\beta)}(q-1)^{\alpha+\beta}\sum_{j=0}^{n-1}(-1)^{n-j-1}f_{j+1}\\
&\quad\times\sum_{k=0}^{n-j-1}\begin{bmatrix}-\alpha\\ k\end{bmatrix}\begin{bmatrix}-\beta\\ n-j-1-k\end{bmatrix}q^{k(k-1)/2+(n-k-1-j)(n-k-2-j)/2}q^{-\beta k}\\
&= q^{(n-1)(\alpha+\beta)}(q-1)^{\alpha+\beta}\sum_{j=0}^{n-1}(-1)^{n-j-1}q^{(n-j-1)(n-j-2)/2}f_{j+1}\\
&\quad\times\sum_{k=0}^{n-j-1}\begin{bmatrix}-\alpha\\ k\end{bmatrix}\begin{bmatrix}-\beta\\ n-j-1-k\end{bmatrix}q^{k^2-k(n-j-1)-\beta k}\\
&= q^{(n-1)(\alpha+\beta)}(q-1)^{\alpha+\beta}\sum_{j=0}^{n-1}(-1)^{n-j-1}q^{(n-j-1)(n-j-2)/2}f_{j+1}\begin{bmatrix}-\alpha-\beta\\ n-j-1\end{bmatrix}\\
&= q^{(n-1)(\alpha+\beta)}(q-1)^{\alpha+\beta}\sum_{j=0}^{n-1}(-1)^{j}q^{j(j-1)/2}f_{n-j}\begin{bmatrix}-\alpha-\beta\\ j\end{bmatrix}\\
&= I^{\alpha+\beta}f_n,
\end{aligned}
$$

which completes the proof. ■

Next we present a fractional $q$-difference operator $\Delta_q^\alpha$, which can be regarded as a $q$-discrete version of Caputo's fractional derivative operator.

**Definition 7** *Let $\alpha$ be a positive real number and $m$ be a positive integer which satisfies $m-1<\alpha\le m$. Then a fractional q-difference operator of order $\alpha>0$ is given by*

$$
\begin{aligned}
\Delta_q^\alpha f_n &= I_q^{m-\alpha}\Delta_q^m f_n\\
&= q^{-(n-1)(\alpha-m)}(q-1)^{-(\alpha-m)}\sum_{k=0}^{n-1}(-1)^k\begin{bmatrix}\alpha-m\\ k\end{bmatrix}q^{\frac{1}{2}k(k-1)}\Delta_q^m f_{n-k}
\end{aligned}
\tag{44}
$$

**Remark 4** *Fractional q-difference operator was first proposed by Al-Salam [1] in 1966. Let $f(x)$ be a given function and $\alpha\in\mathbb{R}\backslash\{1,2,3,\cdots\}$. Then a q-difference operator $K_q^\alpha$ is given by*

$$
K_q^\alpha f(x) = x^{-\alpha}(1-q)^{-\alpha}\sum_{k=0}^{\infty}(-1)^k\begin{bmatrix}\alpha\\ k\end{bmatrix}q^{k(k-1)/2-\alpha(\alpha-1)/2}f(xq^{\alpha-k}) \tag{45}
$$

*Fractional q-difference operator $\Delta_q^\alpha$ presented here is a slight modification of Al-Salam's operator $K_q^\alpha$. The operator $K_q^\alpha$ satisfies the commutative rule,*

$$
K_q^\alpha K_q^\beta = K_q^\beta K_q^\alpha = K_q^{\alpha+\beta} \tag{46}
$$

*for any $\alpha, \beta$, whereas the commutation rule for $\Delta_q^\alpha$ does not always hold. However, as is mentioned in the next section, the operator $\Delta_q^\alpha$ possesses an eigen function, which is regarded as a q-discrete analogue of the Mittag-Leffler function.*

## 4.2 $q$-Mittag-Leffler function

This section provides a $q$-discrete analogue of the Mittag-Leffler function and its relation with the fractional $q$-difference operator $\Delta_q^\alpha$. We first introduce a fundamental function $M_q(a;n)$ defined by

$$M_q(a;n) = (q-1)^{a-1} \begin{bmatrix} n+a-2 \\ n-1 \end{bmatrix} \quad (a > 0, n \in \mathbb{Z}_{\geq 1}). \tag{47}$$

$$M_q(a;0) = \begin{cases} 1 & (a=1) \\ 0 & (a \neq 1) \end{cases} \tag{48}$$

**Remark 5** *In the limit $q \to 1$ and $n \to \infty$ with $t = (q-1)n > 0$ fixed, the above function converges to a monomial,*

$$M_q(a;n) \to K(a;t) = \frac{t^{a-1}}{\Gamma(a)}. \tag{49}$$

*It is a well-known fact that this function $K(a;t)$ plays an essential role in the theory of fractional derivatives.*

The above fundamental function $M_q(a;n)$ satisfies the following two lemmas which states the relation between $M_q(a;n)$ and $q$-difference (or fractional $q$-addition) operator.

**Lemma 3** *If $a > 0$, we have*

$$\Delta_q M_q(a+1;n) = M_q(a;n) \quad (n \in \mathbb{Z}_{\geq 1}). \tag{50}$$

**Lemma 4** *If $\alpha \geq 0$ and $a > 0$, we have*

$$I_q^\alpha M_q(a;n) = M_q(a+\alpha;n). \tag{51}$$

**Proof of Lemma 3.** This is proved essentially by using an addition rule of $q$-binomial coefficient given by eq. (35).

$$\begin{aligned}
\Delta_q M_q(a+1;n) &= \frac{M_q(a+1;n) - M_q(a+1;n-1)}{q^n - q^{n-1}} \\
&= (q-1)^a \left( \begin{bmatrix} n+a-1 \\ n-1 \end{bmatrix} - \begin{bmatrix} n+a-2 \\ n-2 \end{bmatrix} \right) \frac{1}{q^{n-1}(q-1)} \\
&= (q-1)^a q^{n-1} \begin{bmatrix} n+a-2 \\ n-1 \end{bmatrix} \frac{1}{q^{n-1}(q-1)} \\
&= (q-1)^{a-1} \begin{bmatrix} n+a-2 \\ n-1 \end{bmatrix} \\
&= M_q(a;n)
\end{aligned}$$

which completes the proof. ■

**Proof of Lemma 4.** If $\alpha = 0$, it is obvious. We suppose $\alpha > 0$.

$$\begin{aligned}
I_q^\alpha M_q(a;n) &= q^{(n-1)\alpha}(q-1)^\alpha \sum_{k=0}^{n-1} (-1)^k \begin{bmatrix} -\alpha \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} M_q(a;n-k) \\
&= q^{(n-1)\alpha}(q-1)^{a-1+\alpha} \sum_{k=0}^{n-1} (-1)^k \begin{bmatrix} -\alpha \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} \begin{bmatrix} n-k+a-2 \\ n-k-1 \end{bmatrix} \\
&= q^{(n-1)\alpha}(q-1)^{a-1+\alpha} \cdot \\
&\quad \sum_{k=0}^{n-1} (-1)^k \begin{bmatrix} -\alpha \\ k \end{bmatrix} q^{\frac{1}{2}k(k-1)} (-1)^{n-1-k} q^{(n-k-1)a+\frac{1}{2}(n-k-1)(n-k-2)} \begin{bmatrix} -a \\ n-k-1 \end{bmatrix} \\
&= q^{(n-1)\alpha}(q-1)^{a-1+\alpha} q^{(n-1)a+\frac{1}{2}(n-1)(n-2)} (-1)^{n-1} \cdot \\
&\quad \sum_{k=0}^{n-1} \begin{bmatrix} -\alpha \\ k \end{bmatrix} q^{k^2-(n-1)k+k(-a)} \begin{bmatrix} -a \\ n-k-1 \end{bmatrix} \\
&= q^{(n-1)\alpha}(q-1)^{a-1+\alpha} q^{(n-1)a+\frac{1}{2}(n-1)(n-2)} (-1)^{n-1} \begin{bmatrix} -(a+\alpha) \\ n-1 \end{bmatrix} \\
&= q^{(n-1)\alpha}(q-1)^{a-1+\alpha} q^{(n-1)a+\frac{1}{2}(n-1)(n-2)} q^{-(n-1)(a+\alpha)-\frac{1}{2}(n-1)(n-2)} \begin{bmatrix} n+a+\alpha-2 \\ n-1 \end{bmatrix} \\
&= (q-1)^{a+\alpha-1} \begin{bmatrix} n+a+\alpha-2 \\ n-1 \end{bmatrix} = M_q(a+\alpha;n),
\end{aligned}$$

where we have employed an upper negation rule (33) twice and a Vandermonde convolution rule (36). This completes the proof. ■

We next introduce a $q$-analogue of the Mittag-Leffler function.

**Definition 8** *Let $a$ be a positive real number. Then q-Mittag-Leffler function $F_{a,q}(\lambda;n)$ is given by*

$$F_{a,q}(\lambda;n) = \sum_{j=0}^{\infty} \lambda^j M_q(aj+1;n) = \sum_{j=0}^{\infty} \lambda^j (q-1)^{aj} \begin{bmatrix} n+aj-1 \\ n-1 \end{bmatrix} \tag{52}$$

It can be verified easily from eq. (49) that the above function $F_{a,q}(\lambda;n)$ converges to the Mittag-Leffler function $E_a(\lambda t^a)$ in the limit $q \to 1$ and $n \to \infty$ with $t = (q-1)n$ fixed. The following main theorem states that $q$-Mittag-Leffler function serves as an eigen function of the fractional $q$-difference operator $\Delta_q^a$.

**Theorem 2** *If $a > 0$, we have*

$$\Delta_q^a F_{a,q}(\lambda;n) = \lambda F_{a,q}(\lambda;n) \tag{53}$$

**Proof of Theorem 2.** Let $m$ be a positive integer such as $m-1 < a \le m$. Operating $\Delta_q^m$ on $F_{a,q}(\lambda;n)$ and noticing $\Delta_q M_q(1;n) = \Delta_q 1 = 0$, we have from Lemma 3

$$\begin{aligned} \Delta_q^m F_{a,q}(\lambda;n) &= \sum_{j=0}^{\infty} \lambda^j \Delta_q^m M_q(aj+1;n) \\ &= \sum_{j=1}^{\infty} \lambda^j M_q(aj-m+1;n). \end{aligned} \tag{54}$$

Operating fractional $q$-addition operator $I_q^{m-a}$ on both sides of the above equation and employing Lemma 4, we finally obtain

$$\begin{aligned} \Delta_q^a F_{a,q}(\lambda;n) &= I_q^{m-a} \Delta_q^m F_{a,q}(\lambda;n) \\ &= \sum_{j=1}^{\infty} \lambda^j I^{m-a} M_q(aj-m+1;n) \\ &= \sum_{j=1}^{\infty} \lambda^j M_q(aj-a+1;n) \\ &= \sum_{j=0}^{\infty} \lambda^{j+1} M_q(aj+1;n) = \lambda F_{a,q}(\lambda;n), \end{aligned} \tag{55}$$

which completes the proof. ■

# 5 Numerical analysis of fractional differential equation

We here introduce an integrable discretization of eq. (7)

$$\begin{cases} \left(\Delta_{-n} + a\Delta_{*,-n}^{1/2} + b\right) u_n = \beta M(1/2;n) + f_n \\ u_0 = \alpha \end{cases} \tag{56}$$

or equivalently

$$\begin{aligned} &\frac{u_n - u_{n-1}}{\varepsilon} + \frac{a}{\sqrt{\varepsilon}} \sum_{j=0}^{n-1} \begin{pmatrix} -\frac{1}{2} \\ j \end{pmatrix} (-1)^j (u_{n-j} - u_{n-j-1}) + bu_n \\ &\qquad = \beta M(1/2;n) + f_n, \end{aligned} \tag{57}$$

The above discretization preserves the solution given by

$$\begin{aligned} u_n &= \alpha \frac{-\lambda_+\lambda_-}{\lambda_+ - \lambda_-} \left(\lambda_+^{-1} F_{1/2}(\lambda_+, n) - \lambda_-^{-1} F_{1/2}(\lambda_-, n)\right) \\ &+ \beta \frac{1}{\lambda_+ - \lambda_-} \left(F_{1/2}(\lambda_+, n) - F_{1/2}(\lambda_-, n)\right) \\ &+ \frac{\varepsilon}{\lambda_+ - \lambda_-} \sum_{k=1}^{n} f_k \left(\lambda_+ F_{1/2}(\lambda_+, n-k+1) - \lambda_- \tilde{E}_{1/2}(\lambda_-, n-k+1)\right) \end{aligned}$$

Equation (56) gives an explicit and stable difference scheme. Its numerical result is illustrated in Figure 1.

In order to investigate $u(t)$ at large $t$, $q$-difference scheme,

$$\begin{cases} \left(\Delta_q + a\Delta_q^{1/2} + b\right) u_n = \beta M_q(1/2;n) + f_n \\ u_0 = \alpha \end{cases} \tag{58}$$

gives a more powerful tool. Its numerical result is given in Figure 2.

# 6 An integrable mapping with fractional difference

This section provides a new type of integrable mappings equipped with fractional difference. We first consider the mapping [5]

$$\frac{u_{n+1} - u_n}{\varepsilon} = au_n(1 - u_{n+1}) \quad (a > 0) \tag{59}$$

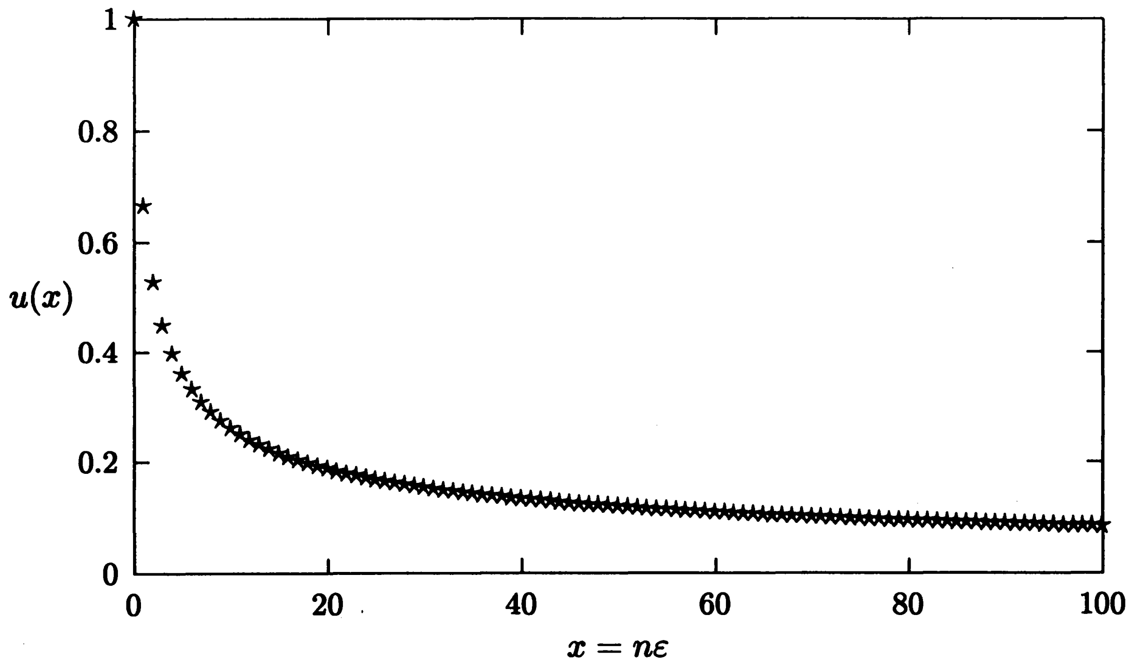


Figure 1: Numerical experiment of eq. (56)

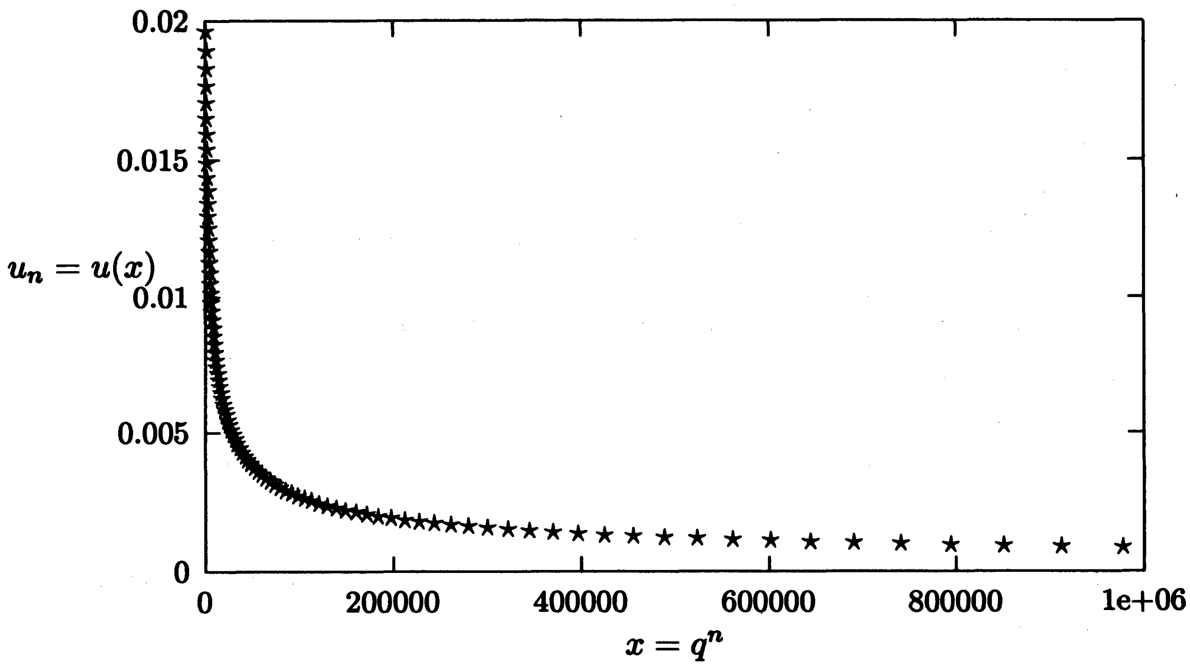


Figure 2: Numerical experiment of eq. (58)

which is a discrete Riccatti equation with constant coefficients. It is also regarded as an integrable discretization of the logistic equation

$$\frac{d}{dt}u = au(1-u) \quad (a>0). \tag{60}$$

A solution to eq. (59) is given by

$$u_n = \frac{u_0}{u_0 + (1-u_0)(1+a\varepsilon)^{-n}} \tag{61}$$

In order to "fractionalize" the mapping (59), we start with

$$u_n = \frac{u_0}{u_0 + (1-u_0)F_p(-a;n)} \tag{62}$$

By making use of Theorem 1, $u_n$ satisfies the following discrete equation,

$$u_n = \frac{1}{1 + \frac{1}{1+a\varepsilon^p}\left\{\frac{1}{u_{n-1}} - \sum_{j=1}^{n-1}\binom{p-1}{j}(-1)^j\left(\frac{1}{u_{n-j}} - \frac{1}{u_{n-j-1}}\right)\right\}}. \tag{63}$$

Putting $p=1$ in eq. (63), we have

$$u_n = \frac{(1+a\varepsilon)u_{n-1}}{1+(1+a\varepsilon)u_{n-1}}, \tag{64}$$

which recovers eq. (59). Figure 3 illustrates time evolutions of the fractional mapping with order parameter $p = n/4(n = 1,2,3,4)$. We have put $u_0 = 0.1$, $a = 1.0$ and $\varepsilon = 0.1$.

Considering the fact that the Mittag-Leffler function has an asymptotic behavior [12],

$$E_p(\lambda t^p) = -\sum_{k=1}^{N-1}\frac{\lambda^{-k}t^{-pk}}{\Gamma(1-pk)} + O(t^{-pN}), \quad t\to\infty, \lambda<0 \tag{65}$$

and that $u_n$ converges to

$$u_n \to \frac{u_0}{u_0 + (1-u_0)E_p(-at^p)} \tag{66}$$

as $n\to\infty, \varepsilon\to 0$ with $t = n\varepsilon$ fixed, we can observe that $u_n$ converges to 1 at the order of $O(1/n^p)$ if $0<p<1$. Table I illustrates a numerical result in which we apply a convergence acceleration algorithm, which is called the $\rho$-algorithm [3],

$$\rho_{k+1}^n = \rho_{k-1}^{n+1} + \frac{(k+n)^p - n^p}{\rho_k^{n+1} - \rho_k^n}$$

$$\rho_0^n = 0,\ \rho_1^n = u_n$$

to the sequence $\{u_n\}$ in the case $p = 1/4$. This table shows that $u_n$ converges to the value near 1.0 at the order $O(1/n^{1/4})$ as $n$ tends to $+\infty$.

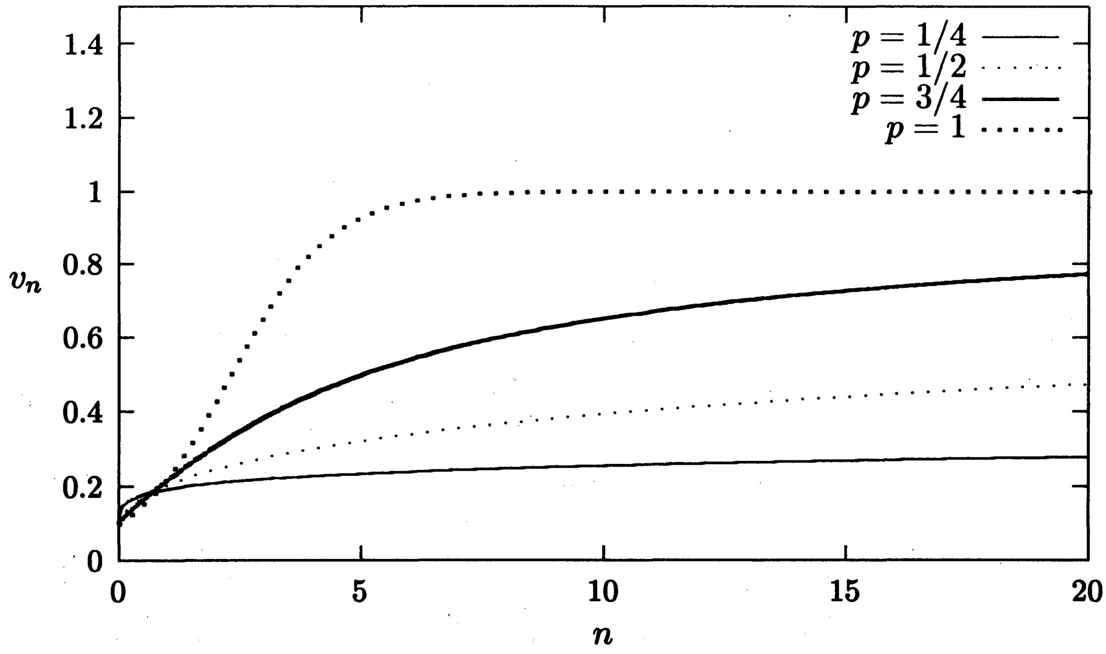


Figure 3: Time evolutions of the fractional mapping (63)

Table 1: The $\rho$-algorithm applied to the sequence $\{u_n\}$ in the case $p = 1/4$

| $n$ | $\rho_1^n (= u_n)$ | $\rho_3^n$ | $\rho_5^n$ | $\rho_7^n$ | $\cdots$ | $\rho_{21}^n$ |
|---|---|---|---|---|---|---|
| 1 | 0.10000 | 0.19745 | 0.84609 | 0.84288 | | 1.00145 |
| 2 | 0.14791 | 0.23921 | 0.86006 | 1.30904 | | 0.99915 |
| 3 | 0.16019 | 0.27802 | 0.99118 | 1.21129 | | 1.00131 |
| 4 | 0.16764 | 0.31406 | 1.07379 | 1.17824 | | 1.00139 |
| 5 | 0.17313 | 0.34778 | 1.11866 | 1.16134 | | 1.00112 |
| 6 | 0.17752 | 0.37941 | 1.14064 | 1.15284 | | 1.00120 |
| 7 | 0.18121 | 0.40908 | 1.14968 | 1.15045 | | 1.00120 |
| 8 | 0.18442 | 0.43692 | 1.15161 | 1.15375 | | 1.00121 |
| 9 | 0.18726 | 0.46304 | 1.14970 | 1.16376 | | 1.00122 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | | $\vdots$ |

## 7 Concluding Remarks

We have presented discrete and $q$-discrete analogues of the Mittag-Leffler function, together with their relations to fractional difference and $q$-difference. However, the Mittag-Leffler function possesses more abundant properties other than its relation to fractional derivative and it is not clear whether its discrete analogues preserve such properties.

It is also interesting to find new types of integrable systems with fractional derivative or difference. Since fractional differential equation describes a system in which a value at $t = t$ depends not only on its local data but also on its historical data from $t = 0$ to $t = t$, it is expected to serve as a model of some physical phenomenon.

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