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TO THE THEORY OF BIORTHOGONAL RATIONAL FUNCTIONS

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ABSTRACT. Some general aspects of the theory of biorthogonal rational functions are considered. A special family of such functions expressed through elliptic hypergeometric series is described in detail.

1. BIORTHOGONAL RATIONAL FUNCTIONS. BASIC RESULTS

In this section, we describe some basics of the theory of biorthogonal rational functions (BRF). A number of statements is taken from [12, 25, 36], however, the Theorems 1 and 2 represent new results.

Let α_i, β_i, $i = 1, 2, \dots$, be two sets of fixed complex numbers. We assume that $\alpha_i \neq \alpha_k$ and $\beta_i \neq \beta_k$ for $i \neq k$. With the sequences α_i, β_i we associate the following polynomials of a free independent variable $z \in \mathbb{C}$

$$A_n(z) = \prod_{i=1}^{n}(z - \alpha_i), \quad B_n(z) = \prod_{i=1}^{n}(z - \beta_i),$$

and we assume that $A_0 = B_0 = 1$. Introduce a linear functional $\mathcal{L}$ defined on the space of all rational functions of z with the prescribed positions of poles at α_i, β_i. The functional $\mathcal{L}$ can be defined by its generalized moments

$$M_{ik} = \mathcal{L}\left\{\frac{1}{B_i(z)A_k(z)}\right\}, \quad i, k \in \mathbb{N}. \tag{1.1}$$

Define the determinants

$$\Delta_n = \det ||M_{ik}||_{i=0,\dots,n}^{k=0,\dots,n},$$
$$\Delta_n^{(01)} = \det ||M_{ik}||_{i=0,\dots,n-1}^{k=1,\dots,n}, \qquad \Delta_n^{(10)} = \det ||M_{ik}||_{i=1,\dots,n}^{k=0,\dots,n-1} \tag{1.2}$$

and assume that $\Delta_n \neq 0$, $\Delta_n^{(01)} \neq 0$, $\Delta_n^{(10)} \neq 0$.

We introduce two sets of rational functions $R_n(z), T_n(z)$ with the help of the following determinants

$$R_n(z) = \begin{vmatrix} M_{00} & M_{01} & \dots & M_{0n} \\ M_{10} & M_{11} & \dots & M_{1n} \\ \dots & \dots & \dots & \dots \\ M_{n-1,0} & M_{n-1,1} & \dots & M_{n-1,n} \\ 1 & 1/A_1(z) & \dots & 1/A_n(z) \end{vmatrix}, \tag{1.3}$$

$$T_n(z) = \begin{vmatrix} M_{00} & M_{10} & \dots & M_{n0} \\ M_{01} & M_{11} & \dots & M_{n1} \\ \dots & \dots & \dots & \dots \\ M_{0,n-1} & M_{1,n-1} & \dots & M_{n,n-1} \\ 1 & 1/B_1(z) & \dots & 1/B_n(z) \end{vmatrix}. \tag{1.4}$$

We have $R_0(z) = T_0(z) = M_{00}$ and $R_1(z) = P_1(z)/(z-\alpha_1)$, $T_1(z) = Q_1(z)/(z-\beta_1)$, where $P_1(z) = M_{00} - M_{01}(z-\alpha_1)$ and $Q_1(z) = M_{00} - M_{10}(z-\beta_1)$. Due to the conditions $\Delta_n \neq 0$, $\Delta_n^{(01)} \neq 0$, $\Delta_n^{(10)} \neq 0$, we have

$$R_n(z) = \frac{P_n(z)}{A_n(z)}, \qquad T_n(z) = \frac{Q_n(z)}{B_n(z)}, \tag{1.5}$$

where $P_n(z)$ and $Q_n(z)$ are some polynomials of the n-th degree in z. Thus, both $R_n(z)$ and $T_n(z)$ are rational functions of the type $[n/n]$, that is they are defined by ratios of two n-th degree polynomials. Evidently, the poles of these rational functions are prescribed: the poles of $R_n(z)$ are located at α_i, $i = 1,\dots,n$, whereas the poles of $T_n(z)$ are located at β_i, $i = 1,\dots,n$.

By construction, we have

$$\mathcal{L}\left\{\frac{R_n(z)}{B_m(z)}\right\} = 0, \qquad \mathcal{L}\left\{\frac{T_n(z)}{A_m(z)}\right\} = 0, \quad m = 0,1,\dots,n-1. \tag{1.6}$$

For example, $\mathcal{L}\{R_n(z)/B_m(z)\}$ equals to the determinant obtained from (1.3) after replacement of the entries $A_i(z)$ from the last row by the moments M_{mi}, $i = 0,1,\dots,n$. Hence, this determinant vanishes as having two coinciding rows. From (1.6), we derive the equality

$$\mathcal{L}\{R_n(z)T_m(z)\} = 0, \quad m \neq n. \tag{1.7}$$

Indeed, if $m < n$ then we can expand $T_m(x) = \sum_{i=0}^{m} \xi_i/B_i(z)$ with some coefficients ξ_i and, hence, (1.7) is valid due to (1.6). If $m > n$, we can expand $R_n(x) = \sum_{i=0}^{n} \eta_i/A_i(z)$ and, again, (1.7) is valid due to (1.6). If $m = n$, we can expand $T_n(z) = \Delta_{n-1}/B_n(z) + \sum_{i=0}^{n-1} \sigma_i/B_i(z)$ and get

$$\mathcal{L}\{R_n(z)T_n(z)\} = \Delta_{n-1}\mathcal{L}\left\{\frac{R_n(z)}{B_n(z)}\right\} = \Delta_{n-1}\,\Delta_n$$

by definitions (1.2), (1.3). We thus have

Theorem 1. *The functions $R_n(z)$ and $T_n(z)$ defined by (1.3) and (1.4) are rational functions of z of the type $[n/n]$ with the prescribed poles at $z = \alpha_i$ and $z = \beta_i$ ($i = 1,2,\dots,n$) respectively. These functions satisfy the biorthogonality relation*

$$\mathcal{L}\{R_n(z)T_m(z)\} = \Delta_{n-1}\,\Delta_n\,\delta_{nm} \tag{1.8}$$

with Δ_n defined in (1.2).

Orthogonality conditions (1.6) can be rewritten in terms of the polynomials $P_n(z)$ and $Q_n(z)$ as follows:

$$\mathcal{L}\left\{\frac{P_n(z)(z-\beta_n)z^m}{A_n(z)B_n(z)}\right\} = 0, \quad m = 0,1,\dots,n-1, \tag{1.9}$$

$$\mathcal{L}\left\{\frac{Q_n(z)(z-\alpha_n)z^m}{A_n(z)B_n(z)}\right\} = 0, \quad m = 0,1,\dots,n-1. \tag{1.10}$$

Relations, similar to (1.9), were considered by Ismail and Masson in [12] in connection to the continued fractions of the R_{II} type. Our functional $\mathcal{L}$ differs from the one in [12] $\mathcal{L}_{IM}$ by a simple transformation $\mathcal{L}\{g(z)\} \equiv \mathcal{L}_{IM}\{g(z)/(z-\beta_0)\}$ for some constant β_0.

There exist non-trivial recurrence relations connecting polynomials $P_n(z)$ and $Q_n(z)$. In order to derive them, we consider the expression $P_{n+1}(z) - b_n(z-\beta_n)P_n(z)$, where b_n are some coefficients. In order to be able to set $n = 0$ in this combination, we add two more constants α_0 and β_0 to the sets $\{\alpha_i\}$ and $\{\beta_i\}$. Assume that $\alpha_i \neq \beta_k$ for all $i, k \in \mathbb{N}$, and, moreover, that zeros of the polynomials $P_n(z)$ do not coincide with α_i, β_k, that is $P_n(\alpha_i)P_n(\beta_k) \neq 0$ for all n, i, k. Then we can choose

$$b_n = \frac{P_{n+1}(\alpha_n)}{(\alpha_n - \beta_n)P_n(\alpha_n)}. \tag{1.11}$$

Such a choice means that

$$P_{n+1}(z) - b_n(z - \beta_n)P_n(z) = (z - \alpha_n)\, q_n(z), \tag{1.12}$$

where $q_n(z)$ is a polynomial of the degree not exceeding n. One can therefore expand

$$q_n(z) = B_n(z)\left(\nu_n^{(0)}T_n(z) + \nu_n^{(1)}T_{n-1}(z) + \cdots + \nu_n^{(n)}\right) \tag{1.13}$$

with some coefficients $\nu_n^{(i)}$. From relation (1.8), we have for $i < n$

$$\begin{aligned}
\nu_n^{(n-i)}\Delta_i\Delta_{i-1} &= \mathcal{L}\left\{\frac{P_{n+1}(z) - b_n(z-\beta_n)P_n(z)}{(z-\alpha_n)B_n(z)}\, R_i(z)\right\} \\
&= \mathcal{L}\left\{\frac{P_{n+1}(z)P_i(z)(z-\beta_{n+1})(z-\alpha_{i+1})\cdots(z-\alpha_{n-1})(z-\alpha_{n+1})}{A_{n+1}(z)B_{n+1}(z)}\right\} \\
&\quad - b_n\mathcal{L}\left\{\frac{P_n(z)P_i(z)(z-\beta_n)(z-\alpha_{i+1})\cdots(z-\alpha_{n-1})}{A_n(z)B_n(z)}\right\}.
\end{aligned} \tag{1.14}$$

Due to (1.9), we see that the right-hand side of (1.14) vanishes: $\nu_n^{(i)} = 0$ for $i = 1, \ldots, n$. Thus, we arrive at the relation

$$P_{n+1}(z) - b_n(z - \beta_n)P_n(z) = \nu_n(z - \alpha_n)Q_n(z) \tag{1.15}$$

with some coefficients $\nu_n = \nu_n^{(0)}$ (cf. [25]).

In the same way, due to the obvious permutational symmetry between $P_n(z)$ and $Q_n(z)$, we get the second relation

$$Q_{n+1}(z) - c_n(z - \alpha_n)Q_n(z) = \mu_n(z - \beta_n)P_n(z), \tag{1.16}$$

where μ_n is some sequence of numbers and

$$c_n = \frac{Q_{n+1}(\beta_n)}{(\beta_n - \alpha_n)Q_n(\beta_n)}.$$

Relations (1.15) and (1.16) are of great importance. They allow us to express one set of polynomials in terms of another. Moreover, one can obtain a three term recurrence relation for polynomials $P_n(z)$ (a similar recurrence relation is valid for the polynomials $Q_n(z)$):

$$\begin{aligned}
&\nu_n P_{n+2}(z) - \left(\nu_n b_{n+1}(z - \beta_{n+1}) + c_n\nu_{n+1}(z - \alpha_{n+1})\right) P_{n+1}(z) \\
&\quad = \nu_{n+1}(z - \beta_n)(z - \alpha_{n+1})(\mu_n\nu_n - c_n b_n)P_n(z), \qquad n \geq 0,
\end{aligned} \tag{1.17}$$

with the initial conditions $P_0(z) = M_{00}$, $P_1(z) = M_{00} - M_{01}(z - \alpha_1)$. Taking into account relations (1.5), we arrive at the three term recurrence relation for the rational functions $R_n(z)$ (a similar relation holds for $T_n(z)$):

$$\begin{aligned}
&\nu_n(z - \alpha_{n+2})R_{n+2}(z) - \left(\nu_n b_{n+1}(z - \beta_{n+1}) + c_n\nu_{n+1}(z - \alpha_{n+1})\right) R_{n+1}(z) \\
&\quad = \nu_{n+1}(z - \beta_n)(\mu_n\nu_n - c_n b_n)R_n(z).
\end{aligned} \tag{1.18}$$

It is seen that (1.18) coincides with the generalized eigenvalue problem (GEVP) [32] for two arbitrary tridiagonal matrices J_1, J_2:

$$J_1 R_n(z) = z J_2 R_n(z), \tag{1.19}$$

where $J_i R_n \equiv \xi_n^{(i)} R_{n+1} + \eta_n^{(i)} R_n + \zeta_n^{(i)} R_{n-1}$ for some coefficients $\xi_n^{(i)}, \eta_n^{(i)}, \zeta_n^{(i)}$.

Recurrence relation (1.17) was a starting point in [12] for studying biorthogonality properties of the polynomials $P_n(z)$. Namely, it was shown in [12] that if $P_n(z)$ satisfy recurrence relation (1.17) with appropriate initial conditions, then there exists a linear functional $\mathcal{L}_{IM}$ providing the orthogonality relations equivalent to (1.9). As shown in [36], GEVP (1.19) for rational functions $R_n(z)$ leads also to the biorthogonality condition in the form (1.7). This gives the first half of an analogue of the Favard theorem for BRF. The results presented above allow us to complete this analogy by the inverse statement.

Theorem 2. *Let there exists a linear functional $\mathcal{L}$ defining finite moments M_{ik} (1.1) which satisfy the conditions $\Delta_n \neq 0$, $\Delta_n^{(01)} \neq 0$, $\Delta_n^{(10)} \neq 0$. Then the pair of rational functions $R_n(z), T_n(z)$ given by (1.3), (1.4) satisfy the biorthogonality condition (1.7) and GEVP (1.19).*

Consider an analogue of the Christoffel transformation for the rational functions $R_n(z), T_n(z)$ [25, 36]. Let λ be a constant such that $R_n(\lambda) \neq 0$. Introduce a new functional $\tilde{\mathcal{L}}$ defined on a set of all rational functions $\tilde{R}_n(z)$ having poles at the points $\tilde{\beta}_k = \beta_k$, $\tilde{\alpha}_k = \alpha_{k+1}$ by the formula

$$\tilde{\mathcal{L}} = \left(\frac{z-\lambda}{z-\alpha_1}\right) \mathcal{L}. \tag{1.20}$$

New generalized moments $\tilde{M}_{nm}$ defined by $\tilde{\mathcal{L}}$ are

$$\begin{aligned} \tilde{M}_{nm} &= \tilde{\mathcal{L}}\left\{\frac{1}{\tilde{B}_n(z)\tilde{A}_m(z)}\right\} = \mathcal{L}\left\{\frac{z-\lambda}{B_n(z)A_{m+1}(z)}\right\} \\ &= M_{nm} + (\alpha_{m+1} - \lambda) M_{n,m+1}. \end{aligned} \tag{1.21}$$

It can be verified that the pair of new rational functions

$$\begin{aligned} \tilde{R}_n(z) &= \frac{z-\alpha_1}{z-\lambda}\left(R_{n+1}(z) - \frac{R_{n+1}(\lambda)}{R_n(\lambda)} R_n(z)\right), \\ \tilde{T}_n(z) &= \frac{1}{z-\lambda}\left((z-\beta_{n+1})T_{n+1}(z) - \frac{(\lambda-\beta_{n+1})T_{n+1}(\lambda)}{(\lambda-\alpha_n)T_n(\lambda)}(z-\alpha_n)T_n(z)\right) \end{aligned} \tag{1.22}$$

satisfies the relations

$$\tilde{\mathcal{L}}\left\{\frac{\tilde{R}_n(z)}{\tilde{B}_m(z)}\right\} = 0, \qquad \tilde{\mathcal{L}}\left\{\frac{\tilde{T}_n(z)}{\tilde{A}_m(z)}\right\} = 0,$$

where $m = 0, 1, \ldots, n-1$.

Remark 1. The parameter α_0 entering the definition of $\tilde{T}_0$ is not defined, it can take arbitrary values except $\alpha_0 \neq \lambda$. Its change influences only the constant $\tilde{T}_0$.

We thus have

Theorem 3. *The functions $\tilde{R}_n(z)$ and $\tilde{T}_n(z)$ defined by (1.22) form a pair of biorthogonal rational functions with respect to the modified functional $\tilde{\mathcal{L}}$ defined by (1.20)*

$$\tilde{\mathcal{L}}\left\{\tilde{R}_n(z)\tilde{T}_m(z)\right\} = 0, \qquad n \neq m. \tag{1.23}$$

Note that in the theory of ordinary orthogonal polynomials the Christoffel transformation corresponds to the transition to kernel polynomials leading to a *linear* modification of the functional $\tilde{\mathcal{L}} = (x - x_0)\mathcal{L}$ (see, e.g. [29]). In the theory of BRF, we have instead *rational* modification (1.20) of the functional. Particular examples of such modifications were first exploited by Wilson [33, 34] for construction of a pair of self-dual BRF expressed through ${}_9F_8$ and ${}_{10}\phi_9$ series.

Möbius transformations of the argument of rational functions is a symmetry of such functions. Namely, if $R_n(z), T_n(z)$ is a pair of BRF, then

$$\tilde{R}_n(z) = R_n\left(\frac{\xi z + \eta}{\zeta z + \sigma}\right), \qquad \tilde{T}_n(z) = T_n\left(\frac{\xi z + \eta}{\zeta z + \sigma}\right)$$

is another pair of BRF. This statement follows from the observation that Möbius transformations of the spectral parameter in a given GEVP (1.19) do not change the form of this eigenvalue problem. Indeed, for $\tilde{J}_1 = \xi J_1 + \eta J_2$, $\tilde{J}_2 = \zeta J_1 + \sigma J_2$ one has the GEVP

$$\tilde{J}_1 R_n(z) = \frac{\xi z + \eta}{\zeta z + \sigma} \tilde{J}_2 R_n(z),$$

which, in turn, generates BRF of the argument $(\xi z + \eta)/(\zeta z + \sigma)$. For appropriate choice of the positions of poles of BRF, it is possible to achieve the equality $R_n(z) = T_n(z)$ [36] and to arrive at the theory of orthogonal rational functions [5].

2. Elliptic hypergeometric functions

A general definition of elliptic hypergeometric functions (including the multivariable case) was proposed in [20]. For functions of one variable, the formal series $\sum_{n=0}^{\infty} c_n$ is called elliptic hypergeometric series if $h(n) = c_{n+1}/c_n$ is an elliptic function of $n \in \mathbb{C}$. Any elliptic function of order $r+1$ admits the factorization [31]:

$$h(n) = z\frac{[u_0 + n, \ldots, u_r + n]}{[v_0 + n, \ldots, v_r + n]}, \tag{2.1}$$

where $[u_0, \ldots, u_k] \equiv [u_0]\cdots[u_k]$ and $[u]$ is the standard θ_1-Jacobi theta function

$$[u] \equiv \theta_1(u) = -i \sum_{n=-\infty}^{\infty} (-1)^n p^{(2n+1)^2/8} q^{(n+1/2)u}$$
$$= p^{1/8} i q^{-u/2}\,(p;p)_\infty\,\theta(q^u;p), \quad u \in \mathbb{C},$$
$$\theta(z;p) = (z;p)_\infty (pz^{-1};p)_\infty, \quad (a;p)_\infty = \prod_{n=0}^{\infty}(1 - ap^n), \tag{2.2}$$

where $p = e^{2\pi i \tau}$, $\mathrm{Im}(\tau) > 0$, $q = e^{2\pi i \sigma}$. Remind some properties of the function $[u]$: (i) $[-u] = -[u]$; (ii) $[u + \sigma^{-1}] = -[u]$, $[u + \tau\sigma^{-1}] = -e^{-\pi i \tau - 2\pi i \sigma u}[u]$; (iii) the Riemann identity [31]:

$$[x+z, x-z, y+w, y-w] - [x+w, x-w, y+z, y-z]$$
$$= [x+y, x-y, z+w, z-w]; \tag{2.3}$$

(iv) $\lim_{Im(\tau)\to+\infty}[u]/[1] = \frac{\sin(\pi\sigma u)}{\sin(\pi\sigma)}$; (v) $\lim_{\sigma\to 0}[u]/[1] = u$; (vi) $[u] = 0$ for $u_{m_1,m_2} = (m_1 + m_2\tau)\sigma^{-1}$, $m_{1,2} \in \mathbb{Z}$. From (ii), it follows that in order for meromorphic function $h(n)$ to be double periodic

$$h(n+\sigma^{-1}) = h(n), \qquad h(n+\tau\sigma^{-1}) = h(n),$$

it is necessary to have

$$\sum_{i=0}^{r} u_i = \sum_{i=0}^{r} v_i. \tag{2.4}$$

Conventions of the theory of hypergeometric series require the choice $v_0 = 1$. After taking that and solving the first order recursion $c_{n+1} = h(n)c_n$ with the initial condition $c_0 = 1$, we get the single variable elliptic hypergeometric series:

$${}_{r+1}E_r\left(\begin{matrix} u_0,\dots,u_r \\ v_1,\dots,v_r \end{matrix};\sigma,\tau;z\right) = \sum_{n=0}^{\infty} \frac{[u_0,u_1,\dots,u_r]_n}{[1,v_1,\dots,v_r]_n} z^n, \tag{2.5}$$

where the elliptic shifted factorials are defined as follows

$$[u_0,\dots,u_k]_n \equiv \prod_{m=0}^{k}\prod_{j=0}^{n-1}[u_m+j].$$

If we drop the ellipticity constraint (2.4), then (2.5) gives a particular example of *theta hypergeometric series* (or Jacobi theta functions extension of the general plain ${}_sF_r$ and basic ${}_s\phi_r$ hypergeometric series) introduced in [20]. In this framework, (2.4) is called the balancing condition and the elliptic hypergeometric series coincide by definition with the *balanced* theta hypergeometric series.

It is natural to demand that the function $h(n)$ is elliptic not only in n but, simultaneously, in all free parameters among u_i, v_i. This is possible only under the constraints [20]: $u_0 + 1 = u_1 + v_1 = \dots = u_r + v_r$, known as the *well-poisedness* conditions for plain and basic hypergeometric series [8]. Series with such a property are called *totally elliptic* hypergeometric series.

The elliptic hypergeometric series are called *very-well-poised*, if, in addition to (2.4) and the well-poisedness conditions, one has

$$\begin{aligned} u_{r-3} &= \frac{1}{2}u_0 + 1, \quad u_{r-2} = \frac{1}{2}u_0 + 1 - \frac{1}{2\sigma}, \\ u_{r-1} &= \frac{1}{2}u_0 + 1 - \frac{\tau}{2\sigma}, \quad u_r = \frac{1}{2}u_0 + 1 + \frac{1+\tau}{2\sigma}. \end{aligned} \tag{2.6}$$

Such series can be represented in the form [20]

$${}_{r+1}E_r = \sum_{n=0}^{\infty} \frac{[u_0+2n]}{[u_0]} \prod_{m=0}^{r-4} \frac{[u_m]_n}{[u_0+1-u_m]_n}(-z)^n, \tag{2.7}$$

where $\sum_{m=1}^{r-4} u_m = u_0(r-5)/2 + (r-7)/2$. It is convenient to use special notation ${}_{r+1}V_r(u_0; u_1,\dots,u_{r-4})$ for this very-well-poised elliptic hypergeometric series at $z = -1$. In the limit $\mathrm{Im}(\tau) \to +\infty$, ${}_{r+1}V_r$ series boil down to the very-well-poised balanced ${}_{r-1}\phi_{r-2}$ basic hypergeometric series [8].

For the first time series of the type (2.7) with $z = -1$ appeared implicitly in the series of papers by Date et al (see [6] and references therein) devoted to solvable statistical mechanics models. Explicitly, they were introduced by Frenkel and Turaev in [7]. The present authors have encountered them in [25, 26, 37] within an independent study of the theory of BRF with the help of techniques of spectral transformation chains (see [22, 24] for a description of our approach to orthogonal polynomials, especially, to Askey-Wilson polynomials [2]). The general theory of series of hypergeometric type built out of Jacobi theta functions was built in [20].

One of the main results of Frenkel and Turaev obtained in [7] consists in a proof (by a rather non-standard technique) of the following summation formula:

$$\sum_{n=0}^{N} \frac{[u_0+2n]}{[u_0]} \prod_{r=0}^{5} \frac{[u_r]_n}{[u_0+1-u_r]_n} = \frac{[u_0+1]_N \prod_{1\le r<s\le 3}[u_0+1-u_r-u_s]_N}{[u_0+1-u_1-u_2-u_3]_N \prod_{r=1}^{3}[u_0+1-u_r]_N}, \tag{2.8}$$

where $\sum_{i=1}^{5} u_i = 2u_0+1$ and $u_4 = -N$, $N \in \mathbb{N}$. According the classification of [20], this formula provides a closed form expression for the terminating very-well-poised balanced ${}_{10}E_9$ theta hypergeometric series at $z=-1$.

An elliptic generalization of the Bailey transformation formula for a terminating very-well-poised balanced ${}_{10}\phi_9$ series was proved in [7]. In our notations, it looks as follows

$$\begin{aligned} {}_{12}V_{11}(u_0;u_1,\ldots,u_6,-n) &= {}_{12}V_{11}(s_0;s_1,\ldots,s_7) \\ &\times \frac{[u_0+1, s_0+1-u_4, s_0+1-u_5, u_0+1-u_4-u_5]_n}{[s_0+1, u_0+1-u_4, u_0+1-u_5, s_0+1-u_4-u_5]_n}, \\ s_0 = 2u_0+1-u_1-u_2-u_3, &\qquad s_j = s_0-u_0+u_j,\ j=1,2,3, \end{aligned} \tag{2.9}$$

and $\{s_4,s_5,s_6,s_7\}$ is an arbitrary permutation of the parameters $u_4,u_5,u_6,u_7 = -n$. A special double use of (2.9) (firstly, with permuted u_1 and u_6 and, secondly, with parameters s_2,s_3,s_6 playing the role of u_1,u_2,u_3) provides another useful transformation

$${}_{12}V_{11}(u_0;u_1,\ldots,u_6,-n) = \zeta_n\, {}_{12}V_{11}(r_0;r_1,\ldots,r_6,r_7), \tag{2.10}$$

where

$$\begin{aligned} &r_0 = u_1-u_6-n, \quad r_1 = u_1, \quad r_6 = u_1-n-u_0, \\ &r_7 = -n, \quad r_i = 1+u_0-u_i-u_6,\ i=2,\ldots,5, \end{aligned} \tag{2.11}$$

$$\zeta_n = \frac{[u_0+1, u_6]_n}{[1+u_0-u_1, u_6-u_1]_n} \prod_{i=2}^{5} \frac{[1+u_0-u_1-u_i]_n}{[1+u_0-u_i]_n}. \tag{2.12}$$

The next two theorems were established in [25]. They describe generalizations of the contiguous relations for terminating very-well-poised balanced ${}_{10}\varphi_9$ basic hypergeometric series from [11]. Denote $\Phi(\mathbf{u}) \equiv {}_{12}V_{11}(u_0;u_1,\ldots,u_7)$ and $\Phi(u_i\pm)$ the function ${}_{12}V_{11}$ with the particular parameter u_i replaced by $u_i\pm1$, other parameters being unchanged. Let also $\Phi_\pm$ represent the functions ${}_{12}V_{11}(u_0\pm2;u_1\pm1,\ldots,u_7\pm1)$.

Theorem 4. *Assume that one of the parameters $u_i=-n$, for some fixed $i=1,\ldots,7$. Then the following identity takes place*

$$\Phi(u_6-,u_7+) - \Phi(\mathbf{u}) = \tag{2.13}$$

$$\Phi_+(u_6-)\, \frac{[u_0+1, u_0+2, u_7-u_6+1, u_7+u_6-u_0-1]}{[1+u_0-u_6, 2+u_0-u_6, u_0-u_7, 1+u_0-u_7]} \prod_{i=1}^{5} \frac{[u_i]}{[1+u_0-u_i]}.$$

Proof. We have

$$\Phi(u_6-,u_7+)-\Phi(\mathbf{u})=\sum_{k=0}^{n} C_k\Big(\frac{[u_6-1,u_7+1]_k}{[2+u_0-u_6,u_0-u_7]_k} \tag{2.14}$$

$$-\frac{[u_6,u_7]_k}{[1+u_0-u_6,1+u_0-u_7]_k}\Big)=\sum_{k=0}^{n} C_k\frac{[u_6-1,u_7]_k}{[1+u_0-u_6,u_0-u_7]_k}Y_k,$$

where

$$C_k=\frac{[u_0+2k]}{[u_0]}\prod_{i=0}^{5}\frac{[u_i]_k}{[1+u_0-u_i]_k},$$

$$Y_k=\frac{[u_0-u_6+1,u_7+k]}{[u_7,u_0-u_6+k+1]}-\frac{[u_0-u_7,u_6-1+k]}{[u_6-1,u_0-u_7+k]}.$$

The expression for Y_k can be simplified using the identity (2.3):

$$Y_k=\frac{[k,k+u_0,u_7-u_6+1,u_7+u_6-u_0-1]}{[u_7,u_0-u_6+k+1,u_6-1,u_0-u_7+k]}.$$

Substituting this into (2.14) and taking into account that $[u]_{k+1}=[u][u+1]_k$, we arrive at (2.13). □

Remark 2. If $u_6=-n$ or $u_7=-n$ we should replace the upper limit of summation in (2.14) to $n+1$ or $n-1$ respectively.

Let us replace the parameters u_6,u_7 in (2.13) by u_4,u_5 and set $u_6+u_7=1+u_0$ which reduces the corresponding ${}_{12}V_{11}$ series to ${}_{10}V_9$. Assume that $u_4=-N, u_5=2u_0+1-u_1-u_2-u_3+N$ and denote as $S_N(u_0,\dots,u_3)$ the ${}_{10}V_9$ series standing on the left-hand side of (2.8). Then contiguous relation (2.13) takes the form

$$S_{N+1}(u_0,\dots,u_3)=S_N(u_0,\dots,u_3)-S_N(u_0+2,u_1+1,u_2+1,u_3+1)$$

$$\times\frac{[u_0+1,u_0+2,u_5+N+1,u_5-N-u_0-1]}{[u_0+1+N,u_0+2+N,u_0-u_5,u_5-u_0-1]}\prod_{r=1}^{3}\frac{[u_r]}{[u_0+1-u_r]}. \tag{2.15}$$

For $N=1$ the sum (2.8) is a simple consequence of (2.3). Suppose that (2.8) is valid for some fixed $N\geq 1$. Substitute the right-hand side of (2.8) into the right-hand side of (2.15). It can be checked that, after an application of the identity (2.3), this gives the formula (2.8) for N replaced by $N+1$, that is we prove inductively the Frenkel-Turaev sum for arbitrary integer N. As shown in [21], the transformation (2.9) can be deduced from (2.8) in an elementary way as well.

Theorem 5. *Under the same assumptions as in the previous theorem, the following contiguous relation holds true*

$$\frac{[u_7]}{[1+u_0-u_6,2+u_0-u_6]}\prod_{i=1}^{5}[1+u_0-u_i-u_6]\Phi_+(u_6-)$$

$$=\frac{[u_6]}{[1+u_0-u_7,2+u_0-u_7]}\prod_{i=1}^{5}[1+u_0-u_i-u_7]\Phi_+(u_7-)$$

$$+\frac{[u_7-u_6]}{[1+u_0,2+u_0]}\prod_{i=1}^{5}[1+u_0-u_i]\Phi(\mathbf{u}). \tag{2.16}$$

Proof. Suppose that $u_7 = -n$. Then, after the application of elliptic Bailey transformation (2.10) to all three ${}_{12}V_{11}$ series in (2.16), we see that this identity is equivalent to the equality

$$\Phi(r_6-, r_1+) - \Phi(\mathbf{r}) = \Phi_+(r_6-) \tag{2.17}$$
$$\times \frac{[r_0+1, r_0+2, r_1-r_6+1, r_1+r_6-r_0-1, r_7]}{[1+r_0-r_6, 2+r_0-r_6, r_0-r_1, 1+r_0-r_1, 1+r_0-r_7]} \prod_{i=2}^{5} \frac{[r_i]}{[1+r_0-r_i]},$$

which coincides with the previous contiguous relation after a change of notations for parameters. Similarly, we can prove identity (2.16) for $u_i = -n$, $i = 1, \ldots, 6$. □

3. A FAMILY OF DISCRETE BIORTHOGONAL FUNCTIONS

Introduce parameters $d_i, i = 1, \ldots, 5$, and $x_{0,1,2}$ satisfying the relations $x_2 = x_0 + x_1$ and $\sum_{i=1}^{5} d_i = 1 + 2(x_0 + x_2)$ (the balancing condition). We shall need the following three sequences of numbers $\alpha_k, \beta_k, \lambda_k$ and a parametrization of the argument of rational functions z in terms of an auxiliary variable u:

$$\alpha_k = \frac{[k-x_2+e_1, k-x_2+e_2]}{[k-x_2+d_1, k-x_2+d_2]}, \quad \beta_k = \frac{[k-e_1+1, k-e_2+1]}{[k-d_1+1, k-d_2+1]},$$
$$\lambda_k = \frac{[k+x_0-e_1, k+x_0-e_2]}{[k+x_0-d_1, k+x_0-d_2]}, \quad z(u) = \frac{[u, u+e_2-e_1]}{[u+d_2-e_1, u+d_1-e_1]}, \tag{3.1}$$

where e_1, e_2 are arbitrary parameters with the restrictions $e_1 + e_2 = d_1 + d_2$ and $e_1 \neq d_{1,2}$. Using (2.3), we derive the following relations

$$z(u) - \alpha_k = \frac{[k+u+e_2-x_2, k-u+e_1-x_2, d_2-e_1, e_1-d_1]}{[u+d_2-e_1, u+d_1-e_1, k-x_2+d_1, k-x_2+d_2]}, \tag{3.2}$$

$$z(u) - \beta_{k-1} = \frac{[k+u-e_1, k-u-e_2, d_2-e_1, e_1-d_1]}{[u+d_2-e_1, u+d_1-e_1, k-d_1, k-d_2]}, \tag{3.3}$$

$$z(u) - \lambda_k = \frac{[k+u+x_0-e_1, k-u+x_0-e_2, d_2-e_1, e_1-d_1]}{[u+d_2-e_1, u+d_1-e_1, k+x_0-d_1, k+x_0-d_2]}. \tag{3.4}$$

Introduce the functions

$$R_n(z(u)) = {}_{12}V_{11}(1-x_1; 1+x_0-d_3, 1+x_0-d_4, 1+x_0-d_5,$$
$$1+u+x_0-e_1, 1-u+x_0-e_2, 1-x_2+n, -n). \tag{3.5}$$

Proposition 6. *The functions $R_n(z)$ defined by (3.5) are rational functions of the type $[n/n]$ of the argument $z(u)$ and the poles of $R_n(z)$ are located at the points α_j, $j = 1, \ldots, n$.*

Proof. By the definition of ${}_{r+1}V_r$ series, we have

$$R_n(z(u)) = \sum_{k=0}^{n} C_k \frac{[1+u+x_0-e_1, 1-u+x_0-e_2]_k}{[1+u-x_2+e_2, 1-u-x_2+e_1]_k}, \tag{3.6}$$

where C_k are some coefficients not depending on u. From (3.2) and (3.4), we have

$$\prod_{i=1}^{k} \frac{z-\lambda_i}{z-\alpha_i} = \frac{[1-x_2+d_1, 1-x_2+d_2, 1+u+x_0-e_1, 1-u+x_0-e_2]_k}{[1+x_0-d_1, 1+x_0-d_2, 1+u-x_2+e_2, 1-u-x_2+e_1]_k}. \tag{3.7}$$

Comparing (3.6) and (3.7), we see that

$$R_n(z(u)) = \sum_{k=0}^{n} \tilde{C}_k \prod_{i=1}^{k} \frac{z(u)-\lambda_i}{z(u)-\alpha_i}, \tag{3.8}$$

where $\tilde{C}_k$ do not depend on z, that is $R_n(z(u))$ is a sum of rational functions of the type $[k/k]$ having poles at $z=\alpha_i$, $i=1,2,\ldots,n$. This proves the proposition. □

Consider the conditions of simplicity of poles α_i. Using (2.3), we find

$$\alpha_k - \alpha_s = \frac{[e_1-d_1, e_1-d_2, k-s, k+s+d_1+d_2-2x_2]}{[k-x_2+d_1, k-x_2+d_2, s-x_2+d_1, s-x_2+d_2]}. \tag{3.9}$$

It is seen that $\alpha_k = \alpha_s$ for $k \neq s$ in the following cases. First, if $e_1 = d_{1,2}$, which is forbidden. Second, if $(m_1+m_2\tau)/\sigma$ is an integer for at least one pair of integers $m_{1,2} \in \mathbb{Z}$, so that $[n]=0$ for some integer n. This is an elliptic analogue of the root of unity situation $q^n=1$ for q-special functions requiring a special treatment (see, e.g. [22]). Finally, if $d_1+d_2-2x_2 = (m_1+\tau m_2)/\sigma - N - 2$ with N a positive integer, $m_{1,2} \in \mathbb{Z}$. In the following, we assume that none of these conditions is satisfied.

Substituting into contiguous relations (2.13) and (2.16) the ${}_{12}V_{11}$ series defining rational functions $R_n(z)$, we get the following three term recurrence relation (for details, see [25, 26, 27])

$$\begin{aligned} &\epsilon_n a_n (z-\alpha_{n+1})(R_{n+1}(z)-R_n(z)) - \epsilon_{n-1} b_n (z-\beta_{n-1})(R_n(z)-R_{n-1}(z)) \\ &\quad = c_n (z-\lambda_1) R_n(z), \qquad n=0,1,2,\ldots, \end{aligned} \tag{3.10}$$

where the second term is equal to zero for $n=0$. The recurrence coefficients have the form

$$\begin{aligned} \epsilon_n &= \frac{[n+2-x_1, n+3-x_1, n-x_0, n-x_0-1]}{[2-x_1, 3-x_1, 2n+2-x_2, -x_0-1]} \prod_{i=1}^{5} \frac{[1-x_2+d_i]}{[1+x_0-d_i]}, \\ a_n &= \frac{[n+1-x_2]}{[n+2-x_1, n+3-x_1]} \prod_{i=1}^{5} [n+1-x_2+d_i], \\ b_n &= \frac{[n]\prod_{i=1}^{5}[n-d_i]}{[n-2-x_0, n-1-x_0]}, \quad c_n = \frac{[2n+1-x_2]}{[2-x_1, 3-x_1]} \prod_{i=1}^{5} [1-x_2+d_i]. \end{aligned}$$

It is convenient to introduce the following combinations of theta functions entering the recurrence coefficients

$$\begin{aligned} G_n &= \frac{[n+1-x_2, n-x_0]\prod_{i=1}^{5}[n+1-x_2+d_i]}{[2n+1-x_2, 2n+2-x_2, n+2-x_1]}, \\ D_n &= \frac{[n, n+1-x_1]\prod_{i=1}^{5}[n-d_i]}{[2n-x_2, 2n+1-x_2, n-x_0-1]}. \\ h_n &= \prod_{i=1}^{n} G_{i-1} D_i = \frac{[1, 1-x_2]_n}{[1-x_2, 2-x_2]_{2n}} \prod_{i=1}^{5} [1-d_i, 1-x_2+d_i]_n. \end{aligned} \tag{3.11}$$

Define also the following polynomials of the n-th degree $P_n(z)$:

$$P_n(z) = \kappa_n A_n(z) R_n(z), \tag{3.12}$$

$$\kappa_n = G_{n-1} \cdots G_1 G_0 = \frac{[1-x_2, -x_0]_n \prod_{i=1}^5 [1-x_2+d_i]_n}{[1-x_2]_{2n}[2-x_1]_n}.$$

Then it is not difficult to see from (3.10) that $P_n(z)$ satisfy the following three term recurrence relation

$$P_{n+1}(z) + (v_n - \rho_n z) P_n(z) + u_n (z-\alpha_n)(z-\beta_{n-1}) P_{n-1}(z) = 0, \tag{3.13}$$

$$u_n = G_{n-1} D_n, \quad \rho_n = G_n + D_n + \frac{[x_0+1] \prod_{i=1}^5 [d_i - x_0 - 1]}{[n+2-x_1, n-1-x_0]},$$

$$v_n = G_n \alpha_{n+1} + D_n \beta_{n-1} + \frac{[x_0+1] \prod_{i=1}^5 [d_i - x_0 - 1]}{[n+2-x_1, n-1-x_0]} \lambda_1.$$

Impose now the constraint

$$d_3 = x_2 - N - 1 + \delta, \qquad \delta \to 0, \tag{3.14}$$

where N is a positive integer. Note that for $\delta \to 0$ the rational function $R_{N+1}(z)$ is not well defined because one of the coefficients in the series (3.5) diverges (there is a simple pole in δ). However, the polynomial $P_{N+1}(z)$ in (3.12) is finite because the coefficient κ_{N+1} contains a simple zero in δ.

Proposition 7. *Zeros of the polynomial $P_{N+1}(z)$ have the following form*

$$z_s = \lambda_{s+1} = \frac{[s+x_0-e_1+1, s+x_0-e_2+1]}{[s+x_0-d_1+1, s+x_0-d_2+1]}, \tag{3.15}$$

where $s = 0, 1, \ldots, N$. These zeros are simple provided $[n] \neq 0$ for some $n \in \mathbb{Z}$ and

$$2x_0 - d_1 - d_2 \neq (m_1 + m_2 \tau)/\sigma - 2 - M \tag{3.16}$$

for some integers $M > 0$ and $m_{1,2} \in \mathbb{Z}$.

Proof. In the limit $\delta \to 0$ the coefficient $\kappa_{N+1} \to 0$ because of the factor $[N+1-x_2+d_3]$. As to the series (3.8), for $n = N+1$ only the last term diverges due to the factor $1/[N+1-x_2+d_3]$. As a result, from (3.8) we get

$$P_{N+1}(z) = \gamma_{N+1} \prod_{i=0}^{N} (z - z_i), \qquad \gamma_{N+1} = \frac{\prod_{i=1}^5 [1+x_0-d_i]_{N+1}}{[2-x_1+N]_{N+1}},$$

where $z_i = \lambda_{i+1}$. The condition of simplicity of zeros z_s (3.16) is established in the same way as for the poles α_i. In what follows, we will assume that the condition (3.16) holds and all zeros z_s are simple. □

Using these zeros z_s, in [25, 26] we have established that rational functions (3.5) and

$$\begin{aligned} T_n(z(u)) = {}_{12}V_{11}(&2+x_0-d_1-d_2; 2+x_0+x_2-d_1-d_2-d_3, \\ &2+x_0+x_2-d_1-d_2-d_4, 2+x_0+x_2-d_1-d_2-d_5, \\ &1-x_2+n, 1+u+x_0-e_1, 1-u+x_0-e_2, -n) \end{aligned} \tag{3.17}$$

with $n = 0, 1, \ldots, N$ and $d_3 = x_2 - N - 1$ satisfy the biorthogonality condition

$$\sum_{s=0}^{N} R_n(z_s) T_m(z_s) \omega_s = f_n \, \delta_{nm}, \tag{3.18}$$

for the following discrete set of values of the argument

$$z = z_s \equiv z(u_s), \quad u_s = s + x_0 + 1 - e_2, \ s = 0, 1, \dots, N, \tag{3.19}$$

and the weight function ω_s and normalization constants f_n

$$\omega_s = \frac{[2x_0 + 2 - d_1 - d_2 + 2s][-N, 2x_0 + 2 - d_1 - d_2]_s}{[2x_0 + 2 - d_1 - d_2][1, 2x_0 + 3 - d_1 - d_2 + N]_s} \times \frac{[x_0, 1 + d_4 - x_2, 1 + d_5 - x_2, 1 + x_0 + x_2 - d_1 - d_2]_s}{[2 - x_1, 3 + x_0 - d_1 - d_2, -N + d_4, -N + d_5]_s}, \tag{3.20}$$

$$f_n = \kappa \frac{[1 - x_2][1, 2 - x_2 + N, 2 - x_1]_n}{[1 - x_2 + 2n][-N, 1 - x_2, -x_0]_n} \times \frac{[3 + x_0 - d_1 - d_2, 1 - d_4, 1 - d_5]_n}{[-1 - x_0 - x_2 + d_1 + d_2, 1 + d_4 - x_2, 1 + d_5 - x_2]_n}, \tag{3.21}$$

$$\kappa = \frac{[2 - x_2, x_2 - d_4 - d_5, 1 + x_0 - d_4, 1 + x_0 - d_5]_N}{[1 - d_4, 1 - d_5, 2 - x_1, x_0 + x_2 - d_4 - d_5]_N}. \tag{3.22}$$

For $\mathrm{Im}(\tau) \to +\infty$ these discrete BRF reduce to the Wilson ${}_{10}\phi_9$ family of functions [33, 34]. For a discussion of self-duality properties of $R_n(z_s), T_n(z_s)$, difference equations for them, and a divided difference operator lowering the "degree" n of these rational functions, see [24, 26, 27].

Biorthogonality conditions for continuous BRF $R_n(z), T_n(z)$ (or elliptic extensions of the Rahman ${}_{10}\phi_9$ family of BRF [16]) and their non-rational functions bilinear generalization have been established in [19]. The elliptic beta integral, discovered in [18], plays a central role in the corresponding considerations. All these functions extend essentially the available set of classical special functions [1].

4. A terminating continued fraction

In this section, we describe the details of derivation of the terminating continued fraction announced in [27]. This fraction is an elliptic generalization of the ${}_{10}\phi_9$ family of terminating continued fractions constructed by Gupta and Masson [10, 11, 14]. The latter represents an extension of the Watson q-hypergeometric series continued fraction [30] which, in turn, is a q-analogue of the famous Ramanujan Entry 40 continued fraction built from a special case of the very-well-poised balanced hypergeometric function ${}_9F_8$ [4, 17].

Suppose we have a three term recurrence relation

$$\psi_{n+1} = \xi_n \psi_n + \eta_n \psi_{n-1}, \quad n \in \mathbb{N}, \tag{4.1}$$

for some nonsingular coefficients ξ_n, η_n. Denote as U_n and V_n two sequences satisfying (4.1) with the initial conditions $U_0 = 0$, $U_1 = 1$ and $V_0 = 1$, $V_1 = \xi_0$. The ratio U_n/V_n is known to be equal to the following continued fraction [13]

$$\frac{U_n}{V_n} = \cfrac{1}{\xi_0 + \cfrac{\eta_1}{\xi_1 + \cfrac{\eta_2}{\xi_2 + \dots + \cfrac{\eta_{n-1}}{\xi_{n-1}}}}}, \qquad n = 1, 2, \dots. \tag{4.2}$$

In the case of orthogonal polynomials (given by the sequence V_n), ξ_n are linear in the argument of polynomials z and η_n do not depend on z. When $\eta_n(z)$ are quadratic in z and $\xi_n(z)$ are linear in z, we get continued fractions named as R_{II}

fractions in [12] (they are known also as osculatory continued fractions [35]). As we know, such three term recurrence relations lead to BRF. We set

$$\xi_n(z) = \rho_n z - v_n, \qquad \eta_n(z) = -u_n(z-\alpha_n)(z-\beta_{n-1}), \tag{4.3}$$

where $\rho_n, v_n, u_n, \alpha_n, \beta_n$ are some sequences of numbers. Then $V_n(z) = P_n(z)$ are the n-th degree polynomials entering numerators of BRF and they satisfy the initial conditions $P_0 = 1$, $P_1(z) = \rho_0 z - v_0$. The polynomials $U_n(z) = P^{(1)}_{n-1}(z)$, called the associated polynomials, have the degree $n-1$. They satisfy the recurrence relation

$$P^{(1)}_n(z) + (v_n - \rho_n z)P^{(1)}_{n-1}(z) + u_n(z-\alpha_n)(z-\beta_{n-1})P^{(1)}_{n-2}(z) = 0 \tag{4.4}$$

with the initial conditions $P^{(1)}_{-1} = 0$, $P^{(1)}_0(z) = 1$.

Substituting recurrence coefficients (4.3) into (4.2), we get

$$F_N(z) \equiv \frac{P^{(1)}_N(z)}{P_{N+1}(z)} = \cfrac{1}{\rho_0 z - v_0 - \cfrac{u_1(z-\alpha_1)(z-\beta_0)}{\rho_1 z - v_1 - \cdots - \cfrac{u_N(z-\alpha_N)(z-\beta_{N-1})}{\rho_N z - v_N}}}. \tag{4.5}$$

Since $F_N(z)$ is a rational function of z, we can expand it into the partial fraction:

$$F_N(z) = \sum_{s=0}^{N} \frac{g_s}{z - z_s}, \tag{4.6}$$

where z_s, $s = 0, 1, \ldots, N$, are zeros of the polynomial $P_{N+1}(z)$ and

$$g_s = \frac{P^{(1)}_N(z_s)}{P'_{N+1}(z_s)} \tag{4.7}$$

We assume that $P_{N+1}(z)$ has only simple zeros, that is $z_s \neq z'_s$ for $s \neq s'$.

Any two solutions U_n, V_n of the recurrence relation (4.1) satisfy the Wronskian type relation

$$U_{n+1}V_n - U_n V_{n+1} = (-1)^n \eta_1 \cdots \eta_n (U_1 V_0 - U_0 V_1),$$

which in our case yields

$$P_n(z)P^{(1)}_n(z) - P_{n+1}(z)P^{(1)}_{n-1}(z) = h_n A_n(z)\tilde{B}_n(z), \tag{4.8}$$

where $h_n = u_1 u_2 \cdots u_n$ and

$$A_n(z) = \prod_{i=1}^{n}(z-\alpha_i), \qquad \tilde{B}_n(z) = \prod_{i=1}^{n}(z-\beta_{i-1}).$$

Taking $n = N$ and $z = z_s$, $s = 0, 1, \ldots, N$, in (4.8), we find $P^{(1)}_N(z_s)$ in terms of $P_N(z_s), h_n, A_n(z_s)$ and $\tilde{B}_N(z_s)$. This results in the following expression for g_s convenient for computations:

$$g_s = \frac{h_N A_N(z_s)\tilde{B}_N(z_s)}{P'_{N+1}(z_s)P_N(z_s)}. \tag{4.9}$$

Consider now the explicit family of elliptic BRF $R_n(z(u))$ defined in (3.5) and calculate g_s for corresponding polynomials (3.12) with $d_3 = x_2 - N - 1$. In this case $u_{N+1} = 0$ and the continued fraction (4.5) terminates automatically. First of all notice that

$$P'_{N+1}(z_s) = \gamma_{N+1}(z_s - z_0) \cdots (z_s - z_{s-1})(z_s - z_{s+1}) \cdots (z_s - z_N). \tag{4.10}$$

This expression can be calculated using the relation

$$z_s - z_k = \mu_s \frac{[k - s, 2 + k + s + 2x_0 - d_1 - d_2]}{[k + x_0 - d_1 + 1, k + x_0 - d_2 + 1]}, \tag{4.11}$$

$$\mu_s = \frac{[d_2 - e_1, e_1 - d_1]}{[s + 1 + x_0 - d_1, s + 1 + x_0 - d_2]}.$$

As a result,

$$P'_{N+1}(z_s) = \gamma_{N+1}\mu_s^N \frac{[s + 1 + x_0 - d_1, s + 1 + x_0 - d_2]}{[2s + 2 + 2x_0 - d_1 - d_2]}$$
$$\times \frac{[s + 2 + 2x_0 - d_1 - d_2]_{N+1}[1]_N[1]_s}{[1 + x_0 - d_1, 1 + x_0 - d_2]_{N+1}[-N]_s}.$$

The polynomial $\tilde{B}_n(z_s)$ is easily found to be

$$\tilde{B}_n(z_s) = \mu_s^n \frac{[s + 2 + x_0 - d_1 - d_2, -s - x_0]_n}{[1 - d_1, 1 - d_2]_n}.$$

Substituting $d_3 = x_2 - N - 1$ and (3.19) into (3.5) for $n = N$, we find

$$R_N(z_s) = {}_{10}V_9(1 - x_1; 1 + x_0 - d_4, 1 + x_0 - d_5,$$
$$N - x_2 + 1, s + 2 + 2x_0 - d_1 - d_2, -s). \tag{4.12}$$

This very-well-poised elliptic hypergeometric series can be summed using the Frenkel-Turaev formula (2.8), which yields

$$R_N(z_s) = \frac{[2 - x_1, d_4 + d_5 - x_0 - x_2, -N + d_4, -N + d_5]_s}{[1 + d_4 - x_2, 1 + d_5 - x_2, 1 + x_0 - N, d_4 + d_5 - 1 - N - x_0]_s}. \tag{4.13}$$

Taking into account that $P_N(z_s)/A_N(z_s) = \kappa_N R_N(z_s)$ and substituting all the necessary entries into (4.9), we find

$$g_s = \frac{t_N[2s + 2 + 2x_0 - d_1 - d_2]}{[s + 1 + x_0 - d_1, s + 1 + x_0 - d_2]} \tag{4.14}$$
$$\times \frac{[x_0 + 1, 2 + 2x_0 - d_1 - d_2, 1 + d_4 - x_2]_s}{[1, 2 + x_0 - d_1 - d_2, 3 + 2x_0 - d_1 - d_2 + N]_s}$$
$$\times \frac{[1 + d_5 - x_2, 1 + x_0 + x_2 - d_1 - d_2, -N]_s}{[2 - x_1, d_4 - N, d_5 - N]_s},$$

$$t_N = \frac{[1 - d_4, 1 - d_5, 2 - x_1, 2 + x_0 - d_1 - d_2]_N}{[2 - x_2, 1 + x_0 - d_4, 1 + x_0 - d_5, 2 + 2x_0 - d_1 - d_2]_{N+1}}.$$

Now we may compute the continued fraction itself:

$$F_N(z(u)) = \sum_{s=0}^{N} \frac{g_s}{z(u) - z_s} = \frac{[u + d_2 - e_1, u + d_1 - e_1]}{[d_2 - e_1, e_1 - d_1]}$$
$$\times \sum_{s=0}^{N} g_s \frac{[s + 1 + x_0 - d_1, s + 1 + x_0 - d_2]}{[s + 1 + u + x_0 - e_1, s + 1 - u + x_0 - e_2]}$$
$$= \frac{t_N[2 + 2x_0 - d_1 - d_2, u + d_2 - e_1, u + d_1 - e_1]}{[d_2 - e_1, d_1 - e_1, u + 1 + x_0 - e_1, u - 1 - x_0 + e_2]} \, {}_{12}V_{11}(u_0; u_1, \ldots, u_7),$$

$u_0 = 2+2x_0-d_1-d_2, \quad u_1 = 1+u+x_0-e_1, \quad u_2 = 1-u+x_0-e_2, \quad u_4 = 1+x_0,$

$u_3 = 1+x_0+x_2-d_1-d_2, \quad u_5 = 1+d_4-x_2, \quad u_6 = 1+d_5-x_2, \quad u_7 = -N.$

Let us apply now to this ${}_{12}V_{11}$ series the elliptic Bailey transformation (2.9). As a result, we get

$$F_N(z(u)) = K_N \frac{[u+d_2-e_1, u+d_1-e_1]}{[u+1+x_0-e_1, u-1-x_0+e_2]} \, {}_{12}V_{11}(2-x_1; 1, 1+x_0,$$
$$1+u-x_2+e_2, 1-u-x_2+e_1, 1+d_4-x_2, 1+d_5-x_2, -N), \quad (4.15)$$
$$K_N = t_N \frac{[2+2x_0-d_1-d_2][3+2x_0-d_1-d_2]_N}{[d_2-e_1, d_1-e_1][3-x_1]_N}$$
$$\times \frac{[2-d_4+x_0, 2-d_5+x_0, x_2-N-1]_N}{[d_4-N, d_5-N, d_1+d_2-x_0-N-1]_N}$$
$$= \frac{[2-x_1]}{[2+N-x_1, 1+x_0-d_4, 1+x_0-d_5, d_2-e_1, d_1-e_1]}.$$

This gives the formula announced in [27]. Giving to the parameters entering the ${}_{12}V_{11}$ series special values, in the same way as it was done in the ${}_{10}\phi_9$ case in [11], we can reduce it to ${}_{10}V_9$ series, sum them, and express corresponding continued fractions as ratios of some products of theta functions. In general, the derived elliptic extension of the Ramanujan-Watson-Gupta-Masson terminating continued fraction is the most general known at present explicit continued fraction.

5. Connections with multipoint Padé approximation

In this section, we show an equivalence between the Cauchy-Jacobi interpolation problem (CJIP) and the theory of BRF. For relevant references see, e.g. [9, 15, 28]. CJIP is a special case of a more general multipoint Padé approximation theory [3].

Consider a special CJIP of the $[(n-1)/n]$ type. Take a meromorphic interpolation function $F(z)$ of a complex argument z together with a fixed set of (distinct) interpolation points $a_i, i = 1, 2, \ldots$. We are interested in the problem of constructing polynomials $P_n(z)$ and $S_n(z)$ such that

(i) both $P_n(z)$ and $S_n(z)$ have the degree n and the polynomials $P_n(z)$ are monic, that is $P_n(z) = z^n + O(z^{n-1})$;

(ii) $P_n(a_i) \neq 0$ for all $n, i \in \mathbb{N}$;

(iii) the following interpolation property

$$F(a_i) = \frac{S_{n-1}(a_i)}{P_n(a_i)}, \quad i = 1, 2, \ldots, 2n, \tag{5.1}$$

holds true for all $n = 1, 2, \ldots$.

It can be shown [3, 15] that this problem has a unique solution under some weak non-degeneracy condition. Its formal solution is based on the technique of divided differences. Recall (see, e.g. [3]) that zero-order divided difference of an arbitrary function $f(z)$ in the point a_i is defined as the value of this function at $z = a_i$: $\mathcal{D}^{(0)}_{a_1} f(z) = f(a_1)$. The first-order divided difference is defined by the formula

$$\mathcal{D}^{(1)}_{a_1,a_2} f(z) = \frac{f(a_1) - f(a_2)}{a_1 - a_2}.$$

The n-th order divided difference is defined by induction. Assume that the $(n-1)$-th order divided difference $\mathcal{D}^{(j-1)}_{a_1,\dots,a_{j-1},a_j} f(z)$ is already defined. Then we set

$$\mathcal{D}^{(j)}_{a_1,\dots,a_j,a_{j+1}} f(z) = \frac{\mathcal{D}^{(j-1)}_{a_1,\dots,a_{j-1},a_j} f(z) - \mathcal{D}^{(j-1)}_{a_1,\dots,a_{j-1},a_{j+1}} f(z)}{a_j - a_{j+1}}.$$

Hermite has found a very convenient formula for divided differences

$$\mathcal{D}^{(j)}_{a_1,\dots,a_{j+1}} f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)d\zeta}{(\zeta - a_1)(\zeta - a_2)\cdots(\zeta - a_{j+1})}, \tag{5.2}$$

where the closed contour Γ on the complex plane encircles all interpolation points $a_1, a_2, \dots, a_{j+1}$ and the function $f(z)$ is analytic inside Γ.

One can show [3] that the following conditions are necessary and sufficient for a solvability of CJIP:

$$\mathcal{D}^{(j)}_{a_{i_1},a_{i_2},\dots,a_{i_{j+1}}} (P_n(z)F(z)) = 0, \quad j = n, n+1, \dots, 2n-1, \tag{5.3}$$

and (a non-degeneracy condition)

$$\mathcal{D}^{(n-1)}_{a_{i_1},a_{i_2},\dots,a_{i_n}} (P_n(z)F(z)) \neq 0, \tag{5.4}$$

where $\{i_1, i_2, \dots, i_{j+1}\}$ is an arbitrary permutation of the numbers $1, 2, \dots, j+1$.

Hermite formula (5.2) allows us to rewrite condition (5.3) in a very convenient form

$$\int_\Gamma \frac{F(\zeta)P_n(\zeta)\zeta^j d\zeta}{(\zeta - a_1)(\zeta - a_2)\cdots(\zeta - a_{2n})} = 0, \quad j = 0, 1, \dots, n-1. \tag{5.5}$$

But (5.5) is nothing else than biorthogonality condition (1.9) for the polynomials $P_n(z)$ defining BRF provided one identifies $\alpha_i = a_{2i}$, $i = 1, 2, \dots, n$, and $\beta_i = a_{2i+1}$, $i = 0, \dots, n-1$. The functional $\mathcal{L}$ is defined as

$$\mathcal{L}\{f(z)\} = \int_\Gamma \frac{f(\zeta)F(\zeta)}{\zeta - \beta_0} d\zeta. \tag{5.6}$$

Non-degeneracy condition (5.4) can be rewritten as

$$\int_\Gamma \frac{F(\zeta)P_n(\zeta)\zeta^n d\zeta}{(\zeta - a_1)(\zeta - a_2)\cdots(\zeta - a_{2n})} \neq 0. \tag{5.7}$$

We see that CJIP is essentially equivalent to the theory of BRF.

It is instructive to see how the pair $R_n(z), T_n(z)$ of BRF appears in CJIP. Let the polynomials $P_n(z), S_n(z)$, $n = 1, 2, \dots$, solve CJIP of the $[(n-1)/n]$ type for an interpolation function $F(z)$ with the interpolation points $a_1, \dots, a_{2n-1}, a_{2n}$. Let polynomials $Q_n(z), U_n(z)$, $n = 1, 2, \dots$, solve CJIP for *the same* function $F(z)$ and a modified set of interpolation points $a_1, \dots, a_{2n-1}, a_{2n+1}$ (i.e. we replace the last point a_{2n} by the new point a_{2n+1}, keeping $a_1, \dots, a_{2n-1}$ intact):

$$F(a_i) = \frac{V_{n-1}(a_i)}{Q_n(a_i)}, \quad i = 1, 2, \dots, 2n-1, 2n+1.$$

We assume that non-degeneracy condition (5.4) is fulfilled for polynomials $P_n(z)$ and $Q_n(z)$. Introduce the corresponding rational functions

$$R_n(z) = \frac{P_n(z)}{(z - a_2)(z - a_4)\cdots(z - a_{2n})} = \frac{P_n(z)}{(z - \alpha_1)\cdots(z - \alpha_n)},$$

$$T_n(z) = \frac{Q_n(z)}{(z - a_3)(z - a_5)\cdots(z - a_{2n+1})} = \frac{Q_n(z)}{(z - \beta_1)\cdots(z - \beta_n)}. \tag{5.8}$$

Clearly, both $R_n(z)$ and $T_n(z)$ are rational functions of the type $[n/n]$.

Theorem 8. *The pair $R_n(z), T_n(z)$ of rational functions satisfies the biorthogonality relation*

$$\int_\Gamma \frac{R_n(\zeta)T_m(\zeta)F(\zeta)d\zeta}{\zeta - a_1} = h_n\delta_{nm}, \tag{5.9}$$

where $h_n \neq 0$ are some normalization constants.

Proof. Assume that $m < n$. Then, equality (5.9) is a simple consequence of biorthogonality relation (5.5) and definition (5.8). Assume now that $m > n$. In this case we have the biorthogonality condition

$$\int_\Gamma \frac{F(\zeta)Q_n(\zeta)\zeta^j d\zeta}{(\zeta-a_1)(\zeta-a_2)\cdots(\zeta-a_{2n-1})(\zeta-a_{2n+1})} = 0, \quad j = 0, 1, \dots, n-1, \tag{5.10}$$

for the polynomials $Q_n(z)$. Then, relation (5.9) is a simple consequence of (5.10) and (5.8). Finally, for $n = m$ we see that $h_n \neq 0$ because of (5.7).

We thus see that biorthogonality condition (1.7) coincides with (5.9) after the identification of the functional $\mathcal{L}$ with the one defined in (5.6). □

As far as we know, despite of the fact that relation (5.5) is well-known in the theory of CJIP [15], the explicit identification of CJIP with the theory of BRF expressed by (5.9) is a new result.

Remark 3. The moments M_{ik} corresponding to BRF (5.8) are defined through the divided differences as follows:

$$M_{ik} = \mathcal{L}\left\{\frac{1}{B_i(z)A_k(z)}\right\} = \mathcal{D}^{(i+k+1)}_{a_2,a_4,\dots,a_{2k},a_1,a_3,\dots,a_{2i+1}} F(z). \tag{5.11}$$

Now we would like to demonstrate how the recurrence relation of R_{II} type for the polynomials $P_n(z)$ can be derived from the theory of CJIP. Assume that the polynomials $P_n(z)$ are monic: $P_n(z) = z^n + O(z^n)$. We denote by r_n the coefficient of the leading term of polynomials $S_n(z)$, that is $S_n(z) = r_n z^n + O(z^{n-1})$. Introduce the function

$$\psi_n(z) = F(z) - \frac{S_{n-1}(z)}{P_n(z)}. \tag{5.12}$$

It has zeros at the points $z = a_1, a_2, \dots, a_{2n}$, which follows from the interpolation property (iii). Similarly, the function

$$\psi_{n+1}(z) = F(z) - \frac{S_n(z)}{P_{n+1}(z)} \tag{5.13}$$

has zeros at the points $z = a_1, a_2, \dots, a_{2n+1}, a_{2n+2}$. Consider the following combination of $\psi_n(z)$

$$\chi_n(z) \equiv \psi_n(z) - \psi_{n+1}(z) = \frac{S_n(z)}{P_{n+1}(z)} - \frac{S_{n-1}(z)}{P_n(z)} = \frac{Y_{2n}(z)}{P_n(z)P_{n+1}(z)}, \tag{5.14}$$

where

$$Y_{2n}(z) = S_n(z)P_n(z) - P_{n+1}S_{n-1}(z)$$

is a polynomial of degree $\leq 2n$. Clearly, the function $\chi_n(z)$ has zeros at $z = a_1, a_2, \dots, a_{2n}$. This is possible if and only if the polynomial $Y_{2n}(z)$ has exactly $2n$ zeros at the same points. Thus

$$Y_{2n}(z) = s_n(z-a_1)(z-a_2)\cdots(z-a_{2n}),$$

where $s_n = r_n - r_{n-1}$ is the leading coefficient of the polynomial $Y_{2n}(z)$. Analogously, from (5.12) we can obtain

$$\rho_n(z) \equiv \frac{S_{n+1}(z)}{P_{n+2}(z)} - \frac{S_{n-1}(z)}{P_n(z)} = \frac{Z_{2n+1}(z)}{P_n(z)P_{n+2}(z)}, \tag{5.15}$$

where

$$Z_{2n+1}(z) = S_{n+1}(z)P_n(z) - P_{n+2}S_{n-1}(z)$$

is a polynomial of degree $\leq 2n+1$ having zeros at $z = a_1, a_2, \ldots, a_{2n}$. This is possible if and only if

$$Z_{2n+1}(z) = (t_n z + \gamma_n)(z - a_1)\cdots(z - a_{2n}), \tag{5.16}$$

where $t_n = r_{n+1} - r_{n-1} = s_n + s_{n+1}$. Observe now that $\rho_n(z) = \chi_n(z) + \chi_{n+1}(z)$, and, simplifying this expression, we arrive at the three term recurrence relation for the polynomials $P_n(z)$:

$$s_n P_{n+2}(z) = (t_n z + \gamma_n)P_{n+1}(z) - s_{n+1}(z - a_{2n+1})(z - a_{2n+2})P_n(z), \tag{5.17}$$

which coincides with recurrence relation (1.17). Note that in (5.17) we deal with monic polynomials and it is easily verified that the leading terms in the left-hand and right-hand sides of (5.17) coincide.

Consider also the role of Christoffel type transformations in the theory of CJIP. Let $P_n(z), S_n(z)$ be a pair of polynomials providing a solution of CJIP of the $[(n-1)/n]$ type for an interpolation function $F(z)$ and the interpolation points $a_1, a_2, \ldots, a_{2n}$. Introduce a new interpolation function

$$\begin{aligned}\tilde{F}(z) &= \mathcal{D}^{(1)}_{z,a_1}((z-\mu)F(z)) \\ &= \frac{(z-\mu)F(z) - (a_1-\mu)F(a_1)}{z - a_1},\end{aligned} \tag{5.18}$$

where μ is an arbitrary parameter. We are seeking a pair of polynomials $\tilde{P}_n(z)$, $\tilde{S}_n(z)$ providing solution of CJIP of the $[(n-1)/n]$ type on the set of interpolation points $a_2, a_3, \ldots, a_{2n}, a_{2n+1}$.

Proposition 9. *Monic polynomials $\tilde{P}_n(z)$ are obtained from the polynomials $P_n(z)$ by the following Christoffel type transformation*

$$\tilde{P}_n(z) = \frac{\xi_n P_{n+1}(z) + (1-\xi_n)(z - a_{2n+1})P_n(z)}{z - \mu}, \tag{5.19}$$

where ξ_n look as follows

$$\xi_n = \frac{(\mu - a_{2n+1})P_n(\mu)}{P_n(\mu)(\mu - a_{2n+1}) - P_{n+1}(\mu)}. \tag{5.20}$$

Proof. From the interpolation conditions, we have two relations

$$\psi_n(z) \equiv F(z) - \frac{S_{n-1}(z)}{P_n(z)} = (z-a_1)\cdots(z-a_{2n})\phi_n(z), \tag{5.21}$$

$$\tilde{\psi}_n(z) \equiv \tilde{F}(z) - \frac{\tilde{S}_{n-1}(z)}{\tilde{P}_n(z)} = (z-a_2)\cdots(z-a_{2n+1})\tilde{\phi}_n(z), \tag{5.22}$$

where the functions $\phi_n(z), \tilde{\phi}_n(z)$ do not have singularities at $z = a_1, \ldots, a_{2n+1}$.

Subtracting (5.21) and (5.22) and taking into account (5.18), we get

$$-\frac{S_{n-1}(z)}{P_n(z)} - \frac{(\mu - a_1)F(a_1)}{z-\mu} + \frac{z-a_1}{z-\mu}\frac{\tilde{S}_{n-1}(z)}{\tilde{P}_n(z)}$$
$$= \frac{\epsilon_n^{(1)}(z-a_1)(z-a_2)\cdots(z-a_{2n})}{(z-\mu)P_n(z)\tilde{P}_n(z)}, \tag{5.23}$$

where $\epsilon_n^{(1)}$ are some constants.

Analogously, subtracting $\psi_{n+1}(z)$ and $\tilde{\psi}_n(z)$, we get

$$-\frac{S_n(z)}{P_{n+1}(z)} - \frac{(\mu - a_1)F(a_1)}{z-\mu} + \frac{z-a_1}{z-\mu}\frac{\tilde{S}_{n-1}(z)}{\tilde{P}_n(z)}$$
$$= \frac{\epsilon_n^{(2)}(z-a_1)(z-a_2)\cdots(z-a_{2n+1})}{(z-\mu)P_{n+1}(z)\tilde{P}_n(z)} \tag{5.24}$$

with different constants $\epsilon_n^{(2)}$. Subtracting (5.23) and (5.24) and taking into account relation (5.14), we arrive the relation (5.19). The coefficients ξ_n are uniquely determined from two properties: (i) both $P_n(z)$ and $\tilde{P}_n(z)$ are monic polynomials; (ii) the right-hand side of (5.19) has no pole at $z = \mu$.

Thus, the Christoffel type transformation corresponds to the transition from initial CJIP to the modified CJIP with the interpolation function $\tilde{F}(z)$ and shifted interpolation points $a_2, a_3, \dots, a_{2n+1}$. □

Note that interpolation functions $F(z)$ and $\kappa F(z)$ correspond to the same CJIP denominator polynomials $P_n(z)$ and the scaled numerator polynomials $S_{n-1}(z) \to \kappa S_{n-1}(z)$. This allows us to take the formal limit $\mu \to \infty$, which corresponds (up to an inessential common factor) to CJIP with the interpolation function

$$\tilde{F}(z) = \mathcal{D}^{(1)}_{z,a_1} = \frac{F(z) - F(a_1)}{z - a_1} \tag{5.25}$$

and the set of interpolation points $a_2, a_3, \dots, a_{2n+1}$. The corresponding CJIP denominator polynomials are

$$\tilde{P}_n(z) = \tau_n \left(P_{n+1}(z) - (z - a_{2n+1})P_n(z)\right), \tag{5.26}$$

where τ_n are normalization constants that guarantee monicity of the polynomials $\tilde{P}_n(z)$. Formula (5.26) is obtained from (5.19) by the limiting process $\mu \to \infty$.

We call transformation (5.26) as an elementary Christoffel type transformation at the point a_{2n+1}. Its importance is illustrated by the following statement.

Theorem 10. *Assume that n-th order polynomials $\tilde{S}_n(z), \tilde{P}_n(z)$ solve CJIP of the type $[n/n]$ with the same interpolation function $F(z)$ as for $P_n(z)$, but with the different set of 2n+1 interpolation points $a_1, a_2, \dots, a_{2n+1}$:*

$$F(a_i) = \frac{\tilde{S}_n(a_i)}{\tilde{P}_n(a_i)}, \quad i = 1, 2, \dots, 2n+1. \tag{5.27}$$

Then the denominator polynomials $\tilde{P}_n(z)$ of CJIP of the $[n/n]$ type are obtained from the denominator polynomials $P_n(z)$ of CJIP of the $[(n-1)/n]$ type with the help of (5.26).

We omit the proof of this theorem (which is quite simple).

Using this result, we can construct a solution of CJIP of the $[(n-1+L)/n]$ type, where L is an arbitrary positive integer. Denote corresponding numerator and denominator polynomials by $S_{n-1}^{(L)}(z)$ and $P_n^{(L)}(z)$ respectively. CJIP of the $[(n-1+L)/n]$ type means that we want to solve the interpolation problem

$$F(a_i) = \frac{S_{n-1+L}^{(L)}(a_i)}{P_n^{(L)}(a_i)}, \quad i = 1, 2, \ldots, a_{2n+L}.$$

The case $L = 0$ corresponds to the considered $[(n-1)/n]$ CJIP. As we know, for $L = 1$ the denominator polynomials $P_n^{(1)}(z)$ are obtained from $P_n^{(0)}(z)$ by the elementary Christoffel type transformation at the point a_{2n+1}. Similarly, polynomials $P_n^{(2)}(z)$ are obtained from $P_n^{(1)}(z)$ by such transformation at the point a_{2n+2}, or, equivalently, from $P_n^{(0)}(z)$ by two transformations at the points a_{2n+1}, a_{2n+2}. Repeating this consideration, we arrive at the following statement.

Proposition 11. *Denominator polynomials $P_n^{(L)}(z)$ for CJIP of the $[(n-1+L)/n]$ type are obtained from the polynomials $P_n^{(0)}(z)$ by means of L successive elementary Christoffel type transformations at the points $a_{2n+1}, a_{2n+2}, \ldots, a_{2n+L}$.*

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REFERENCES

[1] G.E. Andrews, R. Askey, and R. Roy, Special Functions, Cambridge Univ. Press, 1999.

[2] R. Askey and J. Wilson, Some Basic Hypergeometric Orthogonal Polynomials that Generalize Jacobi Polynomials, Mem. Amer. Math. Soc. **319** (1985), no. 54.

[3] G.A. Baker, Jr. and P.R. Graves-Morris, Padé Approximants, Cambridge Univ. Press, 1996.

[4] B.C. Berndt, R.L. Lamphere, and B.M. Wilson, *Chapter 12 of Ramanujan's second notebook: Continued fractions*, Rocky Mount. J. Math. **15** (1985), 235-310.

[5] A. Bultheel, P. González-Vera, E. Hendriksen, and O. Njåstad, Orthogonal rational functions, Cambridge Univ. Press, 1999.

[6] E. Date, M. Jimbo, A. Kuniba, T. Miwa, and M. Okado, *Exactly solvable SOS models, II: Proof of the star-triangle relation and combinatorial identities*, Adv. Stud. Pure Math. **16** (1988), 17-122.

[7] I.B. Frenkel and V.G. Turaev, *Elliptic solutions of the Yang-Baxter equation and modular hypergeometric functions*, The Arnold-Gelfand Mathematical Seminars, Birkhäuser, 1997, pp. 171-204.

[8] G. Gasper and M. Rahman, Basic Hypergeometric Series, Cambridge Univ. Press, 1990.

[9] A.A. Gonchar and G. Lopes, *On Markov's theorem for multipoint Padé approximants*, Math. USSR Sb. **34** (1978), 449-459.

[10] D.P. Gupta and D.R. Masson, *Watson's basic analogue of Ramanujan's Entry 40 and its generalization*, SIAM J. Math. Anal. **25** (1994), 429-440.

[11] ———, *Contiguous relations, continued fractions and orthogonality*, Trans. Amer. Math. Soc. **350** (1998), 769-808.

[12] M.E.H. Ismail and D.R. Masson, *Generalized orthogonality and continued fractions*, J. Approx. Theory **83** (1995), 1-40.

[13] W.B. Jones and W.J. Thron, Continued Fractions: Analytic Theory and Applications, Addison-Wesley, 1980.

[14] D.R. Masson, *A generalization of Ramanujan's best theorem on continued fractions*, C.R. Math. Rep. Acad. Sci. Canada **13** (1991), 167-172.

[15] J. Meinguet, *On the solubility of the Cauchy interpolation problem*, Approximation Theory, Academic Press, 1970, pp. 137-164.

[16] M. Rahman, *An integral representation of a* ${}_{10}\varphi_9$ *and continuous biorthogonal* ${}_{10}\varphi_9$ *rational functions*, Canad. J. Math. **38** (1986), 605-618.

[17] S. Ramanujan, Notebooks, Tata Institute of Fundamental Research, Bombay, 1957.

[18] V.P. Spiridonov, *An elliptic beta integral*, Proc. Fifth Int. Conf. on Difference Eqs and Appl. (Temuco, Chile, January 3–7, 2000), Taylor and Francis, 2001, pp. 273–282; *On the elliptic beta function*, Russ. Math. Surveys **56** (1) (2001), 185-186.

[19] ———, *Elliptic beta integrals and special functions of hypergeometric type*, Proc. NATO ARW *Integrable Structures of Exactly Solvable Two-Dimensional Models of Quantum Field Theory* (Kiev, Ukraine, September 25–30, 2000), Kluwer, 2001, pp. 305-313.

[20] ———, *Theta hypergeometric series*, Asymptotic Combinatorics with Application to Mathematical Physics (St. Petersburg, Russia, July 9–22, 2001), Kluwer, 2002, pp. 307-327.

[21] ———, *An elliptic incarnation of the Bailey chain*, Internat. Math. Res. Notices, no. 37 (2002), 1945–1977.

[22] V. Spiridonov and A. Zhedanov, *Discrete Darboux transformations, the discrete time Toda lattice and the Askey-Wilson polynomials*, Meth. Appl. Anal. **2** (1995), 369-398.

[23] ———, *Zeros and orthogonality of the Askey-Wilson polynomials for q a root of unity*, Duke Math. J. **89** (1997), 283-305.

[24] ———, *Discrete-time Volterra chain and classical orthogonal polynomials*, J. Phys. A: Math. & Gen. **30** (1997), 8727-8737.

[25] ———, *Spectral transformation chains and some new biorthogonal rational functions*, Commun. Math. Phys. **210** (2000), 49-83.

[26] ———, *Classical biorthogonal rational functions on elliptic grids*, C.R. Math. Rep. Acad. Sci. Canada **22** (2) (2000), 70-76.

[27] ———, *Generalized eigenvalue problem and a new family of rational functions biorthogonal on elliptic grids*, Special functions 2000, Kluwer, 2001, pp. 365-388.

[28] H. Stahl, *Existence and uniqueness of rational interpolants with free and prescribed poles*, Approximation Theory, Lect. Notes Math. **1287**, Springer, 1987, pp. 180-208.

[29] G. Szegő, Orthogonal Polynomials, Amer. Math. Soc., 1959.

[30] G.N. Watson, *Ramanujan's continued fraction*, Proc. Camb. Phil. Soc. **31** (1935), 7-17.

[31] E.T. Whittacker and G.N. Watson, A Course of Modern Analysis, Cambridge Univ. Press, 1927.

[32] J.H. Wilkinson, The Algebraic Eigenvalue Problem, Clarendon Press, 1965.

[33] J.A. Wilson, *Hypergeometric series, recurrence relations and some new orthogonal functions*, Ph.D. thesis, Univ. of Wisconsin, Madison, WI, 1978.

[34] ———, *Orthogonal functions from Gram determinants*, SIAM J. Math. Anal. **22** (1991), 1147-1155.

[35] L. Wuytack, *On the osculatory rational interpolation problem*, Math. Comp. **29** (1975), 837-843.

[36] A. Zhedanov, *Biorthogonal rational functions and generalized eigenvalue problem*, J. Approx. Theory **101** (1999), 303-329.

[37] A.S. Zhedanov and V.P. Spiridonov, *Hypergeometric biorthogonal rational functions*, Russ. Math. Surveys **54** (2) (1999), 173-174 (in Russian).

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