<table>
<thead>
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<th>項目</th>
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</thead>
<tbody>
<tr>
<td>タイトル</td>
<td>Determinants and Pfaffians: How to obtain N-soliton solutions from 2-soliton solutions (New Developments in the Research of Integrable Systems: Continuous, Discrete, Ultra-discrete)</td>
</tr>
<tr>
<td>著者(s)</td>
<td>Hirota, Ryogo</td>
</tr>
<tr>
<td>引用</td>
<td>数理解析研究所講究録 (2003), 1302: 220-242</td>
</tr>
<tr>
<td>発行日</td>
<td>2003-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/42749">http://hdl.handle.net/2433/42749</a></td>
</tr>
<tr>
<td>タイプ</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>テキストバージョン</td>
<td>publisher</td>
</tr>
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</table>

Kyoto University
Determinants and Pfaffians

How to obtain N-soliton solutions from 2-soliton solutions

Ryogo Hirota (広田 良吾)
Prof. Emeritus, Waseda Univ. (早稲田大学 名誉教授)

Oct. 19, 2002

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1 Introduction

A method of obtaining $N$-soliton solution from 2-soliton solution is described. $N$-soliton solution of soliton equations are obtained by the following procedures;

1. Transform a soliton equation into a bilinear equation.
2. Solve the bilinear equation using a perturbational method. 2-soliton solutions are easily obtained by using the computer algebra (Mathematica, Reduce etc.).
3. Express the 2-soliton solutions by pfaffians (Determinants) .
4. Rewrite the bilinear equation using pfaffians and confirm that the bilinear equation is nothing but the pfaffian identities using the difference (or differential) formula for pfaffians.
5. Then the 2-soliton solution are easily extended to the $N$-soliton solutions.

To this end we study pfaffians.

2 Pfaffians

We expressed an entry (element) of a pfaffian by $\text{pf}(a_1, a_2)$ of characters $a_1, a_2$. A 4th order pfaffian $\text{pf}(a_1, a_2, a_3, a_4)$ is expanded by 6 entries,

$$\text{pf}(a_1, a_2, a_3, a_4) = \text{pf}(a_1, a_2)\text{pf}(a_3, a_4) - \text{pf}(a_1, a_3)\text{pf}(a_2, a_4) + \text{pf}(a_1, a_4)\text{pf}(a_2, a_3).$$

Pfaffians are antisymmetric functions with respect to characters, 

$$\text{pf}(a, b) = -\text{pf}(b, a), \quad \text{for any } a \text{ and } b,$$

from which we obtain antisymmetric properties of pfaffians, for example,

$$\text{pf}(a_1, a_2, a_3, a_4) = -\text{pf}(a_1, a_3, a_2, a_4).$$

A $2n$-th degree pfaffian is defined by the following expansion rule,

$$\text{pf}(a_1, a_2, \ldots, a_{2n}) = \sum_{j=2}^{n} \text{pf}(a_1, a_j)(-1)^j\text{pf}(a_2, \ldots, \hat{a}_j, \ldots, a_{2n}),$$
where $\overset{\cdot}{a}_j$ represents elimination of character $a_j$.
For example, if $n = 3$, we have

\[
\text{pf}(a_1, a_2, a_3, a_4, a_5, a_6) \\
= \sum_{j=2}^{6} \text{pf}(a_1, a_j)(-1)^j \text{pf}(a_2, \ldots, \overset{\cdot}{a}_j, \ldots, a_6) \\
= \text{pf}(a_1, a_2)\text{pf}(a_3, a_4, a_5, a_6) - \text{pf}(a_1, a_3)\text{pf}(a_2, a_4, a_5, a_6) \\
+ \text{pf}(a_1, a_4)\text{pf}(a_2, a_3, a_5, a_6) - \text{pf}(a_1, a_5)\text{pf}(a_2, a_3, a_4, a_6) \\
+ \text{pf}(a_1, a_6)\text{pf}(a_2, a_3, a_4, a_5).
\]

2.1 Determinants and Pfaffians

Pfaffians are related to determinants.
(i) Let $A$ be a determinant of a $m \times m$ antisymmetric matrix defined by

\[
A = \det |a_{j,k}|_{1 \leq j,k \leq m},
\]
where $a_{j,k} = -a_{k,j}$ for $j, k = 1, 2, \ldots, m$.
If $m$ is odd, $A$ gives 0. On the other hand, if $m$ is even, $A$ gives a square of a pfaffian. This pfaffian has a degree $2m$ and is noted as $\text{pf}(a_1, a_2, a_3, \cdots, a_{2m})$ with the entries $\text{pf}(a_j, a_k) = a_{j,k}$ for $j, k = 1, 2, \ldots, m$,

\[
\det |a_{j,k}|_{1 \leq j,k \leq m} = \text{pf}(a_1, a_2, a_3, \cdots, a_{2m})^2.
\]

For example, if $m = 4$, we have

\[
\begin{vmatrix}
 0 & a_{12} & a_{13} & a_{14} \\
-a_{12} & 0 & a_{23} & a_{24} \\
-a_{13} & -a_{23} & 0 & a_{34} \\
-a_{14} & -a_{24} & -a_{34} & 0
\end{vmatrix}
= [a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}]^2
= [\text{pf}(a_1, a_2, a_3, a_4)]^2.
\]

(ii) Let $E$, $A$ and $B$ be a $m \times m$ unit matrix and $m \times m$ antisymmetric matrices respectively. Then the determinant $\det |E + AB|$ is a square of a pfaffian. This pfaffian is denoted as $\text{pf}(a_1, a_2, a_3, \cdots, a_m, b_1, b_2, b_3, \cdots, b_m)$ with the entries $\text{pf}(a_j, a_k) = a_{j,k}$, $\text{pf}(b_j, b_k) = b_{j,k}$ and $\text{pf}(a_j, b_k) = \delta_{j,k}$ for $j, k = 1, 2, \ldots, m$;

\[
\det |E + AB| = \text{pf}(a_1, a_2, a_3, \cdots, a_m, b_1, b_2, b_3, \cdots, b_m)^2.
\]
This is because
\[
\det |E + AB| = \det \begin{vmatrix} A & E \\ -E & B \end{vmatrix} = \text{pf}(a_1, a_2, a_3, \ldots, a_m, b_1, b_2, b_3, \ldots, b_m)^2.
\]

The pfaffian \(\text{pf}(a_1, a_2, a_3, \ldots, a_m, b_1, b_2, b_3, \ldots, b_m)\) plays a crucial role in expressing N-soliton solutions of coupled soliton equations.

### 2.2 Exterior algebra

Making use of exterior algebra, which is based on a concept of a vector exterior product \(A \times B = -B \times A\), one can give a clearer definition of determinant and pfaffian. Let us introduce a one-form given by

\[
\omega_i = \sum_{j=1}^{n} a_{j, k} x^j \quad (i = 1, 2, \cdots, 2n)
\]

where \(x^j\)'s satisfy the following antisymmetric commutation relations,

\[
x_j \wedge x_k = -x_k \wedge x_j, \quad x_j \wedge x_j = 0, \quad j, k = 1, 2, \ldots, n.
\]

Except the above relations, we obey the normal method of calculation. Coefficients \(a_{j, k}\) are arbitrary complex functions.

A determinant \(\det |a_{j, k}|_{1 \leq j, k \leq n}\) is defined by means of exterior products of \(n\) one-forms.

\[
\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \ldots \wedge \omega_n = \det |a_{j, k}|_{1 \leq j, k \leq n} x^1 \wedge x^2 \wedge x^3 \ldots \wedge x^n.
\]

For example, if \(n = 2\),

\[
\omega_1 \wedge \omega_2 = (a_{1,1}x^1 + a_{1,2}x^2) \wedge (a_{2,1}x^1 + a_{2,2}x^2)
\]

\[
= (a_{1,1}a_{2,2} - a_{1,2}a_{2,1}) x^1 \wedge x^2
\]

\[
= \begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} x^1 \wedge x^2,
\]

which defines the \(2 \times 2\) determinant \(\det |a_{j, k}|_{1 \leq j, k \leq 2}\).

Next let \(\Omega\) be a two-form given by

\[
\Omega = \sum_{1 \leq j, k \leq 2n} b_{j, k} x^j \wedge x^k, \quad b_{j, k} = -b_{k, j}.
\]
A pfaffian with its \((i, j)\) entry given by \(b_{j,k}\) is defined by an \(n\)-tuple exterior product of \(\Omega\) as

\[
\Omega \wedge^n = (n!) \text{pf}(b_{1, b_2, b_3, \ldots, b_{2n}}) x_1 \wedge x_2 \wedge x_3 \ldots \wedge x_{2n},
\]

where \(n! = n(n - 1)(n - 2) \cdots 2 \times 1\).

From the above definition, one obtains an expansion formula of a pfaffian. For example, in the case \(n = 2\), putting

\[
\Omega = b_{1,2}x^1 \wedge x^2 + b_{1,3}x^1 \wedge x^3 + b_{1,4}x^1 \wedge x^4 + b_{2,3}x^2 \wedge x^3 + b_{2,4}x^2 \wedge x^4 + b_{3,4}x^3 \wedge x^4
\]

we have

\[
\Omega \wedge \Omega = \{b_{1,2}x^1 \wedge x^2 + b_{1,3}x^1 \wedge x^3 + b_{1,4}x^1 \wedge x^4
\]

\[
+ b_{2,3}x^2 \wedge x^3 + b_{2,4}x^2 \wedge x^4 + b_{3,4}x^3 \wedge x^4\} \wedge \{b_{1,2}x^1 \wedge x^2 + b_{1,3}x^1 \wedge x^3 + b_{1,4}x^1 \wedge x^4
\]

\[
+ b_{2,3}x^2 \wedge x^3 + b_{2,4}x^2 \wedge x^4 + b_{3,4}x^3 \wedge x^4\} = 2\{b_{1,2}b_{3,4} - b_{1,3}b_{2,4} + b_{1,4}b_{2,3}\} x^1 \wedge x^2 \wedge x^3 \wedge x^4. \tag{1}
\]

On the other hand, from the definition, one has

\[
\Omega \wedge \Omega = 2\text{pf}(b_1, b_2, b_3, b_4) x^1 \wedge x^2 \wedge x^3 \wedge x^4. \tag{2}
\]

From eqs. (1) and (2), we have obtained the expansion expression

\[
\text{pf}(b_1, b_2, b_3, b_4) = b_{1,2}b_{3,4} - b_{1,3}b_{2,4} + b_{1,4}b_{2,3}.
\]

\section*{2.3 Laplace expansions of determinants and Plücker relations}

\subsection*{2.3.1 Laplace expansions of determinants}

An \(n\)-th degree determinant given by \(A = \det |a_{i,j}|_{1 \leq i, j \leq n}\) can be expressed as a summation of products of \(r\)- and \((n - r)\)-th degree determinants. This expansion formula is called the Laplace expansion.
Let us show how the Laplace expansion is derived taking a 4th degree determinant an example. Let $\omega_j (j = 1, 2, 3, 4)$ be one-form,

$$\omega_j = \sum_{k=1}^{4} a_{j,k} x^k \quad (j = 1, 2, 3, 4)$$

Then from the definition, we have

$$\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 = \det |a_{j,k}|_{1 \leq j,k \leq 4} x^1 \wedge x^2 \wedge x^3 \wedge x^4.$$

On the other hand,

$$\omega_1 \wedge \omega_2 = (a_{1,1} x^1 + a_{1,2} x^2 + a_{1,3} x^3 + a_{1,4} x^4) \wedge (a_{2,1} x^1 + a_{2,2} x^2 + a_{2,3} x^3 + a_{2,4} x^4)$$

$$= \begin{vmatrix} a_{1,1} & a_{1,2} & x^1 \wedge x^2 & a_{1,1} & a_{1,3} & x^1 \wedge x^3 \\ a_{2,1} & a_{2,2} & x^1 \wedge x^2 & a_{2,1} & a_{2,3} & x^1 \wedge x^3 \\ a_{1,4} & a_{1,2} & x^2 \wedge x^3 & a_{1,2} & a_{1,3} & x^2 \wedge x^3 \\ a_{2,4} & a_{2,2} & x^2 \wedge x^3 & a_{2,2} & a_{2,3} & x^2 \wedge x^3 \\ a_{1,4} & a_{1,2} & x^3 \wedge x^4 & a_{1,3} & a_{1,4} & x^3 \wedge x^4 \\ a_{2,4} & a_{2,2} & x^3 \wedge x^4 & a_{2,3} & a_{2,4} & x^3 \wedge x^4 \end{vmatrix} x^1 \wedge x^2 + \begin{vmatrix} a_{3,1} & a_{3,2} & x^1 \wedge x^2 & a_{3,1} & a_{3,3} & x^1 \wedge x^3 \\ a_{4,1} & a_{4,2} & x^1 \wedge x^2 & a_{4,1} & a_{4,3} & x^1 \wedge x^3 \\ a_{3,4} & a_{3,2} & x^2 \wedge x^3 & a_{3,2} & a_{3,3} & x^2 \wedge x^3 \\ a_{4,4} & a_{4,2} & x^2 \wedge x^3 & a_{4,2} & a_{4,3} & x^2 \wedge x^3 \\ a_{3,4} & a_{3,2} & x^3 \wedge x^4 & a_{3,3} & a_{3,4} & x^3 \wedge x^4 \\ a_{4,4} & a_{4,2} & x^3 \wedge x^4 & a_{4,3} & a_{4,4} & x^3 \wedge x^4 \end{vmatrix} x^2 \wedge x^3 + \begin{vmatrix} a_{3,1} & a_{3,2} & x^2 \wedge x^4 & a_{3,1} & a_{3,3} & x^2 \wedge x^4 \\ a_{4,1} & a_{4,2} & x^2 \wedge x^4 & a_{4,1} & a_{4,3} & x^2 \wedge x^4 \\ a_{3,4} & a_{3,2} & x^3 \wedge x^4 & a_{3,2} & a_{3,3} & x^3 \wedge x^4 \\ a_{4,4} & a_{4,2} & x^3 \wedge x^4 & a_{4,2} & a_{4,3} & x^3 \wedge x^4 \\ a_{3,4} & a_{3,2} & x^4 \wedge x^4 & a_{3,3} & a_{3,4} & x^4 \wedge x^4 \\ a_{4,4} & a_{4,2} & x^4 \wedge x^4 & a_{4,3} & a_{4,4} & x^4 \wedge x^4 \end{vmatrix} x^3 \wedge x^4.$$

Substituting of the above formulae into

$$\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 = (\omega_1 \wedge \omega_2) \wedge (\omega_3 \wedge \omega_4)$$
\[
\det |a_{j,k}|_{1 \leq j,k \leq 4} x^1 \wedge x^2 \wedge x^3 \wedge x^4 \\
= \begin{vmatrix}
 a_{1,1} & a_{1,2} & a_{3,3} & a_{3,4} \\
 a_{2,1} & a_{2,2} & a_{4,3} & a_{4,4}
\end{vmatrix} - \begin{vmatrix}
 a_{1,1} & a_{1,3} & a_{3,2} & a_{3,4} \\
 a_{2,1} & a_{2,3} & a_{4,2} & a_{4,4}
\end{vmatrix} + \begin{vmatrix}
 a_{1,2} & a_{1,4} & a_{3,1} & a_{3,3} \\
 a_{2,2} & a_{2,4} & a_{4,1} & a_{4,3}
\end{vmatrix} + \begin{vmatrix}
 a_{1,2} & a_{1,4} & a_{3,1} & a_{3,2} \\
 a_{2,2} & a_{2,4} & a_{4,1} & a_{4,2}
\end{vmatrix}
\]
\[
x^1 \wedge x^2 \wedge x^3 \wedge x^4.
\]

From the definition, inside the parenthesis \{\ldots\} is equal to the 4th degree determinant, which completes the proof of the Laplace expansion formula of 4th degree determinant.

From \(N\) one-forms,

\[
\omega_j = \sum_{k=1}^{N} a_{j,k} x^j \quad (j = 1, 2, \ldots, N)
\]

we generate an \(N\)-th degree determinant,

\[
\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \ldots \wedge \omega_N = \det |a_{j,k}|_{1 \leq j,k \leq N} x^1 \wedge x^2 \wedge \ldots \wedge x^N.
\]

Decomposing the left hand side of the above equation into the product,

\[
(\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_r) \wedge (\omega_{r+1} \wedge \omega_{r+2} \wedge \ldots \wedge \omega_N)
\]

and rewriting the above equation into a sum of products of \(r\)-th and \((N-r)\)-th degree determinants, we finally obtain the Laplace expansion theorem.

### 2.3.2 Plücker relations

The following identity holds for a summation of products of 2nd degree determinants.

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c}
 a_0 & a_1 & a_2 & a_3 \\
 b_0 & b_1 & b_2 & b_3
\end{array}
- \begin{array}{c|c|c|c|c|c|c|c|c|c}
 a_0 & a_2 & a_3 \\
 b_0 & b_2 & b_3
\end{array}
+ \begin{array}{c|c|c|c|c|c|c|c|c|c}
 a_0 & a_3 \\
 b_0 & b_3
\end{array}
= 0.
\]
which can be proved through direct expansion of each determinant. However, there is another way of proof. Let us consider a 4th degree determinant,

\[
\begin{vmatrix}
a_0 & a_1 & a_2 & a_3 \\
b_0 & b_1 & b_2 & b_3 \\
0 & a_1 & a_2 & a_3 \\
0 & b_1 & b_2 & b_3 \\
\end{vmatrix} = 0,
\]

which is identically equal to 0. Then by means of the Laplace expansion theorem, the determinant is expanded as

\[
0 = \begin{vmatrix}
a_0 & a_1 & a_2 \\
b_0 & b_1 & b_2 \\
\end{vmatrix} \begin{vmatrix}
a_2 & a_3 \\
b_2 & b_3 \\
\end{vmatrix} - \begin{vmatrix}
a_0 & a_2 \\
b_0 & b_2 \\
\end{vmatrix} \begin{vmatrix}
a_1 & a_3 \\
b_1 & b_3 \\
\end{vmatrix} + \begin{vmatrix}
a_0 & a_3 \\
b_0 & b_3 \\
\end{vmatrix} \begin{vmatrix}
a_1 & a_2 \\
b_1 & b_2 \\
\end{vmatrix},
\]

which is the simplest case of the Plücker relations.

2.4 Expressions of determinants and wronskians in terms of pfaffians

A determinant of \( n \)-degree,

\[
B = \det |b_{j,k}|_{1 \leq j,k \leq n},
\]

is expressed by means of a pfaffian of \( 2n \)-th degree as follows

\[
\det |b_{j,k}|_{1 \leq j,k \leq n} = \text{pf}(b_1, b_2, \ldots, b_n, b_n^*, b_{n-1}^*, \ldots, b_2^*, b_1^*),
\]

whose entries are defined by

\[
\text{pf}(b_j, b_k) = \text{pf}(b_j^*, b_k^*) = 0,
\]

\[
\text{pf}(b_j, b_k^*) = b_{j,k}, \text{ for } j, k = 1, 2, \ldots, n.
\]

For example, if \( n=2 \), we have

\[
\begin{vmatrix}
b_{1,1} & b_{1,2} \\
b_{2,1} & b_{2,2} \\
\end{vmatrix} = \text{pf}(b_1, b_2, b_2^*, b_1^*).
\]

This is because

\[
(r.h.s) = -\text{pf}(b_1, b_2^*)\text{pf}(b_2, b_1^*) + \text{pf}(b_1, b_1^*)\text{pf}(b_2, b_2^*)
\]

\[
= b_{1,1}b_{2,2} - b_{1,2}b_{2,1} = (l.h.s).
\]
Next, we consider a Wronskian, which often appears in the theory of linear ordinary differential equations. An \( n \)-th degree Wronskian \( (f_1, f_2, \cdots, f_n) \) is defined by

\[
Wr(f_1(x), f_2(x), \ldots, f_n(x)) = \det \left| \frac{\partial^{j-1} f_k(x)}{\partial x^{j-1}} \right|_{1 \leq j, k \leq n}.
\]

Let \( f_i^{(m)} \) denote an \( m \)-th differential of \( f_i = f_i(x) \) with respect to \( x \),

\[
f_i^{(m)} = \frac{\partial^m}{\partial x^m} f_i, \quad m = 0, 1, 2, \ldots.
\]

We introduce a pfaffian \( (d_m, i) \), which represents \( f_i^{(m)} \), defined by

\[
\begin{align*}
\text{pf}(d_m, i) &= f_i^{(m)}, \quad i = 1, 2, \ldots, \\
\text{pf}(d_m, d_n) &= 0, \quad m, n = 0, 1, 2, \ldots.
\end{align*}
\]

By employing the above notations, \( n \)-th degree Wronskian is expressed by \( 2n \)-th degree pfaffian as

\[
Wr(f_1(x), f_2(x), \ldots, f_n(x)) = \text{pf}(d_0, d_1, d_2, \ldots, d_{n-1}, f_n, f_{n-1}, \ldots, f_1)
\]

\[
\text{pf}(d_j, f_k) = \frac{\partial^j f_k}{\partial x^j}, \quad \text{for } j = 0, 1, \ldots \text{ and for } k := 1, 2, \ldots, n
\]

\[
\text{pf}(d_j, d_k) = 0, \quad \text{for } j, k = 0, 1, 2, \ldots.
\]

For example, in the case of \( n = 2 \), we have

\[
\text{(l.h.s)} = \begin{vmatrix}
   f_1 & \frac{\partial f_1}{\partial x} \\
   f_2 & \frac{\partial f_2}{\partial x}
\end{vmatrix} = f_1 \frac{\partial f_2}{\partial x} - f_2 \frac{\partial f_1}{\partial x}.
\]

On the other hand,

\[
\text{(r.h.s)} = \text{pf}(d_0, d_1, f_2, f_1) = \text{pf}(d_0, d_1) \text{pf}(f_2, f_1) - \text{pf}(d_0, f_2) \text{pf}(d_1, f_1)
\]

\[
\quad + \text{pf}(d_0, f_1) \text{pf}(d_1, f_2)
\]

\[
= f_1 \frac{\partial f_2}{\partial x} - f_2 \frac{\partial f_1}{\partial x},
\]

which completes the proof.
2.5 Pfaffian identities

There are various kinds of pfaffian identities. Let us derive most fundamental identities among them. We start with an expansion formula for $2m$-th degree pfaffian $\text{pf}(a_1, a_2, a_3, \cdots, a_{2m})$,

$$ \text{pf}(a_1, a_2, a_3, \cdots, a_{2m}) = \sum_{j=2}^{2m} (-1)^j \text{pf}(a_1, a_j) \text{pf}(a_2, \ldots, \hat{a_j}, \cdots, a_{2m}). $$

Appending $2n$ characters $1, 2, 3, \cdots, 2n$ homogeneously to each pfaffian above, we obtain an extended expansion formula,

$$ \text{pf}(a_1, a_2, \cdots, a_{2m}, 1, 2, \ldots, 2n) \text{pf}(1, 2, \ldots, 2n) = \sum_{j=2}^{2m} (-1)^j \text{pf}(a_1, a_j, 1, 2, \ldots, 2n) \text{pf}(a_2, \cdots, a_j, \cdots, a_{2m}, 1, 2, \ldots, 2n). $$

(3)

Next expanding the following zero-valued pfaffian ($m$ is odd),

$$ 0 = \text{pf}(a_1, a_2, a_3, \cdots, \hat{a_j}, \cdots, a_m, 2n, 1), $$

with respect to the final character $1$, we obtain

$$ = \sum_{j=1}^{m} (-1)^{j-1} \text{pf}(a_1, a_2, a_3, \cdots, \hat{a_j}, \cdots, a_m, 2n, 1) \text{pf}(a_j, 1) - \text{pf}(a_1, a_2, a_3, \cdots, a_m, 1)(2n, 1). $$

Therefore we have

$$ \text{pf}(a_1, a_2, a_3, \cdots, a_m, 1)(1, 2n) = \sum_{j=1}^{m} (-1)^{j-1} \text{pf}(a_1, a_2, a_3, \cdots, \hat{a_j}, \cdots, a_m, 1, 2n). $$

Appending $2n-2$ characters $2, 3, \cdots, 2n-1$ homogeneously to each pfaffian again, we obtain an identity,

$$ \text{pf}(a_1, a_2, a_3, \cdots, a_m, 1, 2, 3, \cdots, 2n-1)(1, 2, \cdots, 2n) = \sum_{j=1}^{m} (-1)^{j-1} \text{pf}(a_j, 1, 2, \cdots, 2n-1) \text{pf}(a_1, a_2, a_3, \cdots, \hat{a_j}, \cdots, a_m, 1, 2, \cdots, 2n). $$

(4)
For example, in the case $m = 2$, eq(3) is written as

$$\text{pf}(a_1, a_2, a_3, a_4, 1, 2, \ldots, 2n)\text{pf}(1, 2, \ldots, 2n)$$
$$= \text{pf}(a_1, a_2, 1, 2, \ldots, 2n)\text{pf}(a_3, a_4, 1, 2, \ldots, 2n)$$
$$- \text{pf}(a_1, a_3, 1, 2, \ldots, 2n)\text{pf}(a_2, a_4, 1, 2, \ldots, 2n)$$
$$+ \text{pf}(a_1, a_4, 1, 2, \ldots, 2n)\text{pf}(a_2, a_3, 1, 2, \ldots, 2n). \quad (5)$$

In the case $m = 3$, eq(4) is written as

$$\text{pf}(a_1, a_2, a_3, 1, 2, 3, \ldots, 2n-1)(1, 2, \ldots, 2n)$$
$$= \text{pf}(a_1, 1, 2, \ldots, 2n-1)\text{pf}(a_2, a_3, 1, 2, \ldots, 2n)$$
$$- \text{pf}(a_2, 1, 2, \ldots, 2n-1)\text{pf}(a_1, a_3, 1, 2, \ldots, 2n)$$
$$+ \text{pf}(a_3, 1, 2, \ldots, 2n-1)\text{pf}(a_1, a_2, 1, 2, \ldots, 2n). \quad (6)$$

These are examples of the pfaffian identities which we prove later. We show later that the pfafian identity (5) includes both Jacobi identity and Plücker relation.

### 2.5.1 Jacobi identities for determinants

The Jacobi identity for determinants is expressed as

$$DD\begin{pmatrix} i & j \\ k & l \end{pmatrix} = D\begin{pmatrix} i \\ k \end{pmatrix}D\begin{pmatrix} j \\ l \end{pmatrix} - D\begin{pmatrix} i \\ l \end{pmatrix}D\begin{pmatrix} j \\ k \end{pmatrix},$$

$$i < j, \ k < l,$$

$$\quad (7)$$

where $D$ is a $n$-th degree determinant and the minor determinant $D\begin{pmatrix} j \\ k \end{pmatrix}$ is obtained by eliminating $j$-th row and $k$-th column from $D$. The minor determinant $D\begin{pmatrix} i & j \\ k & l \end{pmatrix}$ is obtained by eliminating $i,j$ row and $k,l$ column from $D$.

For example, if $n = 3$ and $i = 1, j = 2, k = 1, l = 2$ we have

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}$$
$$= \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} \begin{vmatrix} a_{1,1} & a_{1,3} \end{vmatrix} - \begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} \begin{vmatrix} a_{1,2} & a_{1,3} \end{vmatrix},$$

$$10$$
which we express by the pfaffians

\[
\begin{align*}
\text{pf}(a_1, a_2, a_3, a_3^*, a_2^*, a_1^*)\text{pf}(a_3, a_3^*)
&= \text{pf}(a_1, a_3, a_3^*, a_1^*)\text{pf}(a_2, a_3, a_2^*, a_1^*) \\
&- \text{pf}(a_1, a_3, a_2^*, a_2^*)\text{pf}(a_2, a_3, a_1^*, a_1^*)
\end{align*}
\]

Arranging the characters we obtain

\[
\begin{align*}
\text{pf}(a_1, a_2, a_2^*, a_1^*, a_3, a_3^*)\text{pf}(a_3, a_3^*)
&= \text{pf}(a_1, a_2, a_3, a_3^*)\text{pf}(a_2^*, a_1^*, a_3, a_3^*) \\
&- \text{pf}(a_1, a_2^*, a_3, a_3^*)\text{pf}(a_2, a_1^*, a_3, a_3^*) \\
&+ \text{pf}(a_1, a_1^*, a_3, a_3^*)\text{pf}(a_2, a_2^*, a_3, a_3^*)
\end{align*}
\]

which is nothing but the Pfaffian identity (5) for \( n = 1 \), \( a_3 = a_2^*, a_4 = a_1^*, 1 = a_3, 2 = a_3^* \). The first term in the r.h.s is identically equal to zero \( (\text{pf}(a_j, a_k) = 0) \).

The Plücker relation is obtained by putting the l.h.s of eq.(5) to be zero.

### 2.6 Proof of the pfaffian identities

In order to prove the pfaffian identities, we start with the following simple identity after Ohta (Y.Ohta: *Bilinear Theory of Soliton*, PhD Thesis (Faculty of Engineering, Tokyo Univ. 1992)).

\[
\sum_{j=0}^{M}(-1)^{j}\text{pf}(b_0, b_1, \ldots, \hat{b}_j, \ldots, b_M)\text{pf}(b_j, c_0, c_1, \ldots, c_N)
= \sum_{k=0}^{N}(-1)^{k}\text{pf}(b_0, b_1, \ldots, b_M, c_k)\text{pf}(c_0, c_1, \ldots, \hat{c}_k, \ldots, c_N)
\]

The proof of eq.(9) is quite simple. Expanding pfaffian \( \text{pf}(b_j, c_0, c_1, \ldots, c_N) \) on the left hand side with respect to the first character \( b_j \) and \( \text{pf}(b_0, b_1, \ldots, b_M, c_k) \) on the right hand side with respect to the final character \( c_k \), we obtain

\[
\sum_{j=0}^{M}(-1)^{j}\sum_{k=0}^{N}(-1)^{k}\text{pf}(b_0, b_1, \ldots, \hat{b}_j, \ldots, b_M) \\
\times \text{pf}(b_j, c_k)\text{pf}(c_0, c_1, \ldots, \hat{c}_k, \ldots, c_N)
\]

11
\[
= \sum_{k=0}^{N} (-1)^{k} \sum_{j=0}^{M} (-1)^{j} \text{pf}(b_0, b_1, \ldots, \hat{b}_j, \ldots, b_M) \\
\times \text{pf}(b_j, c_k) \text{pf}(c_0, c_1, \ldots, \hat{c}_k, \ldots, c_N)
\]

which is nothing but a trivial identity obtained by interchanging the sums over \( j \) and \( k \).

As a special case of the identity (9), we select \( M = 2n, N = 2n + 2 \) and characters \( b_j, c_k \) as follows:

\[
b_0 = a_1, b_1 = 1, b_2 = 2, b_3 = 3, \ldots, b_M = 2n,
\]
\[
c_0 = a_2, c_1 = a_3, c_2 = a_4, c_3 = 1, c_4 = 2, c_5 = 3, \ldots, c_N = 2n.
\]

Since the above choice makes summands on the left hand side of eq.(9) 0 except \( j=0 \), the left hand side is equal to

\[
= \text{pf}(1, 2, \ldots, 2n) \text{pf}(a_1, a_2, a_3, a_4, 1, 2, 3, \ldots, 2n).
\]

On the other hand, the right hand side of eq.(9) is equal to

\[
= \sum_{k=0}^{N} (-1)^{k} \text{pf}(b_0, b_1, \ldots, b_M, c_k) \text{pf}(c_0, c_1, \ldots, \hat{c}_k, \ldots, c_N)
\]
\[
= \sum_{k=0}^{2} (-1)^{k} \text{pf}(b_0, b_1, \ldots, b_M, c_k) \text{pf}(c_0, c_1, \ldots, \hat{c}_k, \ldots, c_N)
\]
\[
+ \sum_{k_1=1}^{2n} (-1)^{k_1} \text{pf}(a_1, 1, 2, \ldots, 2n, k_1) \text{pf}(a_2, a_3, a_4, 1, 2, \ldots, k_1, \ldots, 2n),
\]
\[
(k = k_1 + 2),
\]
\[
= \text{pf}(b_0, b_1, \ldots, b_M, c_0) \text{pf}(c_1, c_2, c_3, \ldots, c_N)
\]
\[
- \text{pf}(b_0, b_1, \ldots, b_M, c_1) \text{pf}(c_0, c_2, c_3, \ldots, c_N)
\]
\[
- \text{pf}(b_0, b_1, \ldots, b_M, c_2) \text{pf}(c_0, c_1, c_3, \ldots, c_N)
\]
\[
= \text{pf}(a_1, 1, 2, \ldots, 2n, a_2) \text{pf}(a_3, a_4, 1, 2, \ldots, 2n)
\]
\[
- \text{pf}(a_1, 1, 2, \ldots, 2n, a_3) \text{pf}(a_2, a_4, 1, 2, \ldots, 2n)
\]
\[
+ \text{pf}(a_1, 1, 2, \ldots, 2n, a_4) \text{pf}(a_2, a_3, 1, 2, \ldots, 2n)
\]

(12)

Hence, eq.(9) results in eq.(5).
The identity (3) is obtained from eq.(9) by choosing $M = 2n, N = 2n + 2m - 2$ ($m$ is odd) and characters $b_j, c_k$ as follows:

\[
\begin{align*}
\quad b_0 &= a_1, b_1 = 1, b_2 = 2, b_3 = 3, \ldots, b_M = b_{2n} = 2n, \\
c_0 &= a_2, c_1 = a_3, c_2 = a_4, c_3 = a_5, \ldots, c_{2m-2} = a_{2m}, \\
c_{2m-1} &= 1, c_{2m} = 2, c_{2m+1} = 3, \ldots, c_N = c_{2n+2m-2} = 2n.
\end{align*}
\]

Then eq.(9) results in the following equation.

\[
\text{pf}(1, 2, \ldots, 2n)\text{pf}(a_1, a_2, a_3, \ldots, a_{2m}, 1, 2, 3, \ldots, 2n)
= \sum_{k_1=2}^{2m} (-1)^{k_1}\text{pf}(a_1, a_{k_1}, 1, 2, \ldots, 2n) \\
\times \text{pf}(a_2, a_3, \ldots, a_{k_1}, \ldots, a_{2m}, 1, 2, \ldots, 2n) \tag{13}
\]

which is the pfaffian identity (3).

The identity (4) is obtained from eq.(9) by choosing $M = 2n - 2, N = 2n + 2m - 1$ ($m$ is odd) and characters $b_j, c_k$ as follows:

\[
\begin{align*}
\quad b_0 &= 1, b_1 = 2, b_2 = 3, b_3 = 4, \ldots, b_M = b_{2n-2} = 2n - 1, \\
c_0 &= a_1, c_1 = a_2, c_2 = a_3, c_3 = a_4, \ldots, c_{m-1} = a_m, \\
c_m &= 1, c_{m+1} = 2, c_{m+2} = 3, \ldots, c_N = c_{2n+m-1} = 2n.
\end{align*}
\]

Then eq.(9) results in the following equation,

\[
\text{pf}(1, 2, \ldots, 2n)\text{pf}(a_1, a_2, a_3, \ldots, a_m, 1, 2, 3, \ldots, 2n - 1)
= \sum_{j=1}^{m} (-1)^{j-1}\text{pf}(a_j, 1, 2, \ldots, 2n - 1) \\
\times \text{pf}(a_1, a_2, a_3, \ldots, a_j, \ldots, a_m, 1, 2, \ldots, 2n)
\]

which is the pfaffian identity (4).

We have observed that almost all bilinear soliton equations result in the pfaffian identities eqs.(3) and (4).
2.7 Expansion formulae for the pfaffian \((a_1, a_2, 1, 2, \cdots, 2n)\)

The pfaffian \((a_1, a_2, 1, 2, \cdots, 2n)\) is, if \((a_1, a_2) = 0\), expanded in the following forms (i),(ii):

(i) \[
(a_1, a_2, 1, 2, \cdots, 2n) = \sum_{1 \leq j < k \leq 2n} (-1)^{j+k-1}(a_1, a_2, j, k)(1, 2, \cdots, \hat{j}, \cdots, \hat{k}, \cdots, 2n)
\]

(ii) \[
(a_1, a_2, 1, 2, \cdots, 2n)
= \sum_{j=2}^{2n} (-1)^j [(a_1, a_2, 1, j)(2, 3, \cdots, \hat{j}, \cdots, 2n) + (1, j)(a_1, a_2, 2, 3, \cdots, \hat{j}, \cdots, 2n)]
\]

Let us prove (i) first. Expanding the pfaffian \((a_1, a_2, 1, 2, \cdots, 2n)\) first with respect to \(a_1\) and next \(a_2\), we have

\[
(a_1, a_2, 1, 2, \cdots, 2n) = \sum_{j=1}^{2n} \sum_{k=1}^{2n} (a_1, j)(a_2, k)(1, 2, \cdots, \hat{j}, \cdots, \hat{k}, \cdots, 2n)
= \sum_{1 \leq j < k \leq 2n} (-1)^{j+k}[(a_1, j)(a_2, k) - (a_1, k)(a_2, j)](1, 2, \cdots, \hat{j}, \cdots, \hat{k}, \cdots, 2n).
\]

Noticing the relation \((a_1, a_2) = 0\), we obtain

\[
= \sum_{1 \leq j < k \leq 2n} (-1)^{j+k-1}(a_1, a_2, j, k)(1, 2, \cdots, \hat{j}, \cdots, \hat{k}, \cdots, 2n).
\]

In order to prove (ii), we expand the pfaffian \((a_1, a_2, 1, 2, \cdots, 2n)\) with respect to the character 1.

\[
(a_1, a_2, 1, 2, \cdots, 2n) = (1, a_1)(a_2, 2, \cdots, 2n) - (1, a_2)(a_1, 2, \cdots, 2n)
+ \sum_{j=2}^{2n} (-1)^j (1, j)(a_1, a_2, 2, 3, \cdots, \hat{j}, \cdots, 2n).
\]

Next, pfaffians \((a_2, 2, \cdots, 2n)\) and \((a_1, 2, \cdots, 2n)\) are expanded as

\[
= (1, a_1) \sum_{j=2}^{2n} (-1)^j (a_2, j)(2, 3, \cdots, \hat{j}, \cdots, 2n)
- (1, a_2) \sum_{j=2}^{2n} (-1)^j (a_1, j)(2, 3, \cdots, \hat{j}, \cdots, 2n)
+ \sum_{j=2}^{2n} (-1)^j (1, j)(a_1, a_2, 2, 3, \cdots, \hat{j}, \cdots, 2n).
\]
Noticing the relation \( (a_1, a_2) = 0 \), we obtain

\[
= \sum_{j=2}^{2n} (-1)^j [(a_1, a_2, 1, j)(2, 3, \ldots, \hat{j}, \ldots, 2n) + (1, j)(a_1, a_2, 2, 3, \ldots, 2n)]
\]

If we consider a pfaffian \((b_1, b_2, 1, 2, \ldots, 2n)\) instead of \((1, 2, \ldots, 2n)\) in the expansion formula (i), this formula can be generalized as follows;

(iii) \((a_1, a_2, b_1, b_2, 1, 2, \ldots, 2n)\)

\[
= \sum_{j=1}^{2n} \sum_{k=j+1}^{2n} (a_1, a_2, j, k)(b_1, b_2, 1, 2, 3, \ldots, \hat{j}, \ldots, \hat{k}, \ldots, 2n),
\]

where \((a_j, a_k) = (b_j, b_k) = 0\), for \(j, k = 1, 2\).

We make use of these expansion formulae as pfaffian difference (differential) formulae later.

### 2.8 Difference formula for pfaffians

In order to show that bilinear soliton equations result in the pfaffian identities, we study difference formula for pfaffians.

We consider a 2n-th degree pfaffian with special entries,

\[
\text{pf}(a_1, a_2, a_3, \ldots, a_{2n})_{\alpha}
\]

whose entries \(\text{pf}(a_i, a_j)_{\alpha}\) are given by summation of pfaffians.

\[
\text{pf}(a_j, a_k)_{\alpha} = \text{pf}(a_j, a_k) + \lambda \text{pf}(d_0, d_1, a_j, a_k)
\]

where \(\lambda\) is a parameter and \(\text{pf}(d_0, d_1) = 0\).

The pfaffian \((14)\) obeys the usual expansion rule,

\[
\text{pf}(a_1, a_2, a_3, \ldots, a_{2n})_{\alpha} = \sum_{j=2}^{2n} (-1)^j \text{pf}(a_1, a_j)_{\alpha} \text{pf}(a_2, a_3, \ldots, \hat{a}_j, \ldots, a_{2n})_{\alpha}.
\]

Then the following formula holds for arbitrary \(n\),

\[
\text{pf}(a_1, a_2, a_3, \ldots, a_{2n})_{\alpha} = \text{pf}(a_1, a_2, a_3, \ldots, a_{2n}) + \lambda \text{pf}(d_0, d_1, a_1, a_2, a_3, \ldots, a_{2n}).
\]
Let us prove the formula (16) by induction. Obviously, the formula holds if \( n = 1 \). We suppose that the formula holds for an arbitrary \((2n - 2)\)-th degree pfaffian,

\[
pf(a_2, a_3, \cdots, \hat{a}_j, \cdots, 2n) = \sum_{j=2}^{2n} (-1)^j \left[ pf(a_1, a_j) pf(a_2, a_3, \cdots, \hat{a}_j, \cdots, a_{2n}) + \lambda pf(d_0, d_1, a_1, a_2, a_3, \cdots, a_{2n}) \right]
\]

Expanding the left hand side in eq.(16), we have

\[
pf(a_1, a_2, a_3, \ldots, a_{2n}) = \sum_{j=2}^{2n} (-1)^j pf(a_1, a_j) pf(a_2, a_3, \cdots, \hat{a}_j, \cdots, a_{2n}).
\] (18)

Employing eq.(17) we have

\[
= \sum_{j=2}^{2n} (-j)^j \left[ pf(a_1, a_j) + \lambda pf(d_0, d_1, a_1, a_2, a_3, \cdots, a_{2n}) \right] pf(a_2, a_3, \cdots, \hat{a}_j, \cdots, a_{2n}),
\]

whose coefficient in \( \lambda^0 \) is obviously \( pf(a_1, a_2, \cdots, a_{2n}) \). Coefficient in \( \lambda^1 \) is \( pf(d_0, d_1, a_1, a_2, \cdots, a_{2n}) \) due to the expansion formula (ii).

Expanding the following zero-valued pfaffian, we obtain

\[
0 = pf(d_0, d_1, a_1, a_2, \cdots, a_{2n}, d_0, d_1)
\]

\[
= \sum_{j=2}^{2n} pf(d_0, d_1, a_1, a_j) pf(a_2, a_3, \cdots, \hat{a}_j, \cdots, a_{2n}, d_0, d_1),
\]

from which we find that coefficient in \( \lambda^2 \) is zero. Therefore, we have

\[
pf(a_1, a_2, a_3, \ldots, a_{2n}) = pf(a_1, a_2, a_3, \ldots, a_{2n}) + \lambda pf(d_0, d_1, a_1, a_2, a_3, \ldots, a_{2n}),
\]

which completes the proof.

2.9 Difference formula for determinants

We have the pfaffian expression for a determinant,

\[
det |a_{j,k}|_{1 \leq j,k \leq n} = pf(a_1, a_2, \ldots, a_n, a_n^*, \ldots, a_2^*, a_1^*)
\] (19)
where \( \text{pf}(a_j, a_k) = \text{pf}(a_j^*, a_k^*) = 0 \), \( \text{pf}(a_j, a_k^*) = a_{j,k} \). If entries of a determinant are expressed by pfaffians,

\[
\begin{align*}
\text{pf}(a_j, a_k^*)_{\alpha} &= \text{pf}(a_j^*, a_k^*)_{\alpha} + \text{pf}(d_{\gamma}, a_j, a_k^*, d_{\delta}^*)' \\
\text{pf}(a_j, a_k)_{\alpha} &= \text{pf}(a_j^*, a_k^*)_{\alpha} = \text{pf}(d_{\gamma}, d_{\delta}^*)' = 0,
\end{align*}
\]

we obtain, by using the difference formula for pfaffians,

\[
\begin{align*}
\text{pf}(a_1, a_2, \ldots, a_n, a_n^*, \ldots, a_2^*, a_1^*)_{\alpha} &= \text{pf}(a_1, a_2, \ldots, a_n, a_n^*, \ldots, a_2^*, a_1^*)_{\alpha}' \\
&\quad+ \text{pf}(d_{\gamma}, a_1, a_2, \ldots, a_n, a_n^*, \ldots, a_2^*, a_1^*, d_{\delta}^*)' \\
&= \text{pf}(d_{\gamma}, a_1, a_2, \ldots, a_n, a_n^*, \ldots, a_2^*, a_1^*, d_{\delta}^*). \tag{22}
\end{align*}
\]

Suppose that the entries have the following properties:

\[
\begin{align*}
\text{pf}(a_j, a_k^*)' &= \text{pf}(a_j, a_k^*) c_k / c_j, \tag{23}
\text{pf}(d_{\gamma}, a_k^*)' &= \text{pf}(d_{\gamma}, a_k^*) c_k, \tag{24}
\text{pf}(a_j, d_{\delta}^*)' &= \text{pf}(a_j, d_{\delta}^*) / c_j, \tag{25}
\end{align*}
\]

where all \( c_j (\neq 0), j = 1, 2, \ldots, n \) are parameters. Then we have the following relation,

\[
\begin{align*}
\text{pf}(d_{\gamma}, a_1, a_2, \ldots, a_n, a_n^*, \ldots, a_2^*, a_1^*, d_{\delta}^*)' &= \text{pf}(d_{\gamma}, a_1, a_2, \ldots, a_n, a_n^*, \ldots, a_2^*, a_1^*, d_{\delta}^*). \tag{26}
\end{align*}
\]

Accordingly the difference formula for the determinant is expressed by

\[
\begin{align*}
\text{pf}(a_1, a_2, \ldots, a_n, a_n^*, \ldots, a_2^*, a_1^*)_{\alpha} &= \text{pf}(a_1, a_2, \ldots, a_n, a_n^*, \ldots, a_2^*, a_1^*)_{\alpha} \\
&\quad+ \text{pf}(d_{\gamma}, a_1, a_2, \ldots, a_n, a_n^*, \ldots, a_2^*, a_1^*, d_{\delta}^*), \tag{27}
\end{align*}
\]

provided that

\[
\begin{align*}
\text{pf}(a_j, a_k^*)_{\alpha} &= [\text{pf}(a_j, a_k^*) + \text{pf}(d_{\gamma}, a_j, a_k^*, d_{\delta}^*)] c_k / c_j, \tag{28}
\text{pf}(a_j, a_k)_{\alpha} &= \text{pf}(a_j^*, a_k^*)_{\alpha} = 0, \tag{29}
\text{pf}(d_{\gamma}, d_{\delta}^*) &= 0. \tag{30}
\end{align*}
\]
3 Pfaffian Solutions to the Discrete KdV Equation

3.1 Discretization of the KdV equation

The KdV equation

\[
\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0
\]

is transformed into the bilinear form

\[
D_x(D_t + D_x^3)f \cdot f = 0
\] (31)

through the logarithmic transformation

\[
u = 2 \frac{\partial^2}{\partial x^2} \log f
\]

\[
= \frac{D_x^2 f \cdot f}{f^2}
\]

We rewrite the above equation as

\[
(D_t + D_x^3)f \cdot f_x = 0.
\] (32)

A semi-discrete KdV equation is obtained by discretizing the spatial part of the bilinear equation (32),

\[
D_t f_n \cdot f_{n+1} + \frac{1}{\epsilon}(f_{n+1}f_n - f_{n+2}f_{n-1}) = 0
\] (33)

where \(\epsilon\) is a spatial interval.

Replacing the differential bilinear operator \(D_t\) by a corresponding difference operator and taking a gauge-invariance of the bilinear equation into account, we obtain a discrete KdV equation,

\[
f_{n+1} f_n^m - f_n f_{n+1}^m + q_0(f_{n+1} f_n^m - f_{n+2} f_{n-1}^m) = 0
\] (34)

where \(q_0 = \delta/\epsilon\), \(\delta\) being a time-interval.
3.2 Soliton solution to the discrete KdV equation

It is easy to obtain 2-soliton solution to the discrete bilinear KdV equation (34) by using a perturbational method. We find

\[
\begin{align*}
f_n^m &= 1 + a_1 \exp \eta_1 + a_2 \exp \eta_2 + a_{1,2} a_1 a_2 \exp (\eta_1 + \eta_2), \\
\exp \eta_j &= \Omega_j^m P_j^n, \\
\Omega_j &= \frac{1 + q_0/P_j}{1 + q_0 P_j}, \quad \text{for } j = 1, 2, \\
a_{1,2} &= (P_1 - P_2)^2/(P_1 P_2 - 1)^2.\end{align*}
\]

where \(a_1, a_2\) are arbitrary parameters. Hereafter we choose the parameters to be \(a_j = 1/(p_j^2 - 1)\) for \(j = 1, 2\).

We express 2-soliton solution to the discrete KdV equation by a pfaffian,

\[
\begin{align*}
f_n^m &= \text{pf}(a_1, a_2, a_2^*, a_1^*), \\
\text{pf}(a_j, a_k) &= \text{pf}(a_j^*, a_k^*) = 0, \\
\text{pf}(a_j, a_k^*) &= \delta_{j,k} + \exp [(\eta_j + \eta_k)/2]/(P_j P_k - 1),
\end{align*}
\]

which is equal to the following determinant expression,

\[
f_n^m = \begin{vmatrix}
1 + \exp (\eta_1)/(P_1^2 - 1) & \exp [(\eta_1 + \eta_2)/2]/(P_1 P_2 - 1) \\
\exp [(\eta_1 + \eta_2)/2]/(P_1 P_2 - 1) & 1 + \exp (\eta_2)/(P_2^2 - 1)
\end{vmatrix}.
\]

4 Pfaffian identities of the discrete bilinear Kdv equation

We are going to show that the bilinear equation results in the pfaffian identity

\[
\begin{align*}
\text{pf}(a_1, a_2, a_3, a_4, 1, 2, \ldots, 2n)\text{pf}(1, 2, \ldots, 2n) \\
= \text{pf}(a_1, a_2, 1, 2, \ldots, 2n)\text{pf}(a_3, a_4, 1, 2, \ldots, 2n) \\
-\text{pf}(a_1, a_3, 1, 2, \ldots, 2n)\text{pf}(a_2, a_4, 1, 2, \ldots, 2n) \\
+\text{pf}(a_1, a_4, 1, 2, \ldots, 2n)\text{pf}(a_2, a_3, 1, 2, \ldots, 2n). \quad (42)
\end{align*}
\]

In order to show that bilinear discrete KdV eq.(34) results in the pfaffian identity (42), we have to express \(f_{n+1}^m, f_{n-1}^m, f_{n+1}^m, \ etc\) by pfaffians.
To this end we start with the simplest case of $f_n^m$, 1-soliton solution. Let us introduce a pfaffian entry $\text{pf}(j, k^*)$, for $j, k = 1, 2, \cdots$, by

$$\text{pf}(j, k^*) = \delta_{j,k} + \exp [(\eta_j + \eta_k)/2]/(P_j P_k - 1),$$

where $\delta_{j,k}$ is a Kronecker’s delta.

Then 1-soliton solution is expressed by the pfaffian,

$$f_n^m = 1 + \exp (\eta_1)/(P_1^2 - 1) = \text{pf}(1, 1^*).$$

We have

$$\text{pf}(1, 1^*)_{n+1} = f_{n+1}^m = 1 + P_1 \exp (\eta_1)/(P_1^2 - 1),$$

which is rewritten as

$$= 1 + \exp (\eta_1)/(P_1^2 - 1) + (P_1 - 1)/(P_1^2 - 1) \exp (\eta_1)$$

$$= f_n^m + \frac{1}{P_1 + 1} \exp (\eta_1).$$

The last term in the above expression is expressed by a pfaffian,

$$\frac{1}{P_1 + 1} \exp (\eta_1) = \text{pf}(d_p, 1, 1^*, d_0^*),$$

by introducing the following pfaffian entries;

$$\text{pf}(d_p, j) = 0,$$

$$\text{pf}(d_p, j^*) = \frac{1}{P_j + 1} \exp (\eta_j/2),$$

$$\text{pf}(d_p, d_0^*) = 0,$$

$$\text{pf}(j, d_0^*) = - \exp (\eta_j/2),$$

$$\text{pf}(j^*, d_0^*) = 0, \quad \text{for} \quad j = 1, 2, \cdots.$$

Because we have

$$\text{pf}(d_p, 1, 1^*, d_0^*) = -\text{pf}(d_p, 1^*)\text{pf}(1, d_0^*)$$

$$= \frac{1}{P_j + 1} \exp (\eta_j).$$

Accordingly we find that the difference of the pfaffian is expressed by a sum of pfaffians,

$$\text{pf}(1, 1^*)_{n+1} = \text{pf}(1, 1^*) + \text{pf}(d_p, 1, 1^*, d_0^*).$$
Following the same procedure we obtain
\[
\begin{align*}
\text{pf}(1,1^*)_{n-1} &= \text{pf}(1,1^*) + \text{pf}(d_p,1,1^*,d_n^*), \\
\text{pf}(1,1^*)^{m+1} &= \text{pf}(1,1^*) + \text{pf}(d_q,1,1^*,d_0^*)/q_0, \\
\text{pf}(1,1^*)^{m+1} &= \text{pf}(1,1^*), \\
\text{pf}(1,1^*)_{n+1}^{m+1} &= \text{pf}(1,1^*) + \text{pf}(d_p,1,1^*,d_n^*) - \text{pf}(d_q,1,1^*,d_0^*) \\
&= \text{pf}(1,1^*) - \text{pf}(d_q,1,1^*,d_n^*)/q_0 - \text{pf}(d_p,1,1^*,d_n^*),
\end{align*}
\]
where we have introduced pfaffian entries as follow
\[
\begin{align*}
\text{pf}(d_p,d_q) &= 0, \\
\text{pf}(d_p,d_n^*) &= 0, \\
\text{pf}(d_q,j) &= 0, \\
\text{pf}(d_q,j^*) &= \frac{1}{P_j + 1/q_0} \exp(\eta_j/2), \\
\text{pf}(d_q,d_n^*) &= 0, \\
\text{pf}(d_q,d_0^*) &= 0, \\
\text{pf}(j,d_n^*) &= \frac{1}{P_j} \exp(\eta_j/2), \\
\text{pf}(d_n^*,d_0^*) &= 0, \\
\text{pf}(j^*,d_0^*) &= 0, \quad \text{for } j = 1,2,\ldots.
\end{align*}
\]
In the above pfaffian representations \(\text{pf}(1,1^*)^{m+1}_{n+1}\) is not uniquely determined. We fixed it by using 2-soliton solution. We have, for 2-soliton solution,
\[
f_n^m = \text{pf}(1,2,2^*,1^*).
\]
We assume that the pfaffian representations of \(\text{pf}(1,2,2^*,1^*)^{m+1}_{n+1}\) has the following form
\[
\begin{align*}
\text{pf}(1,2,2^*,1^*)^{m+1}_{n+1} &= \text{pf}(1,2,2^*,1^*) + c_1\text{pf}(d_p,1,2,2^*,1^*,d_n^*) \\
&+ c_2\text{pf}(d_q,1,2,2^*,1^*,d_0^*) + c_3\text{pf}(d_p,1,2,2^*,1^*,d_n^*) \\
&+ c_4\text{pf}(d_q,1,2,2^*,1^*,d_n^*) + c_5\text{pf}(d_q,d_q,1,2,2^*,1^*,d_n^*,d_0^*),
\end{align*}
\]
where \(c_1,c_2,\ldots,c_5\) are parameters to be determined. By using a computer algebra (Mathematica,Reduce etc.), we determine the parameters,
\[
\text{pf}(1,2,2^*,1^*)^{m+1}_{n+1}
\]
\[\begin{align*}
&= \text{pf}(1, 2, 2^*, 1^*) + [-\text{pf}(d_p, 1, 2, 2^*, 1^*, d_0^*) + \text{pf}(d_q, 1, 2, 2^*, 1^*, d_n^*)] \\
&+ q_0 \cdot \text{pf}(d_p, 1, 2, 2^*, 1^*, d_n^*) - \text{pf}(d_q, 1, 2, 2^*, 1^*, d_n^*) \\
&- \text{pf}(d_q, d_q, 1, 2, 2^*, 1^*, d_n^*, d_0^*)/(q_0 - 1)
\end{align*}\]

and confirm the pfaffian expressions:

\[
\begin{align*}
\text{pf}(1, 2, 2^*, 1^*)_{n-1} &= \text{pf}(1, 2, 2^*, 1^*) + \text{pf}(d_p, 1, 2, 2^*, 1^*, d_n^*), \\
\text{pf}(1, 2, 2^*, 1^*)^{m+1} &= \text{pf}(1, 2, 2^*, 1^*) + \text{pf}(d_q, 1, 2, 2^*, 1^*, d_n^*), \\
\text{pf}(1, 2, 2^*, 1^*)_{n+2}^{m+1} &= \text{pf}(1, 2, 2^*, 1^*) + \text{pf}(d_q, d_q, 1, 2, 2^*, 1^*, d_n^*, d_0^*)/q_0.
\end{align*}
\]

Substituting these expressions into the discrete bilinear KdV equation (34) we find that eq. (34) is reduced to the pfaffian identity

\[
\begin{align*}
\text{pf}(d_p, d_q, 1, 2, 2^*, 1^*, d_n^*, d_0^*)\text{pf}(1, 2, 2^*, 1^*) \\
= \text{pf}(d_p, d_q, 1, 2, 2^*, 1^*, d_n^*, d_0^*) \\
- \text{pf}(d_p, 1, 2, 2^*, 1^*, d_n^*)\text{pf}(d_q, 1, 2, 2^*, 1^*, d_0^*) \\
+ \text{pf}(d_p, 1, 2, 2^*, 1^*, d_0^*)\text{pf}(d_q, 1, 2, 2^*, 1^*, d_n^*),
\end{align*}
\]

where the first term in the right hand side is identically equal to zero (\(\text{pf}(d_p, d_q) = \text{pf}(1, 2) = 0\)).

Accordingly we have shown that the discrete bilinear KdV equation (34) results in the pfaffian identity for 2-soliton solution.

The pfaffian identity (43) for 2-soliton solution is easily extended to the pfaffian identity for \(N\)-soliton solution,

\[
\begin{align*}
\text{pf}(d_p, d_q, 1, 2, \cdots, N, N^*, \cdots, 2^*, 1^*, d_n^*, d_0^*)\text{pf}(1, 2, \cdots, N, N^*, \cdots, 2^*, 1^*) \\
= \text{pf}(d_p, d_q, 1, 2, \cdots, N, N^*, \cdots, 2^*, 1^*, d_n^*, d_0^*) \\
- \text{pf}(d_p, 1, 2, \cdots, N, N^*, \cdots, 2^*, 1^*, d_n^*)\text{pf}(d_q, 1, 2, \cdots, N, N^*, \cdots, 2^*, 1^*, d_0^*) \\
+ \text{pf}(d_p, 1, 2, \cdots, N, N^*, \cdots, 2^*, 1^*, d_0^*)\text{pf}(d_q, 1, 2, \cdots, N, N^*, \cdots, 2^*, 1^*, d_n^*),
\end{align*}
\]

Thus we have shown that the \(N\)-soliton solution expressed by the pfaffian,

\[f_n^m = \text{pf}(1, 2, \cdots, N, N^*, \cdots, 2^*, 1^*)\]

satisfies the discrete bilinear KdV equation (34).