

Products of $k$-spaces, and questions

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As is well-known, every product of a locally compact space with a $k$-space is a $k$-space, but not every product of a metric space with a $k$-space is a $k$-space. We consider characterizations or conditions for (finite) products of $k$-spaces to be $k$-spaces, and pose related questions. For other topics on the products of $k$-spaces, see [T3], [T4], for example.

We assume that spaces are regular $T_1$, and maps are continuous and onto.

1 Definitions and Preliminaries

Let $X$ be a space, and let $P$ be a (not necessarily open or closed) cover of $X$. Then $X$ is determined by a cover $P$, \(^1\) if $U \subset X$ is open in $X$ if and only if $U \cap P$ is relatively open in $P$ for every $P \in P$. Here, we can replace "open" by "closed". Every space is determined by its open (or hereditarily closure-preserving closed) cover.

Let us recall that a space is a $k$-space (resp. sequential space) if it is determined by a cover of compact (resp. compact metric) subsets. Sequential space are $k$-spaces, and the converse holds if points are $G_\delta$-sets. A space $X$ is called a $k_\omega$-space [M3] (resp. $s_\omega$-space) if $X$ is determined by a countable cover of compact (resp. compact metric) subsets.

A space $X$ is called a bi-$k$-space (resp. bi-quasi-$k$-space) [M3] if, whenever a filter base $\mathcal{F}$ accumulates at $x \in X$, then there exists a $k$-sequence (resp. $q$-sequence) $\{A_n : n \in N\}$ such that $x \in \overline{F \cap A_n}$ for all $n \in N$ and all $F \in \mathcal{F}$. When the filter base $\mathcal{F}$ is a decreasing sequence, then such a space $X$ is a countably bi-$k$-space (resp. countably bi-quasi-$k$-space) [M3]. Here, a $k$-sequence (resp. $q$-sequence) is a decreasing sequence $\{A_n : n \in N\}$ such that $A = \cap \{A_n : n \in N\}$ is compact (resp. countably compact), and any open set $U \supset A$ contains some $A_n$ ([M3]).

Let us recall that a space $X$ is of pointwise countable type (resp. $q$-space) if each point has nbds $\{V_n : n \in N\}$ which is a $k$-sequence (resp. $q$-sequence) $\{A_n : n \in N\}$ such that $x \in \overline{F \cap A_n}$ for all $n \in N$ and all $F \in \mathcal{F}$. When the filter base $\mathcal{F}$ is a decreasing sequence, then such a space $X$ is a countably bi-$k$-space (resp. countably bi-quasi-$k$-space) [M3]. Here, a $k$-sequence (resp. $q$-sequence) is a decreasing sequence $\{A_n : n \in N\}$ such that $A = \cap \{A_n : n \in N\}$ is compact (resp. countably compact), and any open set $U \supset A$ contains some $A_n$ ([M3]).

\(^1\)Following [GMT], we shall use "$X$ is determined by $P$" instead of the usual "$X$ has the weak topology with respect to $P$".
sequence). Also, a space is an $M$-space if and only if it is the inverse image of a metric space under a quasi-perfect map. The following diagrams hold.

(a) Locally compact spaces, or first countable spaces $\rightarrow$ spaces of pointwise countable type $\rightarrow$ bi-$k$-spaces $\rightarrow$ countably bi-$k$-spaces $\rightarrow$ $k$-spaces.

(b) Locally countably compact spaces, or $M$-spaces $\rightarrow$ $q$-spaces $\rightarrow$ bi-quasi-$k$-spaces $\rightarrow$ countably bi-quasi-$k$-spaces.

A space $X$ is called a Tanaka space [My2], if $X$ satisfies the following condition (C) in [T2].

(C) Let $\{A_n : n \in \mathbb{N}\}$ be a decreasing sequence of subsets of $X$ with $x \in \overline{A_n}$ for any $n \in \mathbb{N}$. Then there exist $x_n \in A_n$ such that $\{x_n : n \in \mathbb{N}\}$ converges to some point $y \in X$. If $y = x$, then such a space $X$ is called countably bi-sequential [M3] ($=\$ strongly Fréchet [S]).

Sequentially compact spaces, or sequential countably bi-quasi-$k$-spaces are Tanaka spaces. But, every Tanaka space (actually, sequentially compact space) need not be sequential, not even a $k$-space$^2$.

A space $X$ is strongly sequential [M1] if, whenever $\{A_n : n \in \mathbb{N}\}$ is a decreasing sequence of subsets of $X$ with $x \in \overline{A_n}$ for any $n \in \mathbb{N}$, then the point $x$ belongs to the (idempotent) sequential closure of $A$, where $A$ is the set of all limit points of convergent sequences $\{x_n : n \in \mathbb{N}\}$ with $x_n \in A_n$. Namely, a space $X$ is strongly sequential if and only if it is a sequential space such that if $\{A_n : n \in \mathbb{N}\}$ is a decreasing sequence of subsets of $X$ with $x \in \overline{A_n}$ for any $n \in \mathbb{N}$, then the point $x$ belongs to the (usual) closure of the above set $A$. Strongly Fréchet spaces are strongly sequential. Every strongly sequential space is precisely a sequential Tanaka space ([My2]).

A map $f : X \rightarrow Y$ is called bi-quotient [M2] if, whenever $y \in Y$ and $U$ is a cover of $f^{-1}(y)$ by open subsets of $X$, then finitely many $f(U)$, with $U \in U$, cover some nbd of $y$ in $Y$. If $U$ is countable, then such a map $f$ is called countably bi-quotient [S]. Open maps, or perfect maps are bi-quotient. Every product of bi-quotient maps is bi-quotient, hence quotient ([M2]). A map $f : X \rightarrow Y$ is called a compact (resp. $s$-map) if every $f^{-1}(y)$ is compact (resp. separable).

$^2$This is pointed out by Z. Dolecki or P. Nyikos.
In the following characterizations, (1) is well-known, (2) is routinely shown, and (3) is due to [M3].

Characterization: (1) $X$ is a $k$-space (resp. sequential space) $\iff X$ is the quotient image of a locally compact (resp. locally compact, metric) space.

(2) (a) $X$ is a $k_\omega$-space (resp. $s_\omega$-space) $\iff X$ is the quotient image of a locally compact Lindelöf (resp. locally compact, separable metric) space.

(b) $X$ is a space determined by a point-finite cover of compact (resp. compact metric) subsets $\iff X$ is the quotient compact image of a locally compact paracompact (resp. locally compact metric) space. Here, we can replace “point-finite cover” by “point-countable cover”, but change “quotient compact image” to “quotient s-image”.

(3) (a) $X$ is a bi-$k$-space (resp. bi-quasi-$k$-space) $\iff X$ is the bi-quotient image of a paracompact $M$-space (resp. $M$-space).

(b) $X$ is a countably bi-$k$-space (resp. countably bi-quasi-$k$-space) $\iff X$ is the countably bi-quotient image of a paracompact $M$-space (resp. $M$-space).

In the following results, (1) is well-known (see [M1], for example). (2) (resp. (3)) is due to [M3] (resp. [M2]). (4) holds in view of [My1] and [M2], here note that every product of a first countable space with a strongly sequential space is strongly sequential ([M1]). (5) is due to [T1].

Result: (1) Every product of a locally compact space (resp. locally countably compact, sequential space) with a $k$-space (resp. sequential space) is a $k$-space (resp. sequential space).

(2) Every product of bi-$k$-spaces is a bi-$k$-space, hence a $k$-space.

(3) Every product of $k_\omega$-spaces is a $k_\omega$-space, hence a $k$-space.

(4) Every product of a first countable space with a sequential Tanaka space is a sequential space.

(5) For sequential spaces $X$ and $Y$, $X \times Y$ is sequential if and only if

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$^3$This is an affirmative answer to the author's question (when he prepared [T2]). F. Mynard obtained this result by use of categorical method ([My1] & [My2]). The result is also proved by use of multisequences method ([D]), or directly shown without these methods ([L]).
2. Questions and Comments

**Question 1.** ([T5]) Every product of sequentially compact (or countably compact) $k$-spaces $X$ and $Y$ is a $k$-space?

*Comment:* (1.1) Question 1 is affirmative if $X$ or $Y$ is sequential ([T1]). But, not every product of a countably compact first countable space with a $k$-space is a $k$-space.

(1.2) Every product of a $k$-and-$q$-space with a bi-$k$-space (or sequential $q$-space) is a $k$-space by (2.2) below. If Question 1 is affirmative, then every product of $k$-and-$q$-spaces is a $k$-space.

(1.3) Let $X$ be sequentially compact (countably compact; $q$), and let $Y$ be sequentially compact (resp. countably compact $k$; $q$-and-$k$), then $X \times Y$ is sequentially compact (resp. countably compact; $q$). Note that every sequentially compact space need not be a $k$-space.

**Question 2.** Let $X$ be a $k$-space which is bi-quasi-$k$. Let $Y$ be a sequential space. Then the following are equivalent?

(a) $X \times Y$ is a $k$-space.

(b) $X$ is locally countably compact, or $Y$ is a Tanaka space?

*Comment:* (2.1) Question 2 is affirmative if $X$ is a bi-$k$-space by (2.2) & (2.4) below.

(2.2) In Question 2, (b) $\Rightarrow$ (a) holds. In general, the following case (c$_1$) or (c$_2$) implies that $X \times Y$ is a $k$-space ([T5]).

(c$_1$) $X$ is a $k$-space which is bi-quasi-$k$, and $Y$ is a sequential Tanaka space (in particular, a sequential countably bi-quasi-$k$-space).

(c$_2$) $X$ is a bi-$k$-space, and $Y$ is a $k$-space which is countably bi-quasi-$k$.

(2.3) Every product of sequential countably bi-$k$-spaces (actually, countably bi-sequential, countable spaces) need not be a $k$-space (not a Tanaka space) under $(2^\aleph_0 < 2^\aleph_1)$ ([O]).

(2.4) In Question 2, (a) $\Rightarrow$ (b) holds if $X$ is a first countable space ([T2]), more generally, a bi-$k$-space ([TS], etc.).

(2.5) Every product of sequential Tanaka spaces (actually, countably bi-sequential, countable spaces) need not be a Tanaka space (hence, not strongly sequential). (Also, cf. (2.3)). But, every product $X \times Y$ of
Tanaka spaces is a Tanaka space if $X$ is bi-quasi-$k$. Thus, for sequential spaces $X$ and $Y$, $(c_1)$ or $(c_2)$ in (2.2) implies that $X \times Y$ is a Tanaka space which is sequential by means of (2.2) and Result (5). In view of this and (2.3), the author has following question: For sequential spaces $X$ and $Y$, if $X \times Y$ is a Tanaka space, then $X \times Y$ is sequential?

Let $S = \{\infty\} \cup \{p_n : n \in N\} \cup \{p_{nm} : n, m \in N\}$ be an infinite countable space such that each $p_{nm}$ is isolated in $S$, $K = \{p_n : n \in N\}$ converges to $\infty \notin K$, and each $L_n = \{p_{nm} : m \in N\}$ converges to $p_n \notin L_n$. We recall the following canonical spaces; the Arens’ space $S_2$, and the sequential fan $S_\omega$. $S_2$ is not Fréchet, but $S_\omega$ is Fréchet.

$S_2 = S$, but $\cup\{F_n : n \in N\}$ is closed in $S$ for every finite $F_n \subset L_n (n \in N)$.

$S_\omega = S_2/(K \cup \{\infty\})$ (i.e., the space obtained from the topological sum of countably many convergent sequences by identifying all the limit points).

**Question 3.** ([TS]) Let $X$ be a bi-$k$-space, and let $Y$ be a sequential space. Then the following are equivalent?

(a) $X \times Y$ is a $k$-space.

(b) $X$ is locally countably compact, or $Y$ contains no (closed) copy of $S_\omega$, and no (closed) copy of $S_2$?

Let us recall that a cover $\mathcal{P}$ of a space $X$ is a $k$-network for $X$ if, for any compact subset $K$, and any open set $V$ with $K \subset V$, $K \subset \cup \mathcal{F} \subset V$ for some finite $\mathcal{F} \subset \mathcal{P}$. If $K$ is a single point, then such a cover $\mathcal{P}$ is called a network. Bases are $k$-networks, and $k$-networks are networks. Quotient $s$-images (or closed images) of metric spaces have point-countable $k$-networks. Paracompact $M$-spaces with point-countable $k$-networks are metrizable ([GMT]).

**Comment:** (3.1) In Question 3, (a) $\Rightarrow$ (b) holds ([TS]).

(3.2) Question 3 is reduced to the following question in view of (2.1): For a sequential space $X$, $X$ is a Tanaka space if and only if it contains no (closed) copy of $S_\omega$, and no $S_2$? (The “only if” part holds).

(3.3) Question 3 is affirmative if the sequential space $Y$ is one of the following spaces ([TS]).
(A1) Fréchet space.
(A2) Space in which every point is a $G_\delta$-set.
(A3) Hereditarily normal space.
(A4) Space having a point-countable k-network.
(A5) Closed image of a countably bi-k-space.
(A6) Closed image of an M-space.

(3.4) The author does not know whether Question 3 is affirmative when the sequential space $Y$ is the quotient $s$-image of a paracompact (countably) bi-k-space ([TS]). Question 3 is affirmative if the domain is metric by (A4).

**Question 4.** ([T6]) For a k-space $X$, $X$ is locally countably compact if and only if $X \times Y$ is a k-space for every quotient compact image $Y$ of a locally compact metric space?

Let us recall that a space $X$ is called symmetric if there exists a real valued, non-negative function $d$ defined on $X \times X$ such that (a) $d(x, y) = 0$ iff $x = y$, (b) $d(x, y) = d(y, x)$, and (c) $F \subseteq X$ is closed in $X$ iff $d(x, F) > 0$ for any $x \in X - F$. If we replace (c) by "$d(x, F) = 0$ iff $x \in \overline{F}$", then such a space $X$ is called semi-metric. Semi-metric spaces, or quotient compact images of metric spaces (e.g., the space $S_2$) are symmetric. Symmetric spaces are sequential. Symmetric M-spaces are metrizable ([N]).

**Comment:** (4.1) In Question 4, the "only if" part holds.

(4.2) Question 4 is affirmative if $X$ is one of the following spaces. For (B1), see (5.2) below. For (B4), we can replace "k-space" by "symmetric space" in Question 4.

(B1) Bi-k-space.
(B2) Space having character $\leq 2^\omega$ (in particular, locally separable space).
(B3) Space having a point-countable k-network.
(B4) Symmetric space.

(4.3) Question 4 is affirmative if we omit the locally compactness of the metric domain. Question 4 is also affirmative if we replace "metric space" by "Fréchet space"; or "quotient compact image" by "closed image".

(4.4) A k-space $X$ is locally compact if and only if $X \times Y$ is a k-space for every quotient compact image $Y$ of a locally compact, paracompact
space. Here, we can replace “quotient compact image” by “closed image”.

**Question 5.** ([T6]) For a $k$-space $X$, $X$ is a locally $k_{\omega}$-space if and only if $X \times Y$ is a $k$-space for every $k_{\omega}$-space $Y$?

**Comment:** (5.1) In Question 5, the “only if” part holds by Result (3).

(5.2) If we replace “$k_{\omega}$-space” $Y$ by “$s_{\omega}$-space” $Y$, then Question 5 is negative under (MA + ¬ CH).

(5.3) A bi-$k$-space $X$ is locally compact (resp. locally countably compact) if and only if $X \times Y$ is a $k$-space for every $k_{\omega}$-space (resp. $s_{\omega}$-space) $Y$. Here, the space $Y$ can be chosen to be the quotient compact (or closed) image of a locally compact Lindelöf (resp. locally compact separable metric) space.

**References**

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