Products of $k$-spaces, and questions

東京学芸大学 田中祥雄 (Yoshio Tanaka)

As is well-known, every product of a locally compact space with a $k$-space is a $k$-space, but not every product of a metric space with a $k$-space is a $k$-space. We consider characterizations or conditions for (finite) products of $k$-spaces to be $k$-spaces, and pose related questions. For other topics on the products of $k$-spaces, see [T3], [T4], for example.

We assume that spaces are regular $T_{1}$, and maps are continuous and onto.

1 Definitions and Preliminaries

Let $X$ be a space, and let $P$ be a (not necessarily open or closed) cover of $X$. Then $X$ is determined by a cover $P$, \(^1\) if $U \subset X$ is open in $X$ if and only if $U \cap P$ is relatively open in $P$ for every $P \in P$. Here, we can replace “open” by “closed”. Every space is determined by its open (or hereditarily closure-preserving closed) cover.

Let us recall that a space is a $k$-space (resp. sequential space) if it is determined by a cover of compact (resp. compact metric) subsets. Sequential space are $k$-spaces, and the converse holds if points are $G_{δ}$-sets. A space $X$ is called a $k_{ω}$-space [M3] (resp. $s_{ω}$-space) if $X$ is determined by a countable cover of compact (resp. compact metric) subsets.

A space $X$ is called a bi-$k$-space (resp. bi-quasi-$k$-space) [M3] if, whenever a filter base $F$ accumulates at $x \in X$, then there exists a $k$-sequence (resp. $q$-sequence) $\{A_{n} : n \in N\}$ such that $x \in F \cap \overline{A_{n}}$ for all $n \in N$ and all $F \in F$. When the filter base $F$ is a decreasing sequence, then such a space $X$ is a countably bi-$k$-space (resp. countably bi-quasi-$k$-space) [M3]. Here, a $k$-sequence (resp. $q$-sequence) is a decreasing sequence $\{A_{n} : n \in N\}$ such that $A = \cap \{A_{n} : n \in N\}$ is compact (resp. countably compact), and any open set $U \supset A$ contains some $A_{n}$ ([M3]).

Let us recall that a space $X$ is of pointwise countable type (resp. $q$-space) if each point has nbds $\{V_{n} : n \in N\}$ which is a $k$-sequence (resp. $q$-

\(^1\) Following [GMT], we shall use “$X$ is determined by $P$” instead of the usual “$X$ has the weak topology with respect to $P$".
sequence). Also, a space is an \(M\)-space if and only if it is the inverse image of a metric space under a quasi-perfect map. The following diagrams hold.

(a) Locally compact spaces, or first countable spaces \(\rightarrow\) spaces of point-wise countable type \(\rightarrow\) bi-\(k\)-spaces \(\rightarrow\) countably bi-\(k\)-spaces \(\rightarrow\) \(k\)-spaces.

(b) Locally countably compact spaces, or \(M\)-spaces \(\rightarrow\) \(q\)-spaces \(\rightarrow\) bi-quasi-\(k\)-spaces \(\rightarrow\) countably bi-quasi-\(k\)-spaces.

A space \(X\) is called a Tanaka space [My2], if \(X\) satisfies the following condition (C) in [T2].

(C) Let \(\{A_n : n \in N\}\) be a decreasing sequence of subsets of \(X\) with \(x \in \overline{A_n}\) for any \(n \in N\). Then there exist \(x_n \in A_n\) such that \(\{x_n : n \in N\}\) converges to some point \(y \in X\). If \(y = x\), then such a space \(X\) is called countably bi-sequential [M3] (= strongly Fréchet [S]).

Sequentially compact spaces, or sequential countably bi-quasi-\(k\)-spaces are Tanaka spaces. But, every Tanaka space (actually, sequentially compact space) need not be sequential, not even a \(k\)-space\(^2\).

A space \(X\) is strongly sequential [M1] if, whenever \(\{A_n : n \in N\}\) is a decreasing sequence of subsets of \(X\) with \(x \in \overline{A_n}\) for any \(n \in N\), then the point \(x\) belongs to the (idempotent) sequential closure of \(A\), where \(A\) is the set of all limit points of convergent sequences \(\{x_n : n \in N\}\) with \(x_n \in A_n\).

Namely, a space \(X\) is strongly sequential if and only if it is a sequential space such that if \(\{A_n : n \in N\}\) is a decreasing sequence of subsets of \(X\) with \(x \in \overline{A_n}\) for any \(n \in N\), then the point \(x\) belongs to the (usual) closure of the above set \(A\). Strongly Fréchet spaces are strongly sequential. Every strongly sequential space is precisely a sequential Tanaka space ([My2]).

A map \(f : X \rightarrow Y\) is called bi-quotient [M2] if, whenever \(y \in Y\) and \(U\) is a cover of \(f^{-1}(y)\) by open subsets of \(X\), then finitely many \(f(U)\), with \(U \in U\), cover some nbd of \(y\) in \(Y\). If \(U\) is countable, then such a map \(f\) is called countably bi-quotient [S]. Open maps, or perfect maps are bi-quotient. Every product of bi-quotient maps is bi-quotient, hence quotient ([M2]). A map \(f : X \rightarrow Y\) is called a compact (resp. s-map) if every \(f^{-1}(y)\) is compact (resp. separable).

\(^2\)This is pointed out by Z. Dolecki or P. Nyikos.
In the following characterizations, (1) is well-known, (2) is routinely shown, and (3) is due to [M3].

**Characterization:** (1) $X$ is a $k$-space (resp. sequential space) $\iff X$ is the quotient image of a locally compact (resp. locally compact, metric) space.

(2) (a) $X$ is a $k_\omega$-space (resp. $s_\omega$-space) $\iff X$ is the quotient image of a locally compact Lindelöf (resp. locally compact, separable metric) space.

(b) $X$ is a space determined by a point-finite cover of compact (resp. compact metric) subsets $\iff X$ is the quotient compact image of a locally compact paracompact (resp. locally compact metric) space. Here, we can replace “point-finite cover” by “point-countable cover”, but change “quotient compact image” to “quotient $s$-image”.

(3) (a) $X$ is a bi-$k$-space (resp. bi-quasi-$k$-space) $\iff X$ is the bi-quotient image of a paracompact $M$-space (resp. $M$-space).

(b) $X$ is a countably bi-$k$-space (resp. countably bi-quasi-$k$-space) $\iff X$ is the countably bi-quotient image of a paracompact $M$-space (resp. $M$-space).

In the following results, (1) is well-known (see [M1], for example). (2) (resp. (3)) is due to [M3] (resp. [M2]). (4) holds in view of [My1] and [M2], here note that every product of a first countable space with a strongly sequential space is strongly sequential ([M1]). (5) is due to [T1].

**Result:** (1) Every product of a locally compact space (resp. locally countably compact, sequential space) with a $k$-space (resp. sequential space) is a $k$-space (resp. sequential space).

(2) Every product of bi-$k$-spaces is a bi-$k$-space, hence a $k$-space.

(3) Every product of $k_\omega$-spaces is a $k_\omega$-space, hence a $k$-space.

(4)$^3$ Every product of a first countable space with a sequential Tanaka space is a sequential space.

(5) For sequential spaces $X$ and $Y$, $X \times Y$ is sequential if and only if

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$^3$This is an affirmative answer to the author’s question (when he prepared [T2]). F. Mynard obtained this result by use of categorical method ([My1] & [My2]). The result is also proved by use of *multisequences* method ([D]), or directly shown without these methods ([L]).
it is a $k$-space.

2. Questions and Comments

Question 1. ([T5]) Every product of sequentially compact (or countably compact) $k$-spaces $X$ and $Y$ is a $k$-space?

Comment: (1.1) Question 1 is affirmative if $X$ or $Y$ is sequential ([T1]). But, not every product of a countably compact first countable space with a $k$-space is a $k$-space.

(1.2) Every product of a $k$-and-$q$-space with a bi-$k$-space (or sequential $q$-space) is a $k$-space by (2.2) below. If Question 1 is affirmative, then every product of $k$-and-$q$-spaces is a $k$-space.

(1.3) Let $X$ be sequentially compact (countably compact; $q$), and let $Y$ be sequentially compact (resp. countably compact $k$; $q$-and-$k$), then $X \times Y$ is sequentially compact (resp. countably compact; $q$). Note that every sequentially compact space need not be a $k$-space.

Question 2. Let $X$ be a $k$-space which is bi-quasi-$k$. Let $Y$ be a sequential space. Then the following are equivalent?

(a) $X \times Y$ is a $k$-space.

(b) $X$ is locally countably compact, or $Y$ is a Tanaka space?

Comment: (2.1) Question 2 is affirmative if $X$ is a bi-$k$-space by (2.2) & (2.4) below.

(2.2) In Question 2, (b) $\Rightarrow$ (a) holds. In general, the following case (c$_1$) or (c$_2$) implies that $X \times Y$ is a $k$-space ([T5]).

(c$_1$) $X$ is a $k$-space which is bi-quasi-$k$, and $Y$ is a sequential Tanaka space (in particular, a sequential countably bi-quasi-$k$-space).

(c$_2$) $X$ is a bi-$k$-space, and $Y$ is a $k$-space which is countably bi-quasi-$k$.

(2.3) Every product of sequential countably bi-$k$-spaces (actually, countably bi-sequential, countable spaces) need not be a $k$-space (not a Tanaka space) under $(2^\kappa < 2^{\aleph_1})$ ([O]).

(2.4) In Question 2, (a) $\Rightarrow$ (b) holds if $X$ is a first countable space ([T2]), more generally, a bi-$k$-space ([TS], etc.).

(2.5) Every product of sequential Tanaka spaces (actually, countably bi-sequential, countable spaces) need not be a Tanaka space (hence, not strongly sequential). (Also, cf. (2.3)). But, every product $X \times Y$ of
Tanaka spaces is a Tanaka space if $X$ is bi-quasi-$k$. Thus, for sequential spaces $X$ and $Y$, ($c_1$) or ($c_2$) in (2.2) implies that $X \times Y$ is a Tanaka space which is sequential by means of (2.2) and Result (5). In view of this and (2.3), the author has following question: For sequential spaces $X$ and $Y$, if $X \times Y$ is a Tanaka space, then $X \times Y$ is sequential?

Let $S = \{\infty\} \cup \{p_n : n \in N\} \cup \{p_{nm} : n, m \in N\}$ be an infinite countable space such that each $p_{nm}$ is isolated in $S$, $K = \{p_n : n \in N\}$ converges to $\infty \notin K$, and each $L_n = \{p_{nm} : m \in N\}$ converges to $p_n \notin L_n$. We recall the following canonical spaces; the Arens’ space $S_2$, and the sequential fan $S_\omega$. $S_2$ is not Fréchet, but $S_\omega$ is Fréchet.

$S_2 = S$, but $\bigcup\{F_n : n \in N\}$ is closed in $S$ for every finite $F_n \subset L_n (n \in N)$.

$S_\omega = S_2/(K \cup \{\infty\})$ (i.e., the space obtained from the topological sum of countably many convergent sequences by identifying all the limit points).

**Question 3.** ([TS]) Let $X$ be a bi-$k$-space, and let $Y$ be a sequential space. Then the following are equivalent?

(a) $X \times Y$ is a $k$-space.

(b) $X$ is locally countably compact, or $Y$ contains no (closed) copy of $S_\omega$, and no (closed) copy of $S_2$?

Let us recall that a cover $\mathcal{P}$ of a space $X$ is a $k$-network for $X$ if, for any compact subset $K$, and any open set $V$ with $K \subset V$, $K \subset \cup \mathcal{F} \subset V$ for some finite $\mathcal{F} \subset \mathcal{P}$. If $K$ is a single point, then such a cover $\mathcal{P}$ is called a network. Bases are $k$-networks, and $k$-networks are networks. Quotient $s$-images (or closed images) of metric spaces have point-countable $k$-networks. Paracompact $M$-spaces with point-countable $k$-networks are metrizable ([GMT]).

**Comment:** (3.1) In Question 3, (a) $\Rightarrow$ (b) holds ([TS]).

(3.2) Question 3 is reduced to the following question in view of (2.1): For a sequential space $X$, $X$ is a Tanaka space if and only if it contains no (closed) copy of $S_\omega$, and no $S_2$? (The “only if” part holds).

(3.3) Question 3 is affirmative if the sequential space $Y$ is one of the following spaces ([TS]).
(A₁) Fréchet space.
(A₂) Space in which every point is a $G_\delta$-set.
(A₃) Hereditarily normal space.
(A₄) Space having a point-countable k-network.
(A₅) Closed image of a countably bi-k-space.
(A₆) Closed image of an $M$-space.

(3.4) The author does not know whether Question 3 is affirmative when the sequential space $Y$ is the quotient $s$-image of a paracompact (countably) bi-k-space ([TS]). Question 3 is affirmative if the domain is metric by (A₄).

**Question 4.** ([T6]) For a $k$-space $X$, $X$ is locally countably compact if and only if $X \times Y$ is a $k$-space for every quotient compact image $Y$ of a locally compact metric space?

Let us recall that a space $X$ is called symmetric if there exists a real valued, non-negative function $d$ defined on $X \times X$ such that (a) $d(x, y) = 0$ iff $x = y$, (b) $d(x, y) = d(y, x)$, and (c) $F \subset X$ is closed in $X$ iff $d(x, F) > 0$ for any $x \in X - F$. If we replace (c) by "$d(x, F) = 0$ iff $x \in \overline{F}$", then such a space $X$ is called semi-metric. Semi-metric spaces, or quotient compact images of metric spaces (e.g., the space $S_2$) are symmetric. Symmetric spaces are sequential. Symmetric $M$-spaces are metrizable ([N]).

**Comment:** (4.1) In Question 4, the “only if” part holds.

(4.2) Question 4 is affirmative if $X$ is one of the following spaces. For (B₁), see (5.2) below. For (B₄), we can replace "$k$-space" by "symmetric space" in Question 4.

(B₁) Bi-k-space.
(B₂) Space having character $\leq 2^\omega$ (in particular, locally separable space).
(B₃) Space having a point-countable k-network.
(B₄) Symmetric space.

(4.3) Question 4 is affirmative if we omit the locally compactness of the metric domain. Question 4 is also affirmative if we replace “metric space” by “Fréchet space”; or “quotient compact image” by “closed image”.

(4.4) A $k$-space $X$ is locally compact if and only if $X \times Y$ is a $k$-space for every quotient compact image $Y$ of a locally compact, paracompact
space. Here, we can replace “quotient compact image” by “closed image”.

**Question 5.** ([T6]) For a $k$-space $X$, $X$ is a locally $k_{\omega}$-space if and only if $X \times Y$ is a $k$-space for every $k_{\omega}$-space $Y$?

*Comment:* (5.1) In Question 5, the “only if” part holds by Result (3).
(5.2) If we replace “$k_{\omega}$-space” $Y$ by “$s_{\omega}$-space” $Y$, then Question 5 is negative under (MA + ¬ CH).
(5.3) A bi-$k$-space $X$ is locally compact (resp. locally countably compact) if and only if $X \times Y$ is a $k$-space for every $k_{\omega}$-space (resp. $s_{\omega}$-space) $Y$. Here, the space $Y$ can be chosen to be the quotient compact (or closed) image of a locally compact Lindelöf (resp. locally compact separable metric) space.

**References**

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DEPARTMENT OF MATHEMATICS, TOKYO GAKUEI UNIVERSITY, KOGANEI, TOKYO, 184-8501, JAPAN

E-mail address: ytanaka@u-gakugei.ac.jp