Products of $k$-spaces, and questions

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As is well-known, every product of a locally compact space with a $k$-space is a $k$-space, but not every product of a metric space with a $k$-space is a $k$-space. We consider characterizations or conditions for (finite) products of $k$-spaces to be $k$-spaces, and pose related questions. For other topics on the products of $k$-spaces, see [T3], [T4], for example.

We assume that spaces are regular $T_1$, and maps are continuous and onto.

1 Definitions and Preliminaries

Let $X$ be a space, and let $\mathcal{P}$ be a (not necessarily open or closed) cover of $X$. Then $X$ is determined by a cover $\mathcal{P}$, \footnote{Following [GMT], we shall use "$X$ is determined by $\mathcal{P}$" instead of the usual "$X$ has the weak topology with respect to $\mathcal{P}$".} if $U \subset X$ is open in $X$ if and only if $U \cap P$ is relatively open in $P$ for every $P \in \mathcal{P}$. Here, we can replace "open" by "closed". Every space is determined by its open (or hereditarily closure-preserving closed) cover.

Let us recall that a space is a $k$-space (resp. sequential space) if it is determined by a cover of compact (resp. compact metric) subsets. Sequential space are $k$-spaces, and the converse holds if points are $G_\delta$-sets. A space $X$ is called a $k_\omega$-space [M3] (resp. $s_\omega$-space) if $X$ is determined by a countable cover of compact (resp. compact metric) subsets.

A space $X$ is called a bi-$k$-space (resp. bi-quasi-$k$-space) [M3] if, whenever a filter base $\mathcal{F}$ accumulates at $x \in X$, then there exists a $k$-sequence (resp. $q$-sequence) $\{A_n : n \in N\}$ such that $x \in \bigcap \{A_n : n \in N\}$ for all $n \in N$ and all $F \in \mathcal{F}$. When the filter base $\mathcal{F}$ is a decreasing sequence, then such a space $X$ is a countably bi-$k$-space (resp. countably bi-quasi-$k$-space) [M3]. Here, a $k$-sequence (resp. $q$-sequence) is a decreasing sequence $\{A_n : n \in N\}$ such that $A = \bigcap \{A_n : n \in N\}$ is compact (resp. countably compact), and any open set $U \supset A$ contains some $A_n$ ([M3]).

Let us recall that a space $X$ is of pointwise countable type (resp. $q$-space) if each point has nbds $\{V_n : n \in N\}$ which is a $k$-sequence (resp. $q$-...
sequence). Also, a space is an \( M\)-space if and only if it is the inverse image of a metric space under a quasi-perfect map. The following diagrams hold.

(a) Locally compact spaces, or first countable spaces \( \rightarrow \) spaces of pointwise countable type \( \rightarrow \) bi-\( k\)-spaces \( \rightarrow \) countably bi-\( k\)-spaces \( \rightarrow \) \( k\)-spaces.

(b) Locally countably compact spaces, or \( M\)-spaces \( \rightarrow \) \( q\)-spaces \( \rightarrow \) bi-sequential \( \rightarrow \) countably bi-quasi-\( k\)-spaces.

A space \( X \) is called a Tanaka space [My2], if \( X \) satisfies the following condition (C) in [T2].

(C) Let \( \{ A_n : n \in N \} \) be a decreasing sequence of subsets of \( X \) with \( x \in \overline{A_n} \) for any \( n \in N \). Then there exist \( x_n \in A_n \) such that \( \{ x_n : n \in N \} \) converges to some point \( y \in X \). If \( y = x \), then such a space \( X \) is called countably bi-sequential [M3] (= strongly Fréchet [S]).

Sequentially compact spaces, or sequential countably bi-quasi-\( k\)-spaces are Tanaka spaces. But, every Tanaka space (actually, sequentially compact space) need not be sequential, not even a \( k\)-space\(^2\).

A space \( X \) is strongly sequential [M1] if, whenever \( \{ A_n : n \in N \} \) is a decreasing sequence of subsets of \( X \) with \( x \in \overline{A_n} \) for any \( n \in N \), then the point \( x \) belongs to the (idempotent) sequential closure of \( A \), where \( A \) is the set of all limit points of convergent sequences \( \{ x_n : n \in N \} \) with \( x_n \in A_n \). Namely, a space \( X \) is strongly sequential if and only if it is a sequential space such that if \( \{ A_n : n \in N \} \) is a decreasing sequence of subsets of \( X \) with \( x \in \overline{A_n} \) for any \( n \in N \), then the point \( x \) belongs to the (usual) closure of the above set \( A \). Strongly Fréchet spaces are strongly sequential. Every strongly sequential space is precisely a sequential Tanaka space ([My2]).

A map \( f : X \rightarrow Y \) is called bi-quotient [M2] if, whenever \( y \in Y \) and \( U \) is a cover of \( f^{-1}(y) \) by open subsets of \( X \), then finitely many \( f(U) \), with \( U \in U \), cover some nbd of \( y \) in \( Y \). If \( U \) is countable, then such a map \( f \) is called countably bi-quotient [S]. Open maps, or perfect maps are bi-quotient. Every product of bi-quotient maps is bi-quotient, hence quotient ([M2]). A map \( f : X \rightarrow Y \) is called a compact (resp. \( s\)-map) if every \( f^{-1}(y) \) is compact (resp. separable).

\(^2\)This is pointed out by Z. Dolecki or P. Nyikos.
In the following characterizations, (1) is well-known, (2) is routinely shown, and (3) is due to [M3].

**Characterization:** (1) $X$ is a $k$-space (resp. sequential space) $\iff X$ is the quotient image of a locally compact (resp. locally compact, metric) space.

(2) (a) $X$ is a $k_{\omega}$-space (resp. $s_{\omega}$-space) $\iff X$ is the quotient image of a locally compact Lindelöf (resp. locally compact, separable metric) space.

(b) $X$ is a space determined by a point-finite cover of compact (resp. compact metric) subsets $\iff X$ is the quotient compact image of a locally compact paracompact (resp. locally compact metric) space. Here, we can replace “point-finite cover” by “point-countable cover”, but change “quotient compact image” to “quotient s-image”.

(3) (a) $X$ is a bi-$k$-space (resp. bi-quasi-$k$-space) $\iff X$ is the bi-quotient image of a paracompact $M$-space (resp. $M$-space).

(b) $X$ is a countably bi-$k$-space (resp. countably bi-quasi-$k$-space) $\iff X$ is the countably bi-quotient image of a paracompact $M$-space (resp. $M$-space).

In the following results, (1) is well-known (see [M1], for example). (2) (resp. (3)) is due to [M3] (resp. [M2]). (4) holds in view of [My1] and [M2], here note that every product of a first countable space with a strongly sequential space is strongly sequential ([M1]). (5) is due to [T1].

**Result:** (1) Every product of a locally compact space (resp. locally countably compact, sequential space) with a $k$-space (resp. sequential space) is a $k$-space (resp. sequential space).

(2) Every product of bi-$k$-spaces is a bi-$k$-space, hence a $k$-space.

(3) Every product of $k_{\omega}$-spaces is a $k_{\omega}$-space, hence a $k$-space.

(4) Every product of a first countable space with a sequential Tanaka space is a sequential space.

(5) For sequential spaces $X$ and $Y$, $X \times Y$ is sequential if and only if

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3This is an affirmative answer to the author’s question (when he prepared [T2]). F. Mynard obtained this result by use of categorical method ([My1] & [My2]). The result is also proved by use of *multisequences* method ([D]), or directly shown without these methods ([I]).
it is a \( k \)-space.

2. Questions and Comments

Question 1. ([T5]) Every product of sequentially compact (or countably compact) \( k \)-spaces \( X \) and \( Y \) is a \( k \)-space?

Comment: (1.1) Question 1 is affirmative if \( X \) or \( Y \) is sequential ([T1]). But, not every product of a countably compact first countable space with a \( k \)-space is a \( k \)-space.

(1.2) Every product of a \( k \)-and-\( q \)-space with a bi-\( k \)-space (or sequential \( q \)-space) is a \( k \)-space by (2.2) below. If Question 1 is affirmative, then every product of \( k \)-and-\( q \)-spaces is a \( k \)-space.

(1.3) Let \( X \) be sequentially compact (countably compact; \( q \)), and let \( Y \) be sequentially compact (resp. countably compact \( k \); \( q \)-and-\( k \)), then \( X \times Y \) is sequentially compact (resp. countably compact; \( q \)). Note that every sequentially compact space need not be a \( k \)-space.

Question 2. Let \( X \) be a \( k \)-space which is bi-quasi-\( k \). Let \( Y \) be a sequential space. Then the following are equivalent?

(a) \( X \times Y \) is a \( k \)-space.

(b) \( X \) is locally countably compact, or \( Y \) is a Tanaka space?

Comment: (2.1) Question 2 is affirmative if \( X \) is a bi-\( k \)-space by (2.2) & (2.4) below.

(2.2) In Question 2, (b) \( \Rightarrow \) (a) holds. In general, the following case (\( c_1 \)) or (\( c_2 \)) implies that \( X \times Y \) is a \( k \)-space ([T5]).

(\( c_1 \)) \( X \) is a \( k \)-space which is bi-quasi-\( k \), and \( Y \) is a sequential Tanaka space (in particular, a sequential countably bi-quasi-\( k \)-space).

(\( c_2 \)) \( X \) is a bi-\( k \)-space, and \( Y \) is a \( k \)-space which is countably bi-quasi-\( k \).

(2.3) Every product of sequential countably bi-\( k \)-spaces (actually, countably bi-sequential, countable spaces) need not be a \( k \)-space (not a Tanaka space) under \( (2^{k_0} < 2^{k_1}) \) ([O]).

(2.4) In Question 2, (a) \( \Rightarrow \) (b) holds if \( X \) is a first countable space ([T2]), more generally, a bi-\( k \)-space ([TS], etc.).

(2.5) Every product of sequential Tanaka spaces (actually, countably bi-sequential, countable spaces) need not be a Tanaka space (hence, not strongly sequential). (Also, cf. (2.3)). But, every product \( X \times Y \) of
Tanaka spaces is a Tanaka space if $X$ is bi-quasi-$k$. Thus, for sequential spaces $X$ and $Y$, $(c_1)$ or $(c_2)$ in (2.2) implies that $X \times Y$ is a Tanaka space which is sequential by means of (2.2) and Result (5). In view of this and (2.3), the author has following question: For sequential spaces $X$ and $Y$, if $X \times Y$ is a Tanaka space, then $X \times Y$ is sequential?

Let $S = \{\infty\} \cup \{p_n : n \in \mathbb{N}\} \cup \{p_{nm} : n, m \in \mathbb{N}\}$ be an infinite countable space such that each $p_{nm}$ is isolated in $S$, $K = \{p_n : n \in \mathbb{N}\}$ converges to $\infty \notin K$, and each $L_n = \{p_{nm} : m \in \mathbb{N}\}$ converges to $p_n \notin L_n$. We recall the following canonical spaces; the Arens' space $S_2$, and the sequential fan $S_\omega$. $S_2$ is not Fréchet, but $S_\omega$ is Fréchet.

$S_2 = S$, but $\bigcup\{F_n : n \in \mathbb{N}\}$ is closed in $S$ for every finite $F_n \subset L_n (n \in \mathbb{N})$.

$S_\omega = S_2/(K \cup \{\infty\})$ (i.e., the space obtained from the topological sum of countably many convergent sequences by identifying all the limit points).

**Question 3.** ([TS]) Let $X$ be a bi-$k$-space, and let $Y$ be a sequential space. Then the following are equivalent?

(a) $X \times Y$ is a $k$-space.

(b) $X$ is locally countably compact, or $Y$ contains no (closed) copy of $S_\omega$, and no (closed) copy of $S_2$?

Let us recall that a cover $\mathcal{P}$ of a space $X$ is a $k$-network for $X$ if, for any compact subset $K$, and any open set $V$ with $K \subset V$, $K \cup \mathcal{F} \subset V$ for some finite $\mathcal{F} \subset \mathcal{P}$. If $K$ is a single point, then such a cover $\mathcal{P}$ is called a network. Bases are $k$-networks, and $k$-networks are networks. Quotient $s$-images (or closed images) of metric spaces have point-countable $k$-networks. Paracompact $M$-spaces with point-countable $k$-networks are metrizable ([GMT]).

**Comment:** (3.1) In Question 3, (a) $\Rightarrow$ (b) holds ([TS]).

(3.2) Question 3 is reduced to the following question in view of (2.1): For a sequential space $X$, $X$ is a Tanaka space if and only if it contains no (closed) copy of $S_\omega$, and no $S_2$ ? (The “only if” part holds).

(3.3) Question 3 is affirmative if the sequential space $Y$ is one of the following spaces ([TS]).
(A₁) Fréchet space.
(A₂) Space in which every point is a $G_δ$-set.
(A₃) Hereditarily normal space.
(A₄) Space having a point-countable $k$-network.
(A₅) Closed image of a countably bi-$k$-space.
(A₆) Closed image of an $M$-space.

(3.4) The author does not know whether Question 3 is affirmative when the sequential space $Y$ is the quotient $s$-image of a paracompact (countably) bi-$k$-space ([TS]). Question 3 is affirmative if the domain is metric by (A₄).

**Question 4.** ([T6]) For a $k$-space $X$, $X$ is locally countably compact if and only if $X \times Y$ is a $k$-space for every quotient compact image $Y$ of a locally compact metric space?

Let us recall that a space $X$ is called symmetric if there exists a real valued, non-negative function $d$ defined on $X \times X$ such that (a) $d(x, y) = 0$ iff $x = y$, (b) $d(x, y) = d(y, x)$, and (c) $F \subset X$ is closed in $X$ iff $d(x, F) > 0$ for any $x \in X - F$. If we replace (c) by "$d(x, F) = 0$ iff $x \in \overline{F}$", then such a space $X$ is called semi-metric. Semi-metric spaces, or quotient compact images of metric spaces (e.g., the space $S_2$) are symmetric. Symmetric spaces are sequential. Symmetric $M$-spaces are metrizable ([N]).

**Comment:** (4.1) In Question 4, the "only if" part holds.
(4.2) Question 4 is affirmative if $X$ is one of the following spaces. For (B₁), see (5.2) below. For (B₄), we can replace "$k$-space" by "symmetric space" in Question 4.

(B₁) Bi-$k$-space.
(B₂) Space having character $\leq 2^\omega$ (in particular, locally separable space).
(B₃) Space having a point-countable $k$-network.
(B₄) Symmetric space.

(4.3) Question 4 is affirmative if we omit the locally compactness of the metric domain. Question 4 is also affirmative if we replace "metric space" by "Fréchet space"; or "quotient compact image" by "closed image".

(4.4) A $k$-space $X$ is locally compact if and only if $X \times Y$ is a $k$-space for every quotient compact image $Y$ of a locally compact, paracompact
space. Here, we can replace "quotient compact image" by "closed image".

**Question 5.** ([T6]) For a $k$-space $X$, $X$ is a locally $k_{\omega}$-space if and only if $X \times Y$ is a $k$-space for every $k_{\omega}$-space $Y$?

**Comment:**

(5.1) In Question 5, the "only if" part holds by Result (3).

(5.2) If we replace "$k_{\omega}$-space" $Y$ by "$s_{\omega}$-space" $Y$, then Question 5 is negative under $(\text{MA} + \neg \text{CH})$.

(5.3) A bi-$k$-space $X$ is locally compact (resp. locally countably compact) if and only if $X \times Y$ is a $k$-space for every $k_{\omega}$-space (resp. $s_{\omega}$-space) $Y$. Here, the space $Y$ can be chosen to be the quotient compact (or closed) image of a locally compact Lindelöf (resp. locally compact separable metric) space.

**References**

[D] Z. Dolecki, Strongly sequential convergences (pre-print).


[S] F. Siwiec, Sequence-converging and countably bi-quotient mappings,


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