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Topologies of Hyperspaces
（巾空間のトポロジー）

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1 Introduction

In this note, we survey recent results on hyperspaces with the Wijsman topology and the Attouch-Wets topology.

For a metric space $X = (X, d)$, let $\text{Cld}(X)$ be the hyperspace of non-empty closed sets. By $\text{Fin}(X)$, $\text{Comp}(X)$ and $\text{Bdd}(X)$, we denote the subspaces of $\text{Cld}(X)$ consisting of finite sets, compact sets and bounded sets, respectively. Let $C(X)$ be the set of all continuous real-valued functions on $X$. By identifying each $A \in \text{Cld}(X)$ with the map

$$X \ni x \mapsto d(x, A) = \inf_{a \in A} d(x, a) \in \mathbb{R},$$

we can regard $\text{Cld}(X) \subset C(X)$, whence $\text{Cld}(X)$ has various topologies inherited from $C(X)$. The Hausdorff metric topology on $\text{Cld}(X)$ is the topology of uniform convergence, the Attouch-Wets topology on $\text{Cld}(X)$ is the topology of uniform convergence on bounded sets, and the Wijsman topology on $\text{Cld}(X)$ is the topology of point-wise convergence, which depend on the metric $d$ for $X$.

It should be remarked that the Attouch-Wets topology and the Wijsman topology are equal to the Fell topology on $\text{Cld}(X)$ if $X$ is a finite-dimensional normed linear space (cf. [2, p.142 & p.144]).

2 The Wijsman Topology

When we consider hyperspaces with the Wijsman topology, we denote $\text{Cld}_W(X)$, $\text{Fin}_W(X)$, $\text{Bdd}_W(X)$, etc. It is well-known that $\text{Cld}_W(X)$ is metrizable if and only if $X$ is separable, whence we can define an
admissible metric $d_W$ for $\text{Cld}_W(X)$ by using a countable dense set 
$\{x_i \mid i \in \mathbb{N}\}$ in $X$ as follows:

$$d_W(A, B) = \sup_{i \in \mathbb{N}} \min\{2^{-i}, |d(x_i, A) - d(x_i, B)|\}.$$ 

In [4], the following theorem is proved:

**Theorem 2.1.** If $X$ is an infinite-dimensional separable Banach space, then $\text{Cld}_W(X)$ is homeomorphic to $\approx$ the separable Hilbert space $\ell_2$.

Also, for $\text{Fin}_W(X)$ and $\text{Bdd}_W(X)$, similar results are proved in [4]:

**Theorem 2.2.** If $X$ is an infinite-dimensional separable Banach space, then

$$\text{Fin}_W(X) \approx \text{Bdd}_W(X) \approx \ell_2 \times \ell_2^f,$$

where $\ell_2^f = \{(x_i)_{i \in \mathbb{N}} \in \ell_2 \mid x_i = 0$ except for finitely many $i \in \mathbb{N}\}$.

To prove Theorems 2.1 and 2.2, we need characterizations of $\ell_2$ and $\ell_2 \times \ell_2^f$. The following characterization of $\ell_2$ is due to Toruńczyk [7] (cf. [8]):

**Theorem 2.3.** In order that $X \approx \ell_2$, it is necessary and sufficient that $X$ is a separable completely metrizable AR which has the discrete approximation property, that is,

 GIVEN a map $f : \bigoplus_{n \in \mathbb{N}} I^n \rightarrow X$, there exist maps $g : \bigoplus_{n \in \mathbb{N}} I^n \rightarrow X$ arbitrarily close to $f$ such that $\{g(I^n) \mid n \in \mathbb{N}\}$ is discrete in $X$.

To state the characterization of $\ell_2 \times \ell_2^f$ due to Bestvina and Mogilski [3], we need some notions. A metrizable space $X$ is $\sigma$-completely metrizable if $X$ is a countable union of completely metrizable closed subsets. A closed set $A \subset X$ is a (strong) $Z$-set in $X$ if there are maps $f : X \rightarrow X \setminus A$ arbitrarily close to $\text{id}$ (such that $A \cap \text{cl} f(X) = \emptyset$). A countable union of (strong) $Z$-sets is called a (strong) $Z_{\sigma}$-set. When $X$ itself is a (strong) $Z_{\sigma}$-set in $X$, we call $X$ a (strong) $Z_{\sigma}$-space. For a class $C$ of spaces, $X$ is strongly universal for $C$ if the following condition is satisfied:
Given a map $f : A \to X$ of $A \in C$ such that $f|B$ is a $Z$-embedding of a closed set $B \subset A$, there exist $Z$-embeddings $g : A \to X$ arbitrarily close to $f$ such that $g|B = f|B$.

In these definitions, the phrase 'arbitrarily close' is understood with respect to the limitation topology. In case $X = (X, d)$ is a metric space, given a collection $\mathcal{M}$ of maps from a space $Y$ to $X$, a map $f : Y \to X$ is arbitrarily close to maps in $\mathcal{M}$ if for each $\alpha : X \to (0,1)$ there is $g \in \mathcal{M}$ such that $d(f(y), g(y)) < \alpha(f(y))$ for every $y \in Y$. The following is Corollary 6.3 in [3].

**Theorem 2.4.** In order that $X \approx \ell_{2} \times \ell_{2}^f$, it is necessary and sufficient that $X$ is a separable $\sigma$-completely metrizable AR which is a strong $Z_{\sigma}$-space and is strongly universal for separable completely metrizable spaces.

### 3 The Attouch-Wets Topology

When we consider hyperspaces with the Attouch-Wets topology, we denote $\text{Cld}_{AW}(X)$, $\text{Fin}_{AW}(X)$, $\text{Bdd}_{AW}(X)$, etc. Without the separability of $X$, $\text{Cld}_{AW}(X)$ is always metrizable and has an admissible metric $d_{AW}$ defined as follows:

$$d_{AW}(A, B) = \sup_{n \in \mathbb{N}} \min \left\{ \frac{1}{n}, \sup_{x \in X_n} \{|d(x, A) - d(x, B)|\} \right\},$$

where $x_0 \in X$ is fixed and $X_r = \{x \in X \mid d(x_0, x) \leq r\}$ for each $r \in \mathbb{R}$.

In [1], Banakh, Kurihara and Sakai showed the following theorem:

**Theorem 3.1.** If $X$ is an infinite-dimensional Banach space with weight $\tau$, then $\text{Cld}_{AW}(X) \approx \ell_{2}(2^{\tau})$, where $\ell_{2}(\gamma)$ is the Hilbert space with weight $\gamma$.

In [6], we have a following result which is analogous to Theorem 2.2:

**Theorem 3.2.** For every infinite-dimensional Banach space $X$ with weight $\tau$,

$$\text{Fin}_{AW}(X) \approx \text{Comp}_{AW}(X) \approx \ell_{2}(\tau) \times \ell_{2}^f$$

and

$$\text{Bdd}_{AW}(X) \approx \ell_{2}(2^{\tau}) \times \ell_{2}^f.$$
Theorem 3.2 is based on the following theorem, which is obtained in [5] as the non-separable version of Bestvina-Mogilski's characterization.

**Theorem 3.3.** In order that $X \cong \ell_2(\tau) \times \ell_2^f$, it is necessary and sufficient that $X$ is a $\sigma$-completely metrizable AR with weight $\tau$ which is a strong $Z_\sigma$-space and is strongly universal for $\mathcal{M}_1(\tau)$, where $\mathcal{M}_1(\tau)$ is the space of all completely metrizable spaces with weight $\tau$.

**References**


