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Submetacompactness in countable products

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1 Introduction

A space $X$ is said to be metacompact if every open cover of $X$ has a point finite open refinement and $X$ is said to be submetacompact if for every open cover $U$ of $X$, there is a sequence $(V_n)_{n \in \omega}$ of open refinements of $U$ such that for each $x \in X$, there is an $n \in \omega$ with $\text{Ord}(x,V_n) < \omega$. Here, for $x \in X$ and $n \in \omega$, let $V_{n_x} = \{ V \in V_n : x \in V \}$ and $\text{Ord}(x,V_n) = |V_{n_x}|$. We call this sequence $(V_n)_{n \in \omega}$ a $\theta$-sequence of open refinements of $U$. Clearly, every paracompact space is metacompact and every metacompact space is submetacompact. It is well known that if $X$ is countably compact and submetacompact, then $X$ is compact.

Since the notion of $C$-scattered spaces was introduced by R. Telgársky [Te1], $C$-scattered spaces play the fundamental role in the study of covering properties of products. A space $X$ is said to be scattered if every nonempty subset $A$ of $X$ has an isolated point in $A$, and $X$ is said to be $C$-scattered if for every nonempty closed subset $A$ of $X$, there is a point $x \in A$ which has a compact neighborhood in $A$. Scattered spaces and locally compact spaces are $C$-scattered. R. Telgársky [Te1] proved that if $X$ is a $C$-scattered paracompact (Lindelöf) space, then $X \times Y$ is paracompact (Lindelöf) for every paracompact (Lindelöf) space $Y$.

R. Telgársky [Te2] also introduced the notion of $DC$-like spaces, using topological games. The class of $DC$-like spaces includes all spaces with a $\sigma$-closure-preserving closed cover by compact subsets and all paracompact $C$-scattered spaces. R. Telgársky proved that if $X$ is a paracompact (Lindelöf) $DC$-like space, then $X \times Y$ is paracompact (Lindelöf) for every paracompact (Lindelöf) space $Y$. Furthermore, G. Gruenhage and Y. Yajima [GY] proved that if $X$ is a metacompact (submetacompact) $DC$-like space, then $X \times Y$ is metacompact (submetacompact) for every metacompact (submetacompact) space $Y$ and that if $X$ is a $C$-scattered metacompact (submetacompact) space, then $X \times Y$ is metacompact (submetacompact) for every metacompact (submetacompact) space $Y$. For covering properties of countable products, the author proved the following.

(A) ([T1]) If $Y$ is a perfect paracompact (hereditarily Lindelöf) space and $\{ X_n : n \in \omega \}$ is a countable collection of paracompact (Lindelöf) $DC$-like spaces, then the product $Y \times \prod_{n \in \omega} X_n$ is paracompact (Lindelöf).

(B) ([T2, T3]) If $\{ X_n : n \in \omega \}$ is a countable collection of metacompact (submetacompact) $DC$-like spaces, then the product $\prod_{n \in \omega} X_n$ is metacompact (submetacompact).

(C) ([T2]) If $\{ X_n : n \in \omega \}$ is a countable collection of $C$-scattered metacompact spaces, then the product $\prod_{n \in \omega} X_n$ is metacompact.

The author [T3] asked whether the product $\prod_{n \in \omega} X_n$ is submetacompact whenever $X_n$ is a $C$-scattered submetacompact space for each $n \in \omega$.

Our purpose of this paper is to give an affirmative answer to this problem.
All spaces are assumed to be regular $T_1$. Let $\omega$ denote the set of natural numbers and $|A|$ denote the cardinality of a set $A$. Undefined terminology can be found in R. Engelking [E].

2 Submetacompactness

Let $X$ be a space. For a closed subset $A$ of $X$, let

$$A^* = \{x \in A : x \text{ has no compact neighborhood in } A\}.$$ 

Let $A^{(0)} = A$, $A^{(\alpha+1)} = (A^{(\alpha)})^*$ and $A^{(\alpha)} = \cap_{\beta<\alpha} A^{(\beta)}$ for a limit ordinal $\alpha$. Note that every $A^{(\alpha)}$ is a closed subset of $X$ and if $A$ and $B$ are closed subsets of $X$ such that $A \subseteq B$, then $A^{(\alpha)} \subseteq B^{(\alpha)}$ for each ordinal $\alpha$. Furthermore, $X$ is $C$-scattered if and only if $X^{(\alpha)} = \emptyset$ for some ordinal $\alpha$. Let $X$ be a $C$-scattered space and $A \subseteq X$. Put $\lambda(X) = \inf(\alpha : X^{(\alpha)} = \emptyset)$ and $\lambda(A) = \inf(\alpha : A \cap X^{(\alpha)} = \emptyset) \leq \lambda(X)$.

It is clear that if $A,B$ are subsets of $X$ such that $A \subseteq B$, then $\lambda(A) \leq \lambda(B)$. A subset $A$ of $X$ is said to be topped if there is an ordinal $\alpha(A)$ such that $A \cap X^{(\alpha(A))}$ is a nonempty compact subset of $X$ and $A \cap X^{(\alpha(A)+1)} = \emptyset$. Thus $\lambda(A) = \alpha(A) + 1$. For each $x \in X$, there is a unique ordinal $\alpha$ such that $x \in X^{(\alpha)} - X^{(\alpha+1)}$, which is denoted by $\operatorname{rank}(x) = \alpha$. There is a neighborhood base $B_x$ of $x$ in $X$, consisting of open subsets of $X$, such that for each $B \subseteq B_x$, $cB$ is topped in $X$ and $\alpha(cB) = \operatorname{rank}(x)$. If $A$ is a topped subset of $X$ and $B$ is a subset of $A$ such that $B \cap (A \cap X^{(\alpha(A))}) = B \cap X^{(\alpha(A))} = \emptyset$, then $\lambda(B) \leq \alpha(A) < \lambda(A) = \alpha(A) + 1$.

The following plays the fundamental role in the study of submetacompactness.

**LEMMA 2.1.** (G. Gruenhage and Y. Yajima [GY]) There is a filter $\mathcal{F}$ on $\omega$ satisfying: For every submetacompact space $X$ and every open cover $\mathcal{U}$ of $X$, there is a sequence $(\mathcal{V}_n)_{n \in \omega}$ of open refinements of $\mathcal{U}$ such that for each $x \in X$,

$$\{n \in \omega : \operatorname{Ord}(x, \mathcal{V}_n) < \omega\} \in \mathcal{F}.$$ 

By Lemma 2.1, let $\mathcal{F}^{n+1}$ denote the filter on $\omega^{n+1}$ generated by sets of the form

$$\prod_{i \leq n} F_i,$$ 

where $F_i \in \mathcal{F}$ for each $i \leq n$.

Put

$$\Phi_n = \prod_{i \leq n} \omega^{i+1} \text{ for each } n \in \omega \text{ and } \Phi = \cup \{\Phi_n : n \in \omega\}. $$

For $\mu = (\tau_0, \tau_1, \cdots, \tau_n) \in \Phi_n$, $n \in \omega$ with $n \geq 1$, let $\mu_- = (\tau_0, \tau_1, \cdots, \tau_{n-1}) \in \Phi_{n-1}$. If $\tau \in \Phi_0$, let $\tau_- = \emptyset$. For each $\tau \in \omega^{n+2}$, let $\mu \oplus \tau = (\tau_0, \tau_1, \cdots, \tau_n, \tau) \in \Phi_{n+1}$. Let $\mathcal{U}, \mathcal{V}$ be collections of subsets of a space $X$. Put $\mathcal{U} \wedge \mathcal{V} = \{U \cap V : U \in \mathcal{U} \text{ and } V \in \mathcal{V}\}$.

**THEOREM 2.2.** If $\{X_n : n \in \omega\}$ is a countable collection of $C$-scattered submetacompact spaces, then the product $\prod_{n \in \omega} X_n$ is submetacompact.

**PROOF.** We may assume the following (cf. [A1, Theorem]):

1. $X$ is a $C$-scattered submetacompact space and for each $n \in \omega$, $X_n = X$,
2. $X$ is topped and there is a point $a \in X$ such that $X^{(\alpha(X))} = \{a\}$.

We shall show that $X^\omega$ is submetacompact. Let $B$ be the base of $X^\omega$, consisting of all basic open subsets of $X^\omega$, that is $B = \prod_{n \in \omega} B_n \in B$ if there is an $n \in \omega$ such that for $i < n, B_i$ is an open subset of $X$ and for $i \geq n, B_i = X$. Let
We call $n(B)$ the length of $B$. Let $O$ be an open cover of $X^\omega$, which is closed under finite unions and $O' = \{B \in B : B \subset O \text{ for some } O \in O\}$.

Take a $B = \bigcup_{i \in \omega} B_i \in B$ and let $N(B) = \{i < n(B) : cLB_i \text{ is topped in } X\}$. Take an $i < n(B)$ with $i \notin N(B)$. In case $\lambda(clB_i)$ is an isolated ordinal. Then there is an ordinal $\gamma$ such that $\lambda(clB_i) = \gamma + 1$ and $clB_i \cap X^{(\gamma)}$ is nonempty and locally compact. For each $x \in clB_i \cap X^{(\gamma)}$, there is an open neighborhood $B_x$ of $x$ in $X$ such that $clB_x$ is topped in $X$, $clB_x \cap X^{(\gamma)}$ is compact and $\alpha(clB_x) = rank(x)$. For each $x \in clB_i \cap X^{(\gamma)}$, take an open neighborhood $B_x$ of $x$ in $X$ such that $clB_x$ is topped in $X$, $clB_x \cap X^{(\gamma)}$ is $\emptyset$ and $\alpha(clB_x) = rank(x)$. Then every $clB_i \cap clB_x$ is topped in $X$ and $\alpha(clB_x) = \alpha(clB_x)$. Next, let $i = n(B)$. Since $X^{(\alpha(X))} = \{a\}$, take a proper open neighborhood $B_a$ of $a$ in $X$, and for each $x \in X - \{a\}$, take an open neighborhood $B_x$ of $x$ in $X$ such that $a \notin clB_x, clB_x$ is topped in $X$ and $\alpha(clB_x) = rank(x)$. In case of that $\lambda(clB_i)$ is a limit ordinal. For each $x \in clB_i$, there is an open neighborhood $B_x$ of $x$ in $X$ such that $clB_x$ is topped in $X$ and $\alpha(clB_x) = rank(x)$. Since $B_i(B) = \{B_x : x \in clB_i\}$ is an open cover of $clB_i$ and $X$ is submetacompact, there is a $\theta$-sequence $(V^{j}_{B,i})_{j \in \omega}$ of open (in $X$) refinements of $B_i(B), V^{j}_{B,i} = \{V^{j}_{B,i} : j \in \omega, such that for each $j \in \omega, \cup V^{j}_{B,i} = B_i$ and for each $x \in B_i, \{j \in \omega : \forall x \in X, V^{j}_{B,i} < \omega \} \in \mathcal{F}, where \mathcal{F}$ is the filter on $\omega$ described in Lemma 2.1. For each $j \in \omega$ and $\xi \in \Xi_{B,i}, take an $c(\xi) \in clB_i$ such that $V^{j}_{B,i} \subset \theta(\xi)$. For each $i \in N(B)$ and $j \in \omega, let \Xi_{B,i} = \{c^{j}_{B,i}\}$ and $V^{j}_{B,i} = \{V^{j}_{B,i} : B_i\}$. For each $\eta = (j_0, j_1, \cdots, j_{n(B)}) \in \omega^{n(B)+1}$, put $\Xi_{B,\eta} = \Pi_{i<n(B)} \Xi_{B,i}$. For each $\xi = (\xi(i)) \in \Xi_{B,\eta}$, let $V(\xi) = \Pi_{i \leq n(B)} V^{\xi(i)} \times X \times \cdots$ and $V_n(\xi) = \{V^{\xi} : \xi \in \Xi_{B,\eta}\}$. Then every $V_n(\xi)$ is an open cover of $B$. Take a $\xi = (\xi(i)) \in \Xi_{B,\eta}$ and let $\mathcal{M}(\xi) = \{i \leq n(B) : clV^{\xi(i)} \text{ is topped in } X\}$. Then $N(B) \subset \mathcal{M}(\xi)$. Put $K(\xi) = \Pi_{\xi \in \mathcal{M}(\xi)} (clV^{\xi(i)} \cap X^{(\alpha(clV^{\xi(i)}))}) \times \Pi_{i \leq n(B), i \notin \mathcal{M}(\xi)} V^{\xi(i)} \times \{a\} \times \cdots = \Pi_{i \in \omega} K_{\xi,i}$ and $K(B, \eta) = \{K(\xi) : \xi \in \Xi_{B,\eta}\}$. We consider the following condition for $K(\xi)$.

("*) There is an open set $B' \in O'$ such that $K(\xi) \subset B'$.

If $K(\xi)$ satisfies ("*), define $n(\xi) = inf\{n(O) : K(\xi) \subset O \text{ and } O \in O'\}$. Put $r(\xi) = \max\{n(B), n(\xi)\}$. Then there is an $O(\xi) = \Pi_{i \in \omega} O_{\xi,i} \in O'$ such that:

(3) $K(\xi) \subset O(\xi)$,

(4) $n(\xi) = n(O(\xi))$.

Take an $H(\xi) = \Pi_{i \in n(\xi)} H_{\xi,i} \in O'$ such that:

(5) (a) $\Pi_{i<n(\xi)} H_{\xi,i} \times X \times \cdots \subset O(\xi)$,

(b) for $i$ with $n(\xi) \leq i \leq n(B)$ or $i \leq n(B)$ with $i \notin \mathcal{M}(\xi)$, let $H_{\xi,i} = O_{\xi,i}$,

(c) for $i < n(\xi)$ with $i \in \mathcal{M}(\xi)$, let $H_{\xi,i}$ be an open subset of $X$ such that $K_{\xi,i} = clV^{\xi(i)} \times X^{(\alpha(clV^{\xi(i)}))} \subset H_{\xi,i} \subset clH_{\xi,i} \subset O_{\xi,i}$,

(d) for $i$ with $n(B) < i < n(\xi)$, let $H_{\xi,i}$ be an open subset of $X$ such that $K_{\xi,i} = \{a\} \in H_{\xi,i} \subset clH_{\xi,i} \subset O_{\xi,i}$,

(e) if $r(\xi) = n(B)$, let $H_{\xi,i} = X$ for each $i > n(B)$, and if $r(\xi) = n(\xi) > n(B)$, let $H_{\xi,i} = X$ for $i \geq n(\xi)$. 
Then we have \( K(\xi) \subset H(\xi) \). Let \( \mathcal{P}(B) = \{ P : P \subset \{0, 1, \cdots, n(B)\} \} \) and \( P \in \mathcal{P}(B) \). Define

\[
G(\xi) = \prod_{i \in \omega} G_{\xi,i} \quad \text{and} \quad B(\xi, P) = \prod_{i \in \omega} B_{\xi,P,i}
\]

as follows:

(6) (a) In case of that \( r(\xi) = n(\xi) \). For each \( i \leq n(\xi) \), let \( G_{\xi,i} = V_{\xi(i)} \cap O_{\xi,i} \) and for each \( i > n(\xi) \), let \( G_{\xi,i} = X \).

(b) In case of that \( r(\xi) = n(\xi) > n(\xi) \). For each \( i \in \omega \), let \( G_{\xi,i} = \emptyset \).

(c) In either case, for each \( i \leq n(\xi) \), if \( i \in P \), let \( B_{\xi,P,i} = V_{\xi(i)} - clH_{\xi,i} \) and if \( i \notin P \), let \( B_{\xi,P,i} = V_{\xi(i)} \cap O_{\xi,i} \). For each \( i > n(\xi) \), let \( B_{\xi,P,i} = X \).

Clearly, if \( r(\xi) = n(\xi) \), then \( B(\xi, \emptyset) = G(\xi) \). Notice that for each \( i \in \omega \), \( B_{\xi,P,i} \subset B_i \) and if \( B(\xi, P) \neq \emptyset \), then \( n(B(\xi, P)) = n(B) + 1 \). Let \( i \leq n(\xi) \). If \( i \in P \) and \( i \notin \mathcal{M}(\xi) \), then \( B_{\xi,P,i} = \emptyset \).

Let

\[
B_{\eta,\xi}(\xi) = \{B(\xi, P) : P \in \mathcal{P}(B) - \{\emptyset\} \} \quad \text{if} \quad \eta(\xi) = n(\xi),
\]

\[
B_{\eta,\xi}(\xi) = \{B(\xi, P) : P \in \mathcal{P}(B) \} \quad \text{if} \quad \eta(\xi) = n(\xi) > n(\xi).
\]

CLAIM 1. Let \( K(\xi) \) satisfies the condition \((*)\), \( P \in \mathcal{P}(B) \) and \( B(\xi, P) \in B_{\eta,\xi}(\xi) \) with \( B(\xi, P) \neq \emptyset \). If \( r(\xi) = n(\xi) \), then there is an \( i < n(\xi) \) with \( i \in P \).

Now, assume that \( K(\xi) \) does not satisfy the condition \((*)\). Let \( G(\xi) = \emptyset \). Take a \( P \in \mathcal{P}(B) \) and define \( B(\xi, P) \) as follows: If \( P = \emptyset \), let \( B(\xi, P) = V(\xi) \). If \( P \neq \emptyset \), let \( B(\xi, P) = \emptyset \). Put \( B_{\eta,\xi}(\xi) = \{B(\xi, P) : P \in \mathcal{P}(B)\} = \{V(\xi)\} \).

Then, in each case, we have \( V(\xi) = G(\xi) \cup (\cup B_{\eta,\xi}(\xi)) \).

CLAIM 2. Let \( i \leq n(\xi) \), \( \xi = (\xi(i)) \in \Xi_{B,\eta} \), \( K(\xi) = \prod_{i \in \omega} K_{\xi,i}, P \in \mathcal{P}(B) \) and \( B(\xi, P) = \prod_{i \in \omega} B_{\xi,P,i} \) with \( B_{\xi,P,i} \neq \emptyset \).

(a) If \( i \in P \), then \( K(\xi) \) satisfies \((*)\), \( i \in \mathcal{M}(\xi) \) and \( \lambda(clB_{\xi,P,i}) < \lambda(clB_i) \).

(b) Let \( i \notin P \).

(i) If \( i \in \mathcal{M}(\xi) \), then \( clB_{\xi,P,i} \) is topped in \( X \) such that \( \lambda(clB_{\xi,P,i}) = \lambda(clV_{\xi(i)}) \) and \( K_{\xi,i} = clV_{\xi(i)} \cap X(\lambda(clV_{\xi(i)})) = clB_{\xi,P,i} \cap X(\lambda(clB_{\xi,P,i})) \). Furthermore, if \( i \notin \mathcal{N}(B) \), then \( clB_{\xi,P,i} \) is topped in \( X \) such that \( \lambda(clB_{\xi,P,i}) = \lambda(clB_i) \) and \( K_{\xi,i} = clB_i \cap X(\lambda(clB_i)) = clB_{\xi,P,i} \cap X(\lambda(clB_{\xi,P,i})) \).

(ii) If \( i \notin \mathcal{M}(\xi) \), then \( \lambda(clB_{\xi,P,i}) < \lambda(clB_i) \).

For each \( \eta \in \omega^{\alpha(\xi)+1} \), put

\[
G_{\eta}(\xi) = \{G_{\xi} : \xi \in \Xi_{B,\eta}\} \quad \text{and} \quad B_{\eta}(\xi) = \cup \{B_{\eta,\xi}(\xi) : \xi \in \Xi_{B,\eta}\}.
\]

Then we have

(7) (a) every element of \( G_{\eta}(\xi) \) is contained in some member of \( \mathcal{O}' \),
(b) \( \mathcal{G}_\eta(B) \cup B_\eta(B) \) is a cover of \( B \),
(c) the length of nonempty element of \( B_\eta(B) \) is \( n(B) + 1 \).

For each \( x \in B \), \( \{ \eta \in \omega^{n(B)+1} : \text{Ord}(x, \mathcal{V}_\eta) < \omega \} \in \mathcal{F}^{n(B)+1} \).

For each \( m \in \omega \) and \( \tau \in \Phi_m \), let us define \( \mathcal{G}_\tau \) and \( B_\tau \) of elements of \( B \). For each \( m \in \Phi_0 = \omega \), let \( \mathcal{G}_m = \mathcal{G}_m(X^\omega) \) and \( B_m = B_m(X^\omega) \). Assume that for \( m \in \omega \) and \( \mu \in \Phi_m \), we have already obtained \( \mathcal{G}_\mu \) and \( B_\mu \). Let \( \tau \in \Phi_{m+1} \) and \( \tau = \mu \oplus \eta \), where \( \mu = \tau_{-} \in \Phi_m \) and \( \eta \in \omega^{m+2} \). Let \( B \in B_\mu \). If \( B \neq \emptyset \), then we denote \( \mathcal{G}_\eta(B) \) and \( B_\eta(B) \) by \( \mathcal{G}_\tau(B) \) and \( B_\tau(B) \) respectively. If \( B = \emptyset \), let \( \mathcal{G}_\eta(B) = B_\eta(B) = \emptyset \). Let \( \mathcal{G}_\tau = \mathcal{G}_\mu \cup (\cup \{ \mathcal{G}_\tau(B) : B \in B_\tau \}) \) and \( B_\tau = \cup \{ B_\tau(B) : B \in B_\tau \} \). Then every nonempty element of \( B_\tau \) has the length \( m + 2 \).

**CLAIM 3.** \( \{ \mathcal{G}_\tau \cup (B_\tau \cap \mathcal{O}') : \tau \in \Phi \} \) is a \( \theta \)-sequence of open refinements of \( \mathcal{O}' \).

**PROOF OF CLAIM 3.** By (7) (a) and induction, for each \( \tau \in \Phi \), \( \mathcal{G}_\tau \cup (B_\tau \cap \mathcal{O}') \) is an open refinement of \( \mathcal{O}' \). Take an \( x = (x_1) \in X^\omega \). By (9), take a \( \tau(0) = m(0) \in \omega \) such that
\[
\text{Ord}(x, \mathcal{G}_{\tau(0)}(B_{\tau(0)})) < \omega.
\]
Then \( \tau(0) \in \Phi_0 \). If \( B_{\tau(0)} \neq \emptyset \), then we are done. So, assume that \( B_{\tau(0)} = \emptyset \). By (7) (c), every nonempty element of \( B_{\tau(0)} \) has the length 1. By (9) again, we can take an \( \eta(1) \in \omega^2 \) such that
\[
\eta(1) \in \cap \{ \{ \eta \in \omega^2 : \text{Ord}(x, \mathcal{G}_\eta(B) \cup B_\eta(B)) < \omega \} : x \in B \in B_{\tau(0)} \} \in \mathcal{F}^2.
\]

Let \( \tau(1) = (\eta(0), \eta(1)) \in \Phi_1 \). Then we have \( \text{Ord}(x, \mathcal{G}_{\tau(1)}(B_{\tau(1)})) < \omega \). Assume also that \( B_{\tau(1)} \neq \emptyset \). Continuing in this manner, we can choose a \( \tau(t) = (\eta(0), \eta(1), \cdots, \eta(t)) \in \Phi_t \) such that for each \( t \in \omega \), \( \text{Ord}(x, \mathcal{G}_{\tau(t)}(B_{\tau(t)})) < \omega \) and \( B_{\tau(t)} \neq \emptyset \). Since \( B_{\tau(t)} \neq \emptyset \) and finite for each \( t \in \omega \), it follows from König's lemma (cf. K. Kunen [K]) that there are sequences \( \{ \xi_t : t \in \omega \}, \{ N(t) : t \in \omega \}, \{ M(t) : t \in \omega \}, \{ K(t) : t \in \omega \}, \{ P(t) : t \in \omega \}, \{ B(t) = B(\xi(t), P(t)) : t \in \omega \}, B(\xi(t), P(t)) = \prod_{i \in \omega} K(t)_i \) of elements of \( B \) satisfying: for each \( t \in \omega \),

\[
(10) \quad (a) \ x \in B(t) = \prod_{i \in \omega} B(t)_i \in B_{\eta(t)}(B(t-1)) \text{ and } n(B(t)) = t + 1, \text{ where } B(-1) = X^\omega,
(b) \ [\xi_t \in \Xi_{B(t-1), \eta(t)}],
(c) \ N(t) = N(B(t-1)),
(d) \ M(t) = M(\xi_t),
(e) \ K(t) = K(\xi_t) = \prod_{i \in \omega} K(t)_i \in K(B(t-1), \eta(t)),
(f) \ P(t) \in \mathcal{P}(\{0, 1, \cdots, n(B(t-1))\})
(g) \text{ if } K(t) \text{ satisfies the condition } (*) \text{ and } \tau(\xi_t) = n(B(t-1)), \text{ then there is an } i < n(\xi_t) \text{ with } i \in P(t),
(h) \text{ if } i \in P(t), \text{ then } \lambda(clB(t)_i) < \lambda(clB(t-1)_i),
(i) \text{ for } i \notin P(t),
\]

(1) if \( i \in M(t) \), then \( K(t)_i \in clB(t-1)_i, i \in N(t+1) \) and furthermore, if \( i \in N(t) \), then \( K(t)_i = clB(t-1)_i \cap X^{(\preceq clB(t-1)_i)}) = K(t+1)_i = clB(t) \cap X^{(\preceq clB(t)_i)}) \) and hence \( \lambda(clB(t-1)_i) = \lambda(clB(t)_i) \),
(2) if \( i \notin M(t) \), then \( \lambda(clB(t)_i) < \lambda(clB(t-1)_i) \).
Let $i \in \omega$. By (10)(a), let $t \geq 1$ such that $n(B(t)) > i$. By (10)(h), if $i \in P(m)$ for $m \geq t$, $\lambda(clB(m)) < \lambda(clB(m-1))$. Since there does not exist an infinite decreasing sequence of ordinals, there is an $t_i \geq 1$ such that for each $t \geq t_i$, $i \notin P(t)$. By (10)(i)(2), there is an $m_i$ such that $m_i \geq t_i$ and for each $t \geq m_i$, $i \in \mathcal{M}(t)$. Then, by (10)(i)(1), for each $t \geq m_i$, $clB(t+1)_{i}$ is topped and $clB(t+1)_{i} \cap X^{(\alpha(clB(t+1))}) = K(t+1)_{i} = K(m_i +1)_{i}$.

Let $K = \prod_{i \in \omega} K(m_i +1)_{i}$. Then $K$ is a compact subset of $X^\omega$. Since $O$ is an open cover of $X^\omega$, which is closed under finite unions, there is an $O = \prod_{i \in \omega} O_i \in O'$ such that $K \subset O$. By (10)(a), take an $s \geq 1$ such that:

(11) (a) $n(O) \leq n(B(s-1))$

(b) for each $i < n(O)$, $m_i +1 \leq s$.

For each $i < n(O)$, by (11)(b), $K(s)_{i} = K(m_i +1)_{i} \subset O_i$. Then $K(s) \subset O$ and hence, $K(s)$ satisfies the conditon (*). Since $n(\xi_s) \leq n(O)$, $r(\xi_s) = n(B(s-1))$. By (10)(g), there is an $i < n(\xi_s)$ with $i \in P(s)$, which contradicts the way of taking $s$.

The proof is completed.

References


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