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<td>Author(s)</td>
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<td>Citation</td>
<td>数理解析研究所講究録 (2003), 1303: 56-63</td>
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<tr>
<td>Issue Date</td>
<td>2003-02</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/42759">http://hdl.handle.net/2433/42759</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
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<td>Textversion</td>
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COMMUTATIVITY AND NON-COMMUTATIVITY OF TOPOLOGICAL SEQUENCE ENTROPY ON CONTINUA

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ABSTRACT. Let $h_S(f)$ denote the topological sequence entropy of $f$ respect to the sequence $S$. We will prove the following.

1. $h_S(f) = h_S(g)$ for any sequence $S$ and any graph maps $f, g$.
2. For each $n$-dimensional compact topological manifold $M$ with $n > 1$, there exist two continuous maps $\tilde{F}, \tilde{G} : M \to M$ such that $0 = h_{S_2}(\tilde{F} \circ \tilde{G}) < \log 2 \leq h_{S_2}(\tilde{G} \circ \tilde{F})$ and $0 = h_{S_2}(\tilde{F} \circ \tilde{G}) < \log 2 \leq h_{S_2}(\tilde{F})$, where $S_2 = (2^n)_{n=1}^\infty$ and $\Omega(\tilde{F})$ is the set of nonwandering points of $\tilde{F}$.
3. A graph map $f$ is chaotic in the sense of Li-Yorke if and only if the shift map $\sigma_f : \lim(X, f) \to \lim(X, f)$ is chaotic in the sense of Li-Yorke.
4. For any $n$-dimensional compact topological manifold $M$ with $n \geq 2$, we construct a chaotic map $f_M$ in the sense of Li-Yorke from $M$ to itself such that the shift map $\sigma_{f_M}$ is not chaotic in the sense of Li-Yorke.

(1) and (2) are the affirmative answers of questions in [BCL, Remark 4.7].

1. INTRODUCTION.

T. N. T. Goodman introduced in [G] the notion of topological sequence entropy as an extension of the concept to topological entropy. Let $f$ be a continuous map from a compact metric space $(X, d)$ to itself. Let $h_S(f)$ denote the topological sequence entropy of $f$ respect to the sequence $S$ and $h(f)$ denote the topological entropy of $f$. We know that if $S = (i)_{i=1}^\infty$, then $h_S(f)$ is equal to $h(f)$ for all continuous maps $f$.

A map $f : X \to X$ is said to be chaotic in the sense of Li-Yorke if there exists an uncountable set $D$ such that

$$\limsup_{n \to \infty} d(f^n(x), f^n(y)) > 0 \quad \text{and} \quad \liminf_{n \to \infty} d(f^n(x), f^n(y)) = 0$$

for all $x, y \in D$ with $x \neq y$. This set $D$ is called a scramble set of $f$. When $X$ is a compact interval or the circle to itself, if $h(f) > 0$, then $f$ is chaotic in the sense of Li-Yorke, but the converse is not true, that is, there exists a continuous map $f' : [0, 1] \to [0, 1]$ with $h(f') = 0$ which is chaotic in the sense of Li-Yorke. In [FS] and [H] it was proved that $f$ is chaotic in the sense of Li-Yorke if and only if $h_S(f) > 0$ for some sequence $S$. This shows that chaotic maps can be characterized by the topological sequence entropy.

First, Kolyada and Snoha proved in [KS, Theorem A] that $h(f \circ g) = h(g \circ f)$ for all continuous maps $f, g$ from a compact metric space $X$ to itself. Moreover, it is showed in [BCL, Theorem 3.1 and Proposition 3.2] that $h_S(f \circ g) = h_S(g \circ f)$.
for any sequence $S$ if the maps $f, g$ are onto or $X$ is a compact interval. But, by [BCL, Theorem 4.5], there exist a 0-dimensional compact metric space $X$ and two continuous maps $f, g : X \to X$ such that $0 = h_{S_2}(f \circ g) < h_{S_2}(g \circ f) = \log 2$, where $S_2 = (2^i)_{i=1}^\infty$. The first aim of this paper is to show that $h_{S}(f \circ g) = h_{S}(g \circ f)$ for any sequence $S$ and any continuous maps $f, g$ from a graph to itself. For any $n$-dimensional compact topological manifold $M$ with $n \geq 2$, the second aim of this paper is to construct two continuous maps $\tilde{F}, \tilde{G}$ from $M$ to itself such that $0 = h_{S_2}(\tilde{F} \circ \tilde{G}) < \log 2 < h_{S_2}(\tilde{G} \circ \tilde{F})$. There are the affirmative answers of questions in [BCL, Remark 4.7].

If $\Omega(f)$ denotes the set of nonwandering points of $f$, it is known that $\Omega(f)$ is an invariant set for $f$, $\Omega(f) \subset \bigcap_{n=1}^{\infty} f^n(X)$ and $h(f) = h(f|_{\Omega(f)})$, where $f|_{\Omega(f)} : \Omega(f) \to \Omega(f)$ is the restriction map. Szleuk in [S] first pointed out that the formula $h_S(f) = h_S(f|_{\Omega(f)})$ does not necessarily hold. In [BCL, p.1708], it was shown that $\log 2 = h_{S_2}(f) > h_{S_2}(f|_{\Omega(f)}) = 0$ for some continuous map $f$ from a 0-dimensional compact metric space to itself. And by [C2], there exists a continuous map $f : [0, 1] \to [0, 1]$ such that $h_{S_2}(f) \geq \log 2 > h_{S_2}(f|_{\Omega(f)}) = 0$. We show that for the map $\tilde{F}$ above, $h_{S_2}(\tilde{F}) \geq \log 2 > h_{S_2}(\tilde{G}|_{\Omega(\tilde{F})}) = 0$.

We define the inverse limit space associated to $X$ and $f$ to be the set
\[
\lim_{\leftarrow} (X, f) = \{(x_i)_{i=0}^{\infty} \in X^{\infty} | f(x_i) = x_{i-1} \text{ for each } i = 1, 2, \ldots \}
\]
with a metric $\tilde{d}$ as $\tilde{d}((x_i)_{i=0}^{\infty}, (y_i)_{i=0}^{\infty}) = \sum_{i=0}^{\infty} 2^{-i}d(x_i, y_i)$. And the shift map \( \sigma_f : \lim_{\leftarrow} (X, f) \to \lim_{\leftarrow} (X, f) \) is defined by
\[
\sigma_f((x_i)_{i=0}^{\infty}) = (f(x_0), x_0, x_1, \ldots).
\]

Rongbao in [R] proved that if $f$ is surjective, then $f$ is chaotic in the sense of Li-Yorke if and only if $\sigma_f$ is chaotic in the sense of Li-Yorke. But Cánovas in [C1] showed that the hypothesis that $f$ is surjective can not be removed, that is, there exists a chaotic map $g$ in the sense of Li-Yorke from 0-dimensional compact metric space to itself such that $\sigma_g$ is not chaotic in the sense of Li-Yorke. And he also proved in [C1] that $f : [0, 1] \to [0, 1]$ (whether $f$ is surjective or not) is chaotic in the sense of Li-Yorke if and only if $\sigma_f$ is chaotic in the sense of Li-Yorke. For any $n$-dimensional compact topological manifold $M$ with $n \geq 2$, from the composition method of the map $\tilde{F}$ above, we construct a chaotic map $f_M$ in the sense of Li-Yorke from $M$ to itself such that $\sigma_{f_M}$ is not chaotic in the sense of Li-Yorke. And we show that $f : G \to G$ from a graph to itself is chaotic in the sense of Li-Yorke if and only if $\sigma_f$ is chaotic in the sense of Li-Yorke.

2. Definitions.

**Definition 2.1.** A continuum is a nonempty, compact, connected, metric space. A graph is a continuum which can be written as the union of finitely many arcs any two of which are disjoint or intersect only in one or both of their end points.

**Definition 2.2.** Let $Y$ be a subspace of a metric space $X$. $\text{Cl}(Y)$ and $\text{diam}Y$ denote the closure and the diameter of $Y$ in a space $X$, respectively.
The cardinality of a set $P$ will be denoted by $\text{Card}(P)$. Let $S_k = (k^i)_{i=1}^\infty$ for each positive integer $k > 1$.

Let $f$ be a continuous map from a compact metric space $X$ to itself. We denote the $n$-fold composition $f^n$ of $f$ with itself by $f \circ \cdots \circ f$ and $f^0$ the identity map. Let us denote $f^{-i}(Y)$ the $i$th inverse image of an arbitrary set $Y \subset X$ and $f^\omega(X) = \bigcap_{n=1}^{\infty} f^n(X)$.

Let $A, B$ be finite open covers of $X$. Denote $\{f^{-m}(A) | A \in A\}$ by $f^{-m}(A)$ for each positive integer $m$. The mesh of an open cover $A$ of $X$ is the supremum of the diameter of the elements of $A$, denoted by $\text{mesh} A$. Let us define $A \vee B = \{A \cap B | A \in A, B \in B\}$ and $N(A)$ denotes the minimal possible cardinality of a subcover chosen from $A$.

**Definition 2.3.** Let $f$ be a continuous map from a compact metric space $(X, d)$ to itself and $S = \{s_i | i = 1, 2, \ldots \}$ an increasing unbounded sequence of positive integers. We define the topological sequence entropy of $f$ relative to a finite open cover $A$ of $X$ (respect to the sequence $S$) as

$$h_S(f, A) = \lim_{n \to \infty} \frac{1}{n} \log N(\bigvee_{i=1}^{n-1} f^{-s_i}(A)).$$

And we define the topological sequence entropy of $f$ (respect to the sequence $S$) as

$$h_S(f) = \sup \{h_S(f, A) | A \text{ is a finite open cover of } X \}.$$  

If $s_i = i$ for each $i$, then $h_S(f)$ is equal to the standard topological entropy $h(f)$ of $f$ introduced by Adler, Konheim and McAndrew in [AKM].

3. The Graph Maps Case.

**Lemma 3.1.** Let $f$ be a continuous map from a graph $X$ to itself such that $f^n(X) \neq f^m(X)$ for all $n, m$ the set of all components of $f^n(X) \setminus f^m(X)$ and $E_n = \{\text{Cl}(C) \cap f^\omega(X) | C \in C_n\}$. There exists a positive number $N$ such that $E_n = E_N$ and $\text{Card} C_n = \text{Card} C_N$ for all $n \geq N$, and that $\text{Cl}(C)$ is an arc and $E_N \cap \text{Cl}(C)$ is one point for all $n \geq N$ and all $C \in C_n$ and that $f(E_N) = E_N$.

By making use of Lemma 3.1, we can prove the following.

**Theorem 3.2.** Let $f$ be a continuous map from a graph $X$ into itself. Then $h_S(f) = h_S(f|_{f^\omega(X)})$ for any sequence $S$, where $f|_{f^\omega(X)} : f^\omega(X) \to f^\omega(X)$ is the restriction map.

By Theorem 3.2 and [BCL, Proposition 3.2], we have the following.

**Corollary 3.3.** If $f, g$ are continuous maps from a graph to itself, then $h_S(f \circ g) = h_S(g \circ f)$ for any sequence $S$. 
Let us denote three Cantor sets $\Sigma', \Sigma_1$, and $\Sigma_2$ by $\{-2, -1, 0, 1, 2\}^\infty$, $\{-1, 0, 1\}^\infty$, and $\{(2, \alpha_1, \alpha_2, \ldots) : \alpha_i \in \Sigma'((\alpha_i)_{i=1}^\infty) \in \Sigma_1\}$, respectively. And let $\Sigma = \Sigma_1 \cup \Sigma_2$, $0 = (0, 0, \ldots)$ and $1 = (1, 1, \ldots)$. The shift map $\sigma : \Sigma' \to \Sigma'$ is defined by $\sigma((\alpha_i)_{i=1}^\infty) = (\alpha_{i+1})_{i=1}^\infty$. Let $p_n : \Sigma' \to \{-2, -1, 0, 1, 2\}^n$ be the projection for each $n$ such that $p_n((\alpha_i)_{i=1}^\infty) = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ for any $(\alpha_i)_{i=1}^\infty \in \Sigma'$. Denote $\Sigma^{(n)} = p_n(\Sigma), \Sigma_i^{(n)} = p_n(\Sigma_i)$ and $0^{(n)} = (0, 0, \ldots, 0), 1^{(n)} = (1, 1, \ldots, 1) \in \Sigma^{(n)}$ for each $n \geq 1$ and each $i = 1, 2$. For $\alpha = (\alpha_i)_{i=1}^\infty \in \Sigma$, denote $\alpha|_n = p_n(\alpha) \in \Sigma^{(n)}$ and $\alpha|_{n'} = p_{n'}^{-1}(\alpha|_n)$. For $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ $\in p_n(\Sigma')$ and $\theta' = (\theta_1', \theta_2', \ldots, \theta_n') \in p_{n'}(\Sigma')$ (or $\theta' \in \Sigma'$, respectively), denote $|\theta| = (|\theta_1|, |\theta_2|, \ldots, |\theta_n|)$ and $\theta \ast \theta' = (\theta_1, \theta_2, \ldots, \theta_n, \theta_1', \theta_2', \ldots, \theta_n') \in p_{n+n'}(\Sigma')$ (or $\theta \ast \theta' = (\theta_1, \theta_2, \ldots, \theta_n, \theta_1', \theta_2', \ldots, \theta_n') \in \Sigma'$, respectively).

Now we are going to define a substructuring machine $\mu : \Sigma' \to \Sigma'$. First, define $\mu(0) = 1$. Let $\alpha = (\alpha_i)_{i=1}^\infty \in \Sigma' \setminus \{0\}$ and $k = \min\{i : \alpha_i \neq 0\}$. Define $\mu(\alpha) = (\mu(\alpha)_i)_{i=1}^\infty$ by

$$
\mu(\alpha)_i = \begin{cases} 
 1 & \text{if } 1 \leq i \leq k-1 \\
 1 - |\alpha_k| & \text{if } i = k \\
 \alpha_i & \text{if } i > k 
\end{cases}
$$

We notice that

$$
(4.1) \quad \mu(\alpha|_n) \subset \Sigma\mu(\alpha)|_n \text{ for each } \alpha \in \Sigma \text{ and each } n \geq 1, \text{ thus, } \mu \text{ is continuous.}
$$

Thus, for each $n \geq 1$, we can think of $\mu$ as a map from $\Sigma^{(n)}$ to itself defined by $\theta \mapsto \mu(\theta \ast 0)|_n$. And we have

$$
(4.2) \quad \mu(\theta \ast 0) = \mu(\theta \ast 0) \text{ for all } \theta \in \Sigma^{(n)} \setminus \{0^{(n)}\} \text{ and}
$$

$$
(4.3) \quad \mu^m(\Sigma|_0) = \Sigma|_m, \text{ i.e. } \mu^m(\theta) = |\theta| \text{ for all } m \geq n \text{ and all } \theta \in \Sigma|_0.
$$

**Definition 4.1.**

(a) Let $\alpha, \beta \in \Sigma'$ with $\alpha|_n \neq \beta|_n$ and $k = \min\{i \leq n : \alpha_i \neq \beta_i\}$. Define $\alpha|_n < \beta|_n$ (or $\alpha < \beta$) if $\text{Card}\{1 \leq i < k : |\alpha_i| \leq 0\}$ is even and $\alpha_k < \beta_k$ or $\text{Card}\{1 \leq i < k : |\alpha_i| \leq 0\}$ is odd and $\alpha_k > \beta_k$.

(b) Let $A, B$ be subspaces of $[0, 1]$. If $x < y$ for all $x \in A$ and all $y \in B$, let us denote $A < B$.

Now we construct a family $\{D_\theta|\theta \in \Sigma^{(n)}\} (n = 1, 2, \ldots)$ of pairwise disjoint compact subintervals of $[0, 1]$ satisfying that for any $\alpha \in \Sigma$ and any $n = 1, 2, \ldots$,

$$
(4.4) \quad \text{diam} D_{\alpha|_n} = 9^{-n}
$$

and

$$
(4.5) \quad D_{\alpha|_{n+1}} \subset D_{\alpha|_n}.
$$

Moreover, we have the following property:

$$
(4.6) \quad D_{\alpha|_n} < D_{\beta|_n} \text{ if and only if } \alpha, \beta \in \Sigma \text{ with } \alpha|_n < \beta|_n.
$$

Denote $Y_i = \bigcap_{m=1}^\infty \bigcup\{D_\theta|\theta \in \Sigma_{i}^{(n)}\}$ $(i = 1, 2)$ and $Y = Y_1 \cup Y_2$. We see that $Y_1$ and $Y_2$ are disjoint and Cantor sets. It is known that there exists the homeomorphism $h : Y \to \Sigma$ such that $h^{-1}(\{\alpha\}) = \bigcap_{n=1}^\infty D_{\alpha|_n}$ for each $\alpha \in \Sigma$. Thus, for the sake of convenience, let us regard $Y, Y_1, Y_2$ and $h^{-1} \circ (\mu|_\Sigma) \circ h$ as $\Sigma, \Sigma_1, \Sigma_2$ and $\mu|_\Sigma$, respectively.
Denote $\Sigma(i) = \{ \alpha \in \Sigma | \alpha_i \neq 0 \}$. Let $(a_i)_{i=0}^{\infty}$ be a decreasing sequence of positive real numbers with $\sum_{i=0}^{\infty} 3^i a_i < 9^{-2}$. There exists a family \{ $K_{\alpha} | \alpha \in \bigcup_{i=0}^{\infty} \Sigma(i)$ \} of pairwise disjoint compact subintervals of $[0,1]$ such that $\text{diam} K_{\alpha} < a_i$ for all $\alpha \in \Sigma(i)$ and all $i \geq 0$ and that for $\alpha, \alpha' \in \bigcup_{i=0}^{\infty} \Sigma(i)$, $\alpha < \alpha'$ implies $K_{\alpha} < K_{\alpha'}$.

We have a monotone map $\pi : [0,1] \to [0,1]$ with $\pi(0) = 0$ and $\pi(1) = 1$ such that

$$\pi^{-1}(x) = \begin{cases} K_{\alpha} & \text{if } x = \alpha \in \bigcup_{i=0}^{\infty} \Sigma(i) \\ \text{one point} & \text{if otherwise} \end{cases}$$

Denote $K_{\alpha} = \pi^{-1}(\alpha)$ for each $\alpha \in \Sigma$, $K_{\theta} = \pi^{-1}(\Sigma_{\theta})$ for each $\theta \in \Sigma^{(n)}$, $X_1 = \pi^{-1}(\Sigma_1)$, $X_2 = \pi^{-1}(\Sigma_2)$ and $X = \pi^{-1}(\Sigma)$. By (4.6), we see that one side of $\alpha \in \Sigma$ is mapped by $\mu$ to one side of $\mu(\alpha)$. Thus, there exists the natural continuous map $\tilde{f} : \tilde{X} \to \tilde{X}$ such that $\mu \circ (\pi|_{\tilde{X}}) = (\pi|_{\tilde{X}}) \circ \tilde{f}$ and that for each $\alpha \in \bigcup_{i=1}^{\infty} \Sigma(i)$, $\tilde{f}|_{K_{\alpha}} : K_{\alpha} \to K_{\mu(\alpha)}$ is a linearly homeomorphism.

$$\begin{array}{c}
\tilde{X} \overset{\tilde{f}}{\longrightarrow} \tilde{X} \\
\pi|_{\tilde{X}} \downarrow \quad \downarrow \pi|_{\tilde{X}} \\
\Sigma \overset{\mu|_{\Sigma}}{\longrightarrow} \Sigma
\end{array}$$

Remark 4.2. (1) We can think of $X_i = \bigcup_{\alpha \in K_i} \text{Bd} K_{\alpha}$ (i = 1, 2), $X = X_1 \cup X_2$ and $\tilde{f}$ as $X_i$ (i = 1, 2), $X$ and $\tilde{f}$ in [BCL, p.1704], respectively.

(2) We notice that all fibers of $\pi|_{\tilde{X}} : X \to \Sigma$ have at most two points, that $(\pi|_{\tilde{X}}) \circ f = \mu \circ (\pi|_{\tilde{X}})$ and that $h_{S_2}(\mu|_{\Sigma}) = 0$, but $h_{S_2}(\tilde{f}|_{\tilde{X}}) = \log 2$ by [BCL, Lemma 4.4]. This implies that Bowen's theorem (see [MS, Theorem 7.1, p.165]) for topological sequence entropy does not necessarily hold.

As the proof of [BCL, Lemma 4.3 and 4.4], we have the following.

Lemma 4.3. With the notation above, $0 < h_{S_2}(\tilde{f}|_{X_1}) < \log 2 \leq h_{S_2}(\tilde{f})$.

5. The manifolds case.

Let $a \in [0,1]^2$ and $B$ a subspace of $[0,1]^2$. Denote $C(a,B) = \{ ta + (1-t)b \in [0,1]^2 | b \in B \text{ and } t \in [0,1] \}$. If $B = \{ b \}$, then we write $C(a,b) = C(a,B)$.

Let $\Sigma_1(0) = \{ 0 \} \subset \Sigma_1$, $\Sigma_1(n) = \{ \alpha \in \Sigma_1 | \alpha_n \neq 0 \}$ and $\alpha_k = 0(k > n)$ (n $\geq 1$), $m_{\alpha}$ the middle point of $K_{\alpha}$ and $b_{\alpha}(k) = (m_{\alpha}, 9^{-k}) \in [0,1]^2$ for each $\alpha \in \Sigma_1(n)$ and each $k \geq 0$. We identify $[0,1] \times \{ 0 \}$ with $[0,1]$. Moreover let $\Lambda_{\alpha} = C(b_{\alpha}(n), K_{\alpha})$ and $\tilde{\Lambda}_{\alpha}(t) = \Lambda_{\alpha} \cap ([0,1] \times \{ t \})$ for each $\alpha \in \Sigma_1(n)$ and each $t \in [0,9^{-n}]$.

Next, we are going to define a closed subspace $Z_1 \subset [0,1]^2$ containing $\tilde{X}_1$ and a continuous map $F_1 : Z_1 \to Z_1$ which is an extension of $\tilde{f}|_{X_1}$. Let $I_0 = C(b_{\theta*1*0}(0), b_{\theta*0*0}(0)) \subset [0,1] \times \{ 1 \}$. In general, for each $n$ $\geq 1$ let

$$I_n = \bigcup_{\theta \in \Sigma_1^{(n)}} C(b_{\theta*1*0}(n), b_{\theta*0*0}(n)) \subset [0,1] \times \{ 9^{-n} \}.$$
$Z_1 = \tilde{X}_1 \cup \bigcup_{n \geq 0} (I_n \cup \bigcup_{\alpha \in \Sigma_1(n)} C(b_\alpha(n-1), b_\alpha(n)) \cup \Lambda_\alpha),$

where $b_0(-1) = b_0(0)$. We see that $Z_1$ is a closed subspace and an AR by [M, Theorem 5.5.7, p.237].

Let us define $F_1$ on $\Lambda_0$:

$F_1(\Lambda_0(t9^{-n+1} + 5(1-t)9^{-n})) = \{tb_{1^{(n-1)*0}}(n-1) + (1-t)b_{1^{(n)*0}}(n-1)\}$ and

$F_1(\Lambda_0(t5 \cdot 9^{-n} + (1-t)9^{-n})) = \{tb_{1^{(n)*0}}(n-1) + (1-t)b_{1^{(n)*0}}(n)\},$

where $n \geq 1$, $t \in [0,1]$ and $1^{(0)} * 0 = 0$. We see that $F_1(\Lambda_0)$ is the arc in $Z_1$ connected $b_0(0)$ and $K_1$.

Let us define an embedding $F_1$ on $C(b_\alpha(n-1), b_\alpha(n)) \cup \Lambda_\alpha$ ($\alpha \in \Sigma_1(n)$ and $n \geq 1$):

$F_1(tb_{\alpha}(n-1) + (1-t)b_{\alpha}(n)) = tb_{\mu(\alpha)}(n-1) + (1-t)b_{\mu(\alpha)}(n)$ and

$F_1(tb_{\alpha}(n) + (1-t)x) = tb_{\mu(\alpha)}(n) + (1-t)\tilde{f}(x),$

where $t \in [0,1]$ and $x \in K_\alpha$.

Let us define $F_1$ on $I_0$: $F_1(I_0) = \{b_0(0)\}$.

Let us define $F_1$ on $C(b_{\theta^{*}(0)*0}(n), b_{\theta^{*}(\delta)*0}(n))$ ($n \geq 1$, $\delta = -1, 1$ and $\theta \in \Sigma_1(n)$):

$F_1(tb_{\theta^{*}(0)*0}(n) + (1-t)b_{\theta^{*}(\delta)*0}(n)) = tb_{\mu(\theta^{*}(0)*0)}(n) + (1-t)b_{\mu(\theta^{*}(\delta)*0)}(n),$

where $t \in [0,1]$. 
Final, let us define $F_1$ on $A = C(b_{\theta*0}(n), b_{\theta*0}(n))$ ($n \geq 1, \delta = -1, 1$ and $\theta \in \Sigma_1(n)$ with $\theta_n = 0$). If $\theta_i = 0 (1 \leq i \leq n)$, define $F_1 (A) = \{ F_1 (b_{\theta*0}(n)) \}$. Let $\theta_i \neq 0$ for some $i$. Since $F_1$ is defined on $(A \cap \Lambda_{\theta*0} \cup \{ b_{\theta*0}(n) \})$, we can naturally extend $F_1$ on $A$ which is an embedding.

Denote $K_{\theta,m} = (K_{\theta} \times [0, 9^{-m}]) \cap Z_1$ for each $\theta \in \Sigma_1(n)$ and each $m \geq 0$. By the definition of $F_1$, we have

(5.1) $F_1 (Z_1 \cap [0, 1] \times [9^{-m} - 1, 9^m]) \subset Z_1 \cap [0, 1] \times [9^{-m} - 1, 9^m]$ for each $m \geq 0$ and

(5.2) $F_1 (K_{\theta,m}) \subset K_{\mu(\theta),m}$ for each $\theta \in \Sigma_1(n)$ and each $m \geq 1$.

**Lemma 5.1.** With the notation above, $h_{S_2} (F_1) = 0$.

Let $Z_{-1}$ be the closure of the component of $Z_1 \setminus \{ b_{0}(0) \}$ containing $K_{(-1)*0}$. We can construct a closed subspace $Z_2 \subset [0, 1]^2$ containing $\bar{X}_2$ and a homeomorphism $F_2 : Z_2 \rightarrow Z_{-1}$ which is an extension of $\tilde{f}|_{\tilde{X}_2}$ such that $Z_2 \cap Z_1 = \{ b_{1*0}(0) \}$. Define $Z = Z_1 \cup Z_2$, $F = F_1 \cup F_2 : Z \rightarrow Z$ and $G : Z \rightarrow Z$ by $G|_{Z_1} = F_1$ and $G(Z_2) = \{ b_{0}(0) \}$. As the proof of [BCL, Theorem 4.5], we obtain the following.

**Theorem 5.2.** With the notation above, $h_{S_2} (F) \geq \log 2$ and $0 = h_{S_2} (F \circ G) < \log 2 \leq h_{S_2} (G \circ F)$.

Since $Z$ is an AR, by Theorem 5.2, we can prove the following.

**Theorem 5.3.** For each $n$-dimensional compact topological manifold $M$ with $n > 1$, there exist two continuous maps $\tilde{F}, \tilde{G} : M \rightarrow M$ such that $0 = h_{S_2} (\tilde{F} \circ \tilde{G}) < \log 2 \leq h_{S_2} (\tilde{G} \circ \tilde{F})$ and $0 = h_{S_2} (\tilde{F}|_{\Omega(F)}) < \log 2 \leq h_{S_2} (\tilde{F})$.

6. SOME APPLICATIONS TO INVERSE LIMIT SPACES

By making use of Lemma 3.1, we can prove the following.

**Lemma 6.1.** Let $X$ be a graph and $f$ a continuous map from $X$ to itself. Then $f$ is chaotic in the sense of Li-Yorke if and only if $f|_{f^\omega(X)} : f^\omega(X) \rightarrow f^\omega(X)$ is chaotic in the sense of Li-Yorke.

As in proof of [C1, Theorem 2.2], by Lemma 6.1 we can show the following.

**Theorem 6.2.** Let $X$ be a graph and $f$ a continuous map from $X$ to itself. Then $f$ is chaotic in the sense of Li-Yorke if and only if $\sigma_f$ is chaotic in the sense of Li-Yorke.

**Remark 6.3.** Let $f$ be a continuous map from a compact metric space $X$ to itself. The proof of [C1, Theorem 2.2] implies that if $\sigma_f$ is chaotic in the sense of Li-Yorke, then $f|_{f^\omega(X)}$ is chaotic in the sense of Li-Yorke, thus, $f$ is chaotic in the sense of Li-Yorke.

**Theorem 6.4.** For each $n$-dimensional compact topological manifold $M$ with $n > 1$, there exists a continuous maps $f_M : M \rightarrow M$ such that $f_M$ is chaotic in the sense of Li-Yorke and that $\sigma_{f_M}$ is not chaotic in the sense of Li-Yorke.
Thorem 6.4 shows the possibility of the existence of a map which is not chaotic in the sense of Li-Yorke with positive topological entropy. But, recently, F. Blanchard, E. Glasner, S. Kolyada, and A. Maass [BGKM] prove that every continuous map with positive topological entropy is chaotic in the sense of Li-Yorke.

REFERENCES

[C] N. Chinen, Topological sequence entropy of monotone maps on one-dimensional continua, to submitted.