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Cardinal invariants associated with some combinatorial statements

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Abstract

T. Bartoszyński [1] characterized the uniformity non($\mathcal{M}$) of the meager ideal on the real line as the smallest size of a family $X \subseteq \omega^\omega$ such that $\forall y \in \omega^\omega \exists x \in X \exists n < \omega \ y(n) = x(n)$. By replacing $\omega^\omega$ by certain restricted subsets, we can get weaker combinatorial statements and define cardinal invariants. In this talk, we study these cardinal invariants.

0 Introduction

We use standard notion and notations in set theory (see e.g. [3]). Set

$$\mathcal{F} = \{ f \in (\omega \setminus \{0\})^\omega | f \text{ is non-decreasing and } \lim_{n<\omega} f(n) = \omega \}.$$ 

For each $f \in \mathcal{F}$, define the cardinal $\theta_f$ by

$$\theta_f = \min \{ |X| \ | X \subseteq \prod_{n<\omega} f(n) \text{ and } \forall y \in \prod_{n<\omega} f(n) \exists n < \omega \ y(n) = x(n) \}.$$ 

By the Bartoszyński's characterization of non($\mathcal{M}$), it holds that $\theta_f \leq \text{non}(\mathcal{M})$ for all $f \in \mathcal{F}$. Also, it is easy to see that $\theta_{f_1} \leq \theta_{f_2}$ if $f_1$, $f_2 \in \mathcal{F}$ and $f_1 \preceq f_2$. In the next section, we show that, in a certain generic model which is obtained by adjoining random reals, $\theta_{f_1} < \theta_{f_2}$ holds for some $f_1$, $f_2 \in \mathcal{F}$. Put $\theta = \min \{ \theta_f | f \in \mathcal{F} \}$. Let me introduce another cardinal invariant $\theta^*$ which is associated with a weaker combinatorial statement. For this, we need some definitions. Set

$$\mathcal{H} = \{ h \in \omega^\omega | h \text{ is strictly increasing and } \lim_{n<\omega} h(n+1) - h(n) = \omega \}.$$ 

For each $h \in \mathcal{H}$ and $n < \omega$, $a^h_n$ denotes the interval $[h(n), h(n+1))$ of $\omega$. Define $\theta^*$ by

$$\theta^* = \min \{ |W| \ | W \subseteq 2^\omega \times \mathcal{H} \text{ and } \forall y \in 2^\omega \exists (x,h) \in W \exists n < \omega \ y \cap a^h_n = x \cap a^h_n \}.$$ 

It is easy to check that $\omega_1 \leq \theta^* \leq \theta$. Furthermore, we have:

Theorem 0.1 Assume that $\text{cof}([d]^\omega, \subseteq) = d$. Then, it holds that $\theta^* \leq d$. 
Proof. Take a sufficiently large regular cardinal $\rho$. By using the assumption, take an elementary substructure $M$ of $H(\rho)$ such that

$$M \cap \omega^\omega$$

is a dominating family and $|M| = d$ and $M \cap [M]^\omega$ is $\subset$-cofinal in $[M]^\omega$.

Since $M \cap \omega^\omega$ is a dominating family, it holds that

$$\forall h \in \mathcal{H} \exists h' \in M \cap \mathcal{H} \forall^\infty n < \omega \exists m < \omega \ a^n_m \subset a^n_h.$$  

We show that $W = M \cap (2^\omega \times \mathcal{H})$ satisfy the definition of $\theta^*$. To get a contradiction, assume that there is $y \in 2^\omega$ such that

$$\forall^\infty n < \omega \ y \upharpoonright a^n_m \neq a^n_h, \text{ for all } (x, h) \in W.$$  

Put $X = 2^\omega \cap M$. The next claim is easily verified by using (*).

Claim 0.2  \hspace{1cm} \forall x \in X \exists k < \omega \forall^\infty m < \omega \ x \upharpoonright [m, m + k) \neq x \upharpoonright [m, m + k). \quad \Box$

By Claim 0.2, define $\varphi : X \to \omega$ by

$$\varphi(x) = \text{ the largest } k < \omega \text{ such that } \exists^\infty m < \omega \ x \upharpoonright [m, m + k) \subset y.$$  

It is easy to check that $\sup \{ \varphi(x) | x \in X \} = \omega$. By this, since $[M]^\omega \cap M$ is $\subset$-cofinal in $[M]^\omega$, we can take $A = \{ a_i | i < \omega \} \in M$ such that $\sup \{ \varphi(a_i) | i < \omega \} = \omega$. Take $\psi : \omega \times \omega \to \omega$ such that, for each $(i, n) \in \omega \times \omega$,

$$i + n + \varphi(a_i) \leq \psi(i, n) \text{ and } \exists m \in [n, \psi(i, n) - \varphi(a_i)) \ a_i \upharpoonright [m, m + \varphi(a_i)) \subset y.$$  

Without loss of generality, we may assume that $\psi \in M$. Define $\langle k_i | i < \omega \rangle \in M$ by

$$\begin{cases} 
  k_0 = 0 \\
  k_{i+1} = \psi(i, k_i), \text{ for } i < \omega 
\end{cases}$$

and set $x = \bigcup_{i < \omega} a_i \upharpoonright [k_i, k_{i+1}) \in X$. Then, it holds that

$$\forall k < \omega \exists m < \omega \ x \upharpoonright [m, m + k) \subset y.$$  

But this contradicts Claim 0.2 \hspace{1cm} \Box

Let $C_\omega$ be the notion of forcing which adds a Cohen real. Then, it holds that

$$\vdash_{C_\omega} \forall y \in 2^\omega \exists x \in 2^\omega \forall n < \omega \ x \upharpoonright [n^2, n^2 + n) = y \upharpoonright [n^2, n^2 + n).$$

So, $\theta^* < d$ holds in a certain Cohen generic model.

It is known that the assumption $\text{cof}([d]^\omega, \subset) = d$ is followed from the non-existence of $0^\#$. So, it seems to prove Theorem 0.1 without this assumption. But I failed to find a proof.

Question 0.1  \hspace{1cm} Is $\theta^* \leq d$ proved in ZFC?

In sections 2, 3, 4, we show that the cardinals $\omega_1$, $\theta$, $\theta^*$ can be separated for certain generic models.
1 Generic extensions by random reals

For each infinite cardinal $\kappa$, we denote by $\mathbf{B}(\kappa)$ the measure algebra which adds a random function from $\kappa$ to 2 and by $\mu_\kappa : \mathbf{B}(\kappa) \to [0,1]$ the measure function. In this section, we prove the following theorem.

**Theorem 1.1** Assume CH. Let $\kappa > \omega_1$ be a regular cardinal. Then, there are $f_1, f_2 \in \mathcal{F}$ such that
\[ \langle (\mu_\kappa)_\kappa \rangle \theta_{f_1} = \omega_1 \text{ and } \theta_{f_2} = \kappa. \]

Set $f_2 = \langle 2^n \mid n < \omega \rangle \in \mathcal{F}$. The next well-known lemma guarantees that this $f_2$ is as required in Theorem 1.1.

**Lemma 1.2** (Forklore) $\exists y \in \prod_{n<\omega} f_2(n) \forall x \in \prod_{n<\omega} \mathcal{F} \forall^\infty n < \omega x(n) \neq y(n)$.

**Proof** Define $k_n < \omega$ (for $n < \omega$) by

\[ k_0 = 0 \text{ and } k_{n+1} = k_n + n \text{ for } n < \omega. \]

For each $n < \omega$, put $I_n = [k_n, k_{n+1})$ and take a bijections from $I_n^2$ to $f_2(n)$. Using these bijections, we identify $\prod_{n<\omega} f_2(n)$ with $\prod_{n<\omega}^{I_n^2} 2$. Let $\dot{g}$ be the canonical $\mathbf{B}(\omega)$-name of generic real. Define $\dot{y}$ by

\[ \langle \dot{y} \mid n < \omega \rangle = (\dot{g} \upharpoonright I_n) \]

It holds that, for each $n < \omega$ and $s : I_n \to 2$,
\[ \mu_\omega (\| s = \dot{g} \upharpoonright I_n \|) = 2^{|I_n|} = 2^{-n}. \]

So, $\mu_\omega (\| \exists^\infty n < \omega x \mid I_n = \dot{y}(n) \|) = 0$ for all $x \in 2^\omega$. This implies that
\[ \langle \forall^\infty n < \omega x(n) \neq \dot{y}(n) \rangle, \text{ for all } x \in \prod_{n<\omega} I_n^2. \]

**Lemma 1.3** Let $0 < K, M < \omega$. Suppose that $\{ b_i^m \mid i < K \text{ and } m < M \} \subset \mathbf{B}(\omega)$ and $b \in \mathbf{B}(\omega)$ satisfy

\[ b = \sum_{i<K} b_i^m, \text{ for all } m < M. \]

Then, there is a function $\varphi : M \to K$ such that
\[ \mu_\omega (\sum_{m<M} b_{\varphi(m)}^m) \geq \mu_\omega (b) - \left( \frac{K-1}{K} \right)^M \mu_\omega (b). \]

**Proof** By induction on $M \in [1, \omega)$. The case $M = 1$ is clear. Let $M = M_0 + 1 > 1$. Using the induction hypothesis, take $\varphi_0 : M_0 \to K$ such that
\[
\mu_\omega\left(\sum_{m<M_0} b_{\varphi_0(m)}^m \right) \geq \mu_\omega(b) - \left(\frac{K-1}{K}\right)^{M_0} \mu_\omega(b).
\]

Put \( c = \sum_{m<M_0} b_{\varphi_0(m)}^m \). Since \( b-c = \sum_{i<K} (b_i^{M_0} - c) \), there exists \( j < K \) such that \( \mu_\omega(b_j^{M_0}) \geq \frac{1}{K} \mu_\omega(b-c) \). Then, \( \varphi = \varphi_0(j) \) is as required.

For each \( n < \omega \), let
\[
M_n = \min\{ M < \omega | \left(\frac{n}{n+1}\right)^M < 2^{-n}\}.
\]
Define \( f_1 \in F \) by
\[
|\{ k < \omega \mid f_1(k) = n+1 \}| = M_n \text{, for all } n < \omega.
\]
The next lemma implies that \( f_1 \) satisfies the condition in Theorem 1.1.

**Lemma 1.4** \( \|	ext{B}(\omega) \forall y \in \prod_{k<\omega} f_1(k) \exists x \in \prod_{k<\omega} f_1(k) \cap \mathcal{V} \exists^\infty k < \omega x(k) = y(k) \).

**Proof** For each \( n < \omega \), put \( J_n = \{ k < \omega \mid f_1(k) = n+1 \} \). To show this lemma, let \( \dot{y} \) be a \( \text{B}(\omega) \)-name such that \( \Vdash \dot{y} \in \prod_{k<\omega} f_1(k) \). For each \( n < \omega \), using Lemma 1.3, take \( s_n \in \prod_{k \in J_n} f_1(k) \) such that
\[
\mu_\omega\left(\sum_{k \in J_n} |s_n(k) = \dot{y}(k)|\right) \geq 1 - \left(\frac{n}{n+1}\right)^{M_n}.
\]
Put \( x = \bigcup_{n<\omega} s_n \). It is easy to check that
\[
\mu_\omega(\| \forall^\infty n < \omega \exists k \in J_n x(k) = \dot{y}(k) \|) = 0.
\]
So, it holds that \( \Vdash \exists^\infty k < \omega x(k) = \dot{y}(k) \).

\[\square\]

**2 A forcing notion with the ccc which lifts up \( \theta^* \)**

Define the forcing notion \((Q, \leq)\) by
\[
Q \subset 2^{<\omega} \times [2^\omega \times \mathcal{H}]^{<\omega}
\]
and, for any \((s, u) \in 2^{<\omega} \times [2^\omega \times \mathcal{H}]^{<\omega}\),
\[
(s, u) \in Q
\]
if and only if, for all \((x, h) \in u\) and all \( k < \omega \),
if \( a_k^h \setminus \text{dom}(s) \neq \phi \) then \( |a_k^h \setminus \text{dom}(s)| \geq |u| \) or \( \exists i \in a_k^h \cap \text{dom}(s) \) \( x(i) \neq s(i) \),
and, for any \((s, u), (s', u') \in Q\),
\[
(s', u') \leq (s, u)
\]
if and only if
\( s' \supset s \) and \( u' \supset u \) and, for all \((x, h) \in u\) and all \( k < \omega \), if \( a_k^h \cap (\text{dom}(s') \setminus \text{dom}(s)) \neq \phi \) then \( |a_k^h \setminus \text{dom}(s')| \geq |u'| \) or \( \exists i \in a_k^h \cap \text{dom}(s') \) \( x(i) \neq s'(i) \).
We show that a finite support iteration by the above forcing notion lifts up the value $\theta^*$. For this, we need several lemmas.

**Lemma 2.1** Let $n < \omega$. Then, for every $(s, u) \in Q$, there is $s' \in 2^{<\omega}$ such that $(s', u) \in Q$ and $(s', u) \leq (s, u)$ and $n \in \text{dom}(s)$.

**Proof** For each $j < \omega$, define $\varphi_j : \mathcal{H} \rightarrow \omega$ by

\[
\varphi_j(h) = \text{the unique } k < \omega \text{ such that } j \in a_k^h.
\]

For each $t \in 2^{<\omega}$, define $\psi_t : 2^{\omega} \times \mathcal{H} \rightarrow \omega$ by

\[
\psi_t(x, h) = \begin{cases} 
|a_{\varphi_{\text{dom}(t)}(h)}^h \setminus \text{dom}(t)|, & \text{if } t \upharpoonright a_{\varphi_{\text{dom}(t)}(h)}^h \subset x, \\
|a_{\varphi_{\text{dom}(t)}(h)+1}^h|, & \text{otherwise.}
\end{cases}
\]

To show this lemma, let $n < \omega$ and $(s, u) \in Q$. Put $m = \text{dom}(s)$. Take $M < \omega$ such that

\[
n, m < M \text{ and } |a_{\varphi_M(h)}^h \setminus M| \geq |u|, \text{ for all } (x, h) \in u.
\]

By induction on $j \in [m, M]$, take $s_j : j \rightarrow 2$ as follows:

Put $s_m = s$. Suppose that $j \in [m, M]$ and $s_j$ has been defined. Let $l_j$ be the smallest element of $\{ \psi_{s_j}(x, h) \mid (x, h) \in u \}$. Take $(x_j, h_j) \in u$ such that $\psi_{s_j}(x_j, h_j) = l_j$. Set $s_{j+1} = s_j \langle 1 - x_j(j) \rangle$.

**Claim 2.2** $|\{(x, h) \in u \mid \psi_{s_j}(x, h) < l\}| < l$, for all $0 < l < \omega$ and $j \in [m, M]$.

By induction on $j \in [m, M]$. The case $j = m$ is followed from the fact $(s, u) \in Q$. The case $j = j_0 + 1 > m$ is followed from the fact $\psi_{s_j}(x_{j_0}, h_{j_0}) \geq |u|$. △

By Claim 2.2, it holds that $l_j > 0$, for all $j \in [m, M]$. So, it holds that $(s_M, u) \in Q$ and $(s_M, u) \leq (s, u)$.

**Lemma 2.3** For each $(x, h) \in 2^{\omega} \times \mathcal{H}$,

\[
\{(s, u) \in Q \mid (x, h) \in u\}
\]

is dense in $Q$.

**Proof** Let $(x, h) \in 2^{\omega} \times \mathcal{H}$ and $(s, u) \in u$. Take $M < \omega$ such that

1. $|s| \leq M$,
2. if $a_k^{h'} \setminus M \neq \phi$ then $|a_k^{h'} \setminus M| > |u|$, for all $k < \omega$ and $(x', h') \in u$.
3. if $a_k^h \setminus M \neq \phi$ then $|a_k^h \setminus M| > |u|$, for all $k < \omega$.

Using Lemma 2.1, take $(t, u) \leq (s, u)$ such that $\text{dom}(t) = M$. Then, it holds that $(t, u \cup \{(x, h)\}) \in Q$ and $(t, u \cup \{(x, h)\}) \leq (s, u)$.

**Lemma 2.4** $Q$ satisfies the countable chain condition.
Proof Let $W$ be an uncountable subset of $Q$. Using Lemma 2.1, replace $W$ by certain stronger conditions if necessary, we may assume that, for all $(s, u) \in W$,

for all $(x, h) \in u$ and $k < \omega$, if $a_k^h \setminus k \neq \phi$ then $|a_k^h \setminus k| \geq 2|u|$.

Take $s_0 \in 2^{<\omega}$ and $m < \omega$ such that $W' = \{(s, u) \in W \mid s = s_0 \text{ and } |u| = m\}$ is uncountable. Then, every elements in $W'$ are compatible. \hfill \Box

Let $\dot{G}$ be the canonical generic $Q$-name. Define $\dot{g}$ by

$\Vdash_{Q} \dot{g} = \bigcup \{ s \mid (s, u) \in \dot{G}, \text{ for some } u \}.$

Lemma 2.5  \( \Vdash_{Q} \dot{g} \in 2^\omega \text{ and } \forall x \in 2^\omega \cap V \forall h \in H \cap V \forall n < \omega \exists m < \omega \dot{g} \upharpoonright a_n^h \neq x \upharpoonright a_n^h. \)

Proof This is directly followed from Lemmas 2.1 and 2.3. \hfill \Box

Let $\kappa$ be a regular uncountable cardinal and $P$ the $\kappa$-stage finite support iteration by the above forcing $Q$. Then, by the above arguments, it holds that $\theta^* = \kappa$ in the generic model $\mathbf{V}^P$. Since $P$ is finite support, it adds cofinally many Cohen reals. So, in $\mathbf{V}^P$, the covering number $\text{cov}(\mathcal{M})$ of the meager ideal on the real line lifts up to $\kappa$. Furthermore, the next lemma shows that the unbounding number $\mathfrak{b}$ of $\omega^\omega$ lifts up to $\kappa$, too.

Lemma 2.6 There is a $Q$-name $\dot{d}$ such that

$\Vdash_{Q} \dot{d} \in \omega^\omega \text{ dominates } \omega^\omega \cap V.$

Proof For each set $X$, denote by $0_X$ the constantly zero function from $X$ to $2$.

Claim 2.7 For any $n < \omega$,

$\{ (s, u) \in Q \mid \exists m < \omega ( 0_{[m, m+n]} \subset s ) \} \text{ is dense in } Q.$

\( \vdash \) Let $n < \omega$ and $(s, u) \in Q$. Take $(t, u) \leq (s, u)$ such that, for all $(x, h) \in u$ and $k < \omega$,

if $a_k^h \setminus \text{dom}(t) \neq \phi$ then $|a_k^h \setminus \text{dom}(t)| \geq |u| + n.$

Define $t' : |t| + n \rightarrow 2$ by $t \subset t'$ and $t'(|t| + j) = 0$, for $j < n$. It is easy to check that $(t', u) \in Q$ and $(t', u) \leq (s, u)$. \hfill \triangle

By Claim 2.7, it holds that

$\Vdash_{Q} \forall n < \omega \exists m < \omega \dot{g} \upharpoonright [m, m+n] = 0_{[m, m+n]}.$

So, in $\mathbf{V}^Q$, define $\dot{d} \in \omega^\omega$ by

$\dot{d}(n) = \text{the smallest } m < \omega \text{ such that } n \leq m \text{ and } \dot{g} \upharpoonright [m, m+2n] = 0_{[m, m+2n]}.$

To show $\dot{d}$ is a required one, let $f \in \omega^\omega$ and $(s, u) \in Q$. Without loss of generality, we may assume that $f$ is strictly increasing. Take $h \in H$ such that
\[|a_k^h| \leq |a_{k+1}^h|, \text{ for all } k < \omega \text{ and } |\{ k < \omega \mid |a_k^h| = n \}| \geq f(n) + 1, \text{ for all } n < \omega.\]

By Lemma 2.3, take \((t, v) \leq (s, u)\) such that \((0_\omega, h) \in v\). Let \(k_0\) be the smallest \(k < \omega\) such that \(|t| \geq h(k)\) and set \(n_0 = |a_{k_0}^h| + |t|\). The next claim completes the proof of the lemma.

**Claim 2.8** \((t, v) \Vdash_Q \forall n > n_0 \ f(n) < \check{d}(n)\).

\(\blacklozenge\) To get a contradiction, assume that there are \((t', v') \leq (t, v)\) and \(n > n_0\) such that \((t', v') \Vdash_Q \check{d}(n) \leq f(n)\). Replace \((t', v')\) by a stronger condition if necessary, we may assume that \((t', v')\) decides the value of \(\check{d}(n)\). Let \(m < \omega\) be such that \((t', v') \Vdash_Q \check{d}(n) = m\). Without loss of generality, we may assume that \(m + 2n \subseteq \text{dom}(t')\). Let \(k\) be the unique \(k < \omega\) such that \(m \in a_k^h\). By the choice of \(h\), it holds that \(|a_k^h|, |a_{k+1}^h| \leq n\).

\[a_{k+1}^h \subseteq [m, m+2n).\] Since \((t', v') \Vdash \check{d} \upharpoonright [m, m+2n) = 0_{[m, m+2n]}\), it holds that \(t' \upharpoonright a_{k+1}^h = 0_{a_{k+1}^h}\). This contradicts the facts that \((t', v') \leq (t, v)\) and \((0_\omega, h) \in v\) and \(\text{dom}(t) \cap [m, m+2n) = \emptyset\).

\(\square\)

In section 4, we give a generic model which holds \(\theta^* = \omega_2\) and \(\text{cov}(\mathcal{M}) = \omega_1\). But I do not know whether there is a model which satisfies \(b < \theta^*\).

**Question 2.1** Is \(b < \theta^*\) consistent with ZFC?

### 3 A forcing notion which lifts up \(\theta\)

In this section, we give a forcing notion which gives a generic model of \(\theta^* = \omega_1\) and \(\theta = \omega_2\). The forcing notion which we give here is constructed by the \(\omega_2\)-stage countable support iteration. We begin with the definition of a forcing notion \(\text{BT}_f\) for \(f \in \mathcal{F}\) which will be used each stage in the iteration.

Let \(f \in \mathcal{F}\). For each \(n < \omega\), denote \(\prod_{m < n} f(m)\) by \(S_n^f\). Put \(S^f = \bigcup_{n<\omega} S_n^f\). Note that \((S^f, \subseteq)\) is a tree. Define the forcing notion \((\text{BT}_f, \leq)\) by

\[q \in \text{BT}_f\]

if and only if

1. \(q\) is a subtree of \(S^f\).
2. there is a function \(f' \in \mathcal{F}\) such that \(|\text{succ}_q(s)| \geq f'(|s|)\) for every \(s \in q\).

\(q' \leq q\) if and only if \(q' \subseteq q\).

For each \(q \in \text{BT}_f\), define \(\pi_q \in \omega^\omega\) by

\[\pi_q(n) = \max \{ k < \omega \mid \forall n' \geq n \forall s \in q \cap S_{n'}^f \ |\text{succ}_q(s)| \geq k \}.\]
Note that $\pi_q \in \mathcal{F}$ for all $q \in \mathbf{B}\mathbf{T}_f$. For each $k < \omega$, define the ordering $\leq_k$ on $\mathbf{B}\mathbf{T}_f$ by

$q' \leq_k q$ if and only if $q' \leq q$ and $\pi_q \downarrow m_k = \pi_{q'} \downarrow m_k$,

where $m_k$ denotes the smallest $m < \omega$ such that $\pi_q(m) > k$.

In [2], Bartoszyński, Judah and Shelah have used similar but more complicated forcing notions $Q_{f,g}$. The proof of the next lemma is similar to, but quite easier than the proof of Claim 2.6 in [2].

**Lemma 3.1** Let $\dot{e}$ be a $\mathbf{B}\mathbf{T}_f$-name such that $\Vdash \dot{e} \in \mathbf{V}$. Then, for each $k < \omega$ and $q \in \mathbf{B}\mathbf{T}_f$, there are $q' \leq_k q$ and a finite set $E$ such that $q' \Vdash \dot{e} \in E$.

**Proof** Let $\dot{e}$, $k < \omega$, $q \in \mathbf{B}\mathbf{T}_f$ be as in the lemma. For each $s \in q$, denote by $q[s]$ the condition $\{ t \in q \mid s \subset t \text{ or } t \subset s \}$. Take $M < \omega$ such that $\pi_q(M) \geq 2k$. Set

$T = \{ s \in q \mid |s| \geq M \text{ and } \exists q' \leq_k q[s] \exists E (E \text{ is finite and } q' \Vdash \dot{e} \in E) \}$.

Note that, whenever $s \in q \setminus T$ and $|s| \geq M$, $|\text{succ}_q(s) \cap T| < k$.

**Claim 3.2** $q \cap S_M^f \subset T$.

\[ \vdash \] To get a contradiction, assume that $s \in q \cap S_M^f \setminus T$. Let $U = \{ t \in q \setminus T \mid s \subset t \}$. Then, it holds that

\[ \forall t \in U \ ( |\{ u \in U \mid t \subset u \text{ and } |u| = |t| + 1 \}| > \pi_q(|u| - k) \).

This implies that $r = \{ s \upharpoonright j \mid j < |s| \} \cup U \in \mathbf{B}\mathbf{T}_f$ and $r \leq_k q[s]$. Take $r' \leq r$ such that $r'$ decides $\dot{e}$. Take $t \in r'$ such that $\pi_{r'}(|t|) \geq k$. Since $r'[t] \leq_k q[t]$, we have that $t \in T$. This contradicts that $U \cap T = \phi$. \[ \triangle \]

By Claim 3.2, for each $s \in q \cap S_M^f$, take $q_s \leq_k q[s]$ and a finite set $E_s$ such that $q_s \Vdash \dot{e} \in E_s$. Then $q' = \bigcup_{s \in q \cap S_M^f} q_s$ and $E = \bigcup_{s \in q \cap S_M^f} E_s$ satisfy this lemma. \[ \square \]

**Corollary 3.3** ($\mathbf{B}\mathbf{T}_f, (\leq_k)_{k<\omega}$) satisfies Axiom A and $\mathbf{B}\mathbf{T}_f$ is $\omega^\omega$-bounding. \[ \square \]

Let $\dot{G}$ be the canonical generic $\mathbf{B}\mathbf{T}_f$-name. Define $\mathbf{B}\mathbf{T}_f$-name $\dot{g}$ by

\[ \Vdash \dot{g} = \bigcup (\bigcap \dot{G}) \in \prod_{n<\omega} f(n). \]

Then, it is easy to check that

\[ \Vdash \forall x \in \prod_{n<\omega} f(n) \cap \mathbf{V} \forall^{\infty} n < \omega \ \dot{g}(n) \neq x(n). \]

Now we can describe how to construct a model which satisfies $\theta = \omega_2$ and $\theta^* = \omega_1$. Start with a ground model with CH. Let $\{ f_\alpha \mid \alpha < \omega_2 \} \subset \mathcal{F}$ be such that

$\{ \alpha < \omega_2 \mid f_\alpha = f \}$ is cofinal in $\omega_2$ for each $f \in \mathcal{F}$. 
Define the \( \omega_2 \)-stage countable support iteration \( P_\alpha \) (for \( \alpha \leq \omega_2 \)), \( \dot{Q}_\alpha \) (for \( \alpha < \omega_2 \)) by
\[
\forces \dot{Q}_\alpha = \text{BT}_{f_\alpha}.
\]

Let \( P = P_{\omega_2} \). Then, by the above arguments, it holds that, in \( V^P \), \( \theta = \omega_2 \) and \( d = \omega_1 \).

Since \( \text{cof}(\omega_1^\omega, \subset) = \omega_1 \) does always hold, it holds that, in \( V^P \), \( \theta^* \leq d = \omega_1 \).

## 4 A generic model of \( \theta = \omega_2 \) and \( \text{cov}(M) = \omega_1 \)

In the previous section, we show that \( \text{BT}_f \) does not lift up \( \theta^* \). But, if we first add a dominating real then we get a certain function \( f \in F \) such that \( \text{BT}_f \) lifts up \( \theta^* \). In this section, we show that \( \theta^* \) can be separated from \( \text{cov}(M) \) by using it.

**Lemma 4.1** Let \( V, W \) be transitive models of ZFC such that \( V \subset W \). Assume that \( d \in W \cap \omega^\omega \) dominates \( V \cap \omega^\omega \). In \( W \), define \( h \in H \) by
\[
|a_k^h| \leq |a_{k+1}^h|, \text{ for all } k < \omega \text{ and } \{|k < \omega \mid |a_k^h| = n\}| = d(n) + 1, \text{ for all } n < \omega.
\]

Then, it holds that \( \forall^\infty m < \omega \exists k < \omega \ a_k^h \subset a_{k'}^h \text{ for all } h' \in V \cap H \).

**Proof** Let \( h' \in V \cap H \). In \( V \), define \( f_0, f_1 \in \omega^\omega \) by
\[
f_0(n) = \text{the smallest } m < \omega \text{ such that } \forall m' \geq m \ |a_{m'}^{h'}| \geq 2n, \text{ and } \]
\[
f_1(n) = \max a_{f_0(n+1)}^{h'}.
\]

Since \( d \) dominates \( f_0, f_1 \), there is \( n_0 < \omega \) such that \( \forall n \geq n_0 \ f_0(n), f_1(n) < d(n) \). Put \( k_0 = f_0(n_0) \). To show that \( \forall k \geq k_0 \exists j < \omega \ a_j^h \subset a_k^h \), let \( k \geq k_0 \). Take \( n \) such that \( f_0(n) \leq k < f_0(n+1) \). Then, it holds that \( |a_k^h| \geq 2n \) and \( \max a_k^h < \max a_{f_0(n+1)}^{h'} = f_1(n) \leq d(n) \). Since \( [0,d(n)) \) is covered by \( \{a_j^h \mid j < d(n)\} \) and \( |a_j^h| \leq n \) for all \( j < d(n) \), there is \( j < d(n) \) such that \( a_j^h \subset a_k^h \).

**Lemma 4.2** Let \( V, W, d \) and \( h \) be as in Lemma 4.1. Working in \( W \). Define \( f \in F \) by
\[
f(k) = 2k|a_k^h|, \text{ for all } k < \omega.
\]

Then, there is a \( \text{BT}_f \)-name \( \dot{y} \) such that
\[
\forces \dot{y} \in 2^\omega \text{ and } \forall^\infty k < \omega \ \dot{y} \upharpoonright a_k^h \neq x \upharpoonright a_k^h, \text{ for all } x \in 2^\omega \cap V \text{ and } h' \in H \cap V.
\]

**Proof** Working in \( W \). Considering bijections from \( f(k) \) to \( ^k2 \) for \( k < \omega \), we may identify \( \prod_{k<\omega} f(k) \) with \( \prod_{k<\omega} ^k2 \). Let \( \dot{G} \) be the canonical generic \( \text{BT}_f \)-name. Define \( \text{BT}_f \)-names \( \dot{g} \) and \( \dot{y} \) by
\[
\forces \dot{g} = \bigcup_{k<\omega} \dot{G} \text{ and } \dot{y} = \bigcup_{k<\omega} \dot{g}(k).
\]
Note that $\vdash j \in \prod_{k<\omega} a_k^h 2$ and $\dot{y} \in 2^\omega$. It is easy to check that

$$\vdash \forall x \in 2^\omega \cap W \forall^\infty k < \omega \ 2^\omega \dot{y} \upharpoonright a_k^h \neq x \upharpoonright a_k^h.$$  

To show $\dot{y}$ is as required, let $x \in V \cap 2^\omega$ and $h' \in V \cap H$. Since it holds that $x \in W$ and $\forall^\infty m < \omega \exists k < \omega a_k^h \subset a_m^{h'}$, we have that

$$\vdash \forall^\infty m < \omega \ 2^\omega \dot{y} \upharpoonright a_m^{h'} \neq x \upharpoonright a_m^{h'}.$$

\[\square\]

**Corollary 4.3** Assume that CH holds. There are a forcing notion $R$ and $R$-name $\dot{y}$ such that

1. $R$ is proper and does not add a Cohen real and $|R| = \omega_1$.
2. $\models_R \dot{y} \in 2^\omega$ and $\forall x \in 2^\omega \cap V \forall h \in H \cap V \forall^\infty k < \omega \dot{y} \upharpoonright a_k^h \neq x \upharpoonright a_k^h$.

\[\square\]

Using Corollary 4.3, we can construct a generic model which satisfies $\text{cov}(\mathcal{M}) = \omega_1$ and $\theta^* = \omega_2$. Start with a ground model with CH. Take an $\omega_2$-stage countable support iteration by the forcing notion as in Corollary 4.3. Since the iteration does not add a Cohen real, $\text{cov}(\mathcal{M})$ remains $\omega_1$. On the other hand, since functions $\dot{y} \in 2^\omega$ which satisfy (2) in the corollary is added cofinally, $\theta^*$ must be lifted up.

**References**

