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Cardinal invariants associated with some combinatorial statements

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Abstract

T. Bartoszyński [1] characterized the uniformity non$(\mathcal{M})$ of the meager ideal on the real line as the smallest size of a family $X \subset \omega^\omega$ such that $\forall y \in \omega^\omega \exists x \in X \exists n < \omega y(n) = x(n)$. By replacing $\omega^\omega$ by certain restricted subsets, we can get weaker combinatorial statements and define cardinal invariants. In this talk, we study these cardinal invariants.

0 Introduction

We use standard notion and notations in set theory (see e.g. [3]). Set

$$\mathcal{F} = \{ f \in (\omega \setminus \{0\})^\omega | f \text{ is non-decreasing and } \lim_{n<\omega} f(n) = \omega \}. $$

For each $f \in \mathcal{F}$, define the cardinal $\theta_f$ by

$$\theta_f = \min \{ |X| | X \subset \prod_{n<\omega} f(n) \text{ and } \forall y \in \prod_{n<\omega} f(n) \exists n < \omega y(n) = x(n) \}. $$

By the Bartoszyński's characterization of non$(\mathcal{M})$, it holds that $\theta_f \leq \text{non}(\mathcal{M})$ for all $f \in \mathcal{F}$. Also, it is easy to see that $\theta_{f_1} \leq \theta_{f_2}$ if $f_1, f_2 \in \mathcal{F}$ and $f_1 \leq^* f_2$. In the next section, we show that, in a certain generic model which is obtained by adjoining random reals, $\theta_{f_1} < \theta_{f_2}$ holds for some $f_1, f_2 \in \mathcal{F}$. Put $\theta = \min \{ \theta_f | f \in \mathcal{F} \}$. Let me introduce another cardinal invariant $\theta^*$ which is associated with a weaker combinatorial statement. For this, we need some definitions. Set

$$\mathcal{H} = \{ h \in \omega^\omega | h \text{ is strictly increasing and } \lim_{n<\omega} h(n+1) - h(n) = \omega \}. $$

For each $h \in \mathcal{H}$ and $n < \omega$, $a^h_n$ denotes the interval $[h(n), h(n+1))$ of $\omega$. Define $\theta^*$ by

$$\theta^* = \min \{ |W| | W \subset 2^\omega \times \mathcal{H} \text{ and } \forall y \in 2^\omega \exists (x, h) \in W \exists n < \omega y | a^h_n = x | a^h_n \}. $$

It is easy to check that $\omega_1 \leq \theta^* \leq \theta$. Furthermore, we have:

**Theorem 0.1** Assume that $\text{cof}([d]^\omega, \subset) = d$. Then, it holds that $\theta^* \leq d$. 
Proof. Take a sufficiently large regular cardinal $\rho$. By using the assumption, take an elementary substructure $M$ of $H(\rho)$ such that

$M \cap \omega^\omega$ is a dominating family and $|M| = d$ and $M \cap [M]^\omega$ is $\subseteq$-cofinal in $[M]^\omega$.

Since $M \cap \omega^\omega$ is a dominating family, it holds that

(*) $\forall h \in H \exists h' \in M \cap H \forall \infty n < \omega \exists m < \omega \ a^h_m \subset a^h_n$.

We show that $W = M \cap (2^\omega \times H)$ satisfy the definition of $\theta^*$. To get a contradiction, assume that there is $y \in 2^\omega$ such that

$\forall \infty n < \omega \ y \upharpoonright a^h_n \neq x \upharpoonright a^h_n$, for all $(x, h) \in W$.

Put $X = 2^\omega \cap M$. The next claim is easily verified by using (*).

Claim 0.2 $\forall x \in X \exists k < \omega \forall \infty m < \omega \ y \upharpoonright [m, m + k) \neq x \upharpoonright [m, m + k)$. △

By Claim 0.2, define $\varphi : X \to \omega$ by

$\varphi(x) =$ the largest $k < \omega$ such that $\exists \infty m < \omega \ x \upharpoonright [m, m + k) \subset y$.

It is easy to check that sup\{ $\varphi(x) \mid x \in X$ \} = $\omega$. By this, since $[M]^\omega \cap M$ is $\subseteq$-cofinal in $[M]^\omega$, we can take $A = \{ a_i \mid i < \omega \} \in M$ such that sup\{ $\varphi(a_i) \mid i < \omega$ \} = $\omega$. Take $\psi : \omega \times \omega \to \omega$ such that, for each $(i, n) \in \omega \times \omega$,

$i + n + \varphi(a_i) \leq \psi(i, n)$ and $\exists m \in [n, \psi(i, n) - \varphi(a_i)) \ a_i \upharpoonright [m, m + \varphi(a_i)) \subset y$.

Without loss of generality, we may assume that $\psi \in M$. Define $\{ k_i \mid i < \omega \} \in M$ by

\[
\begin{cases}
  k_0 &= 0 \\
  k_{i+1} &= \psi(i, k_i), \text{ for } i < \omega
\end{cases}
\]

and set $x = \bigcup_{i<\omega} a_i \upharpoonright [k_i, k_{i+1}) \in X$. Then, it holds that

$\forall k < \omega \ \exists m < \omega \ x \upharpoonright [m, m + k) \subset y$.

But this contradicts Claim 0.2 □

Let $C_\omega$ be the notion of forcing which adds a Cohen real. Then, it holds that

$\Vdash_{C_\omega} \forall y \in 2^\omega \ \exists x \in 2^\omega \ V \exists \infty n < \omega \ x \upharpoonright [n^2, n^2 + n) = y \upharpoonright [n^2, n^2 + n)$.

So, $\theta^* < d$ holds in a certain Cohen generic model.

It is known that the assumption cof($([d]^\omega, \subseteq) = d$ is followed from the non-existence of $0^\#$. So, it seems to prove Theorem 0.1 without this assumption. But I failed to find a proof.

Question 0.1 Is $\theta^* \leq d$ proved in ZFC?

In sections 2, 3, 4, we show that the cardinals $\omega_1$, $\theta$, $\theta^*$ can be separated for certain generic models.
1 Generic extensions by random reals

For each infinite cardinal $\kappa$, we denote by $B(\kappa)$ the measure algebra which adds a random function from $\kappa$ to 2 and by $\mu_{\kappa}: B(\kappa) \to [0,1]$ the measure function. In this section, we prove the following theorem.

**Theorem 1.1** Assume CH. Let $\kappa > \omega_1$ be a regular cardinal. Then, there are $f_1, f_2 \in F$ such that
\[ \Vdash_{B(\kappa)} \theta_{f_1} = \omega_1 \text{ and } \theta_{f_2} = \kappa. \]

Set $f_2 = \langle 2^n \mid n < \omega \rangle \in F$. The next well-known lemma guarantees that this $f_2$ is as required in Theorem 1.1.

**Lemma 1.2** (Forklore) $\Vdash_{B(\omega)} \exists y \in \prod_{n<\omega} f_2(n) \forall x \in \prod_{n<\omega} f_2(n) \cap V \forall^\infty n < \omega x(n) \neq y(n)$.

**Proof** Define $k_n < \omega$ (for $n < \omega$) by
\[ k_0 = 0 \text{ and } k_{n+1} = k_n + n \text{ for } n < \omega. \]
For each $n < \omega$, put $I_n = [k_n, k_{n+1})$ and take a bijections from $I_n 2$ to $f_2(n)$. Using these bijections, we identify $\prod_{n<\omega} f_2(n)$ with $\prod_{n<\omega} I_n 2$. Let $\dot{g}$ be the canonical $B(\omega)$-name of generic real. Define $\dot{y}$ by
\[ \Vdash \dot{y} = \langle j \mathrm{ r} I_n \mid n < \omega \rangle. \]
It holds that, for each $n < \omega$ and $s : I_n \to 2$,
\[ \mu_{\omega}(\| s = \dot{g} \mid I_n \|) = 2^{-|I_n|} = 2^{-n}. \]
So, $\mu_{\omega}(\exists^\infty n < \omega x \mid I_n = \dot{y}(n) \|) = 0$ for all $x \in 2^\omega$. This implies that
\[ \Vdash \forall^\infty n < \omega x(n) \neq \dot{y}(n), \text{ for all } x \in \prod_{n<\omega} I_n 2. \]

**Lemma 1.3** Let $0 < K, M < \omega$. Suppose that $\{ b_i^m \mid i < K \text{ and } m < M \} \subset B(\omega)$ and $b \in B(\omega)$ satisfy
\[ b = \sum_{i<K} b_i^m, \text{ for all } m < M. \]
Then, there is a function $\varphi : M \to K$ such that
\[ \mu_{\omega}(\sum_{m<M} b_{\varphi(m)}^m) \geq \mu_{\omega}(b) - \left( \frac{K-1}{K} \right)^M \mu_{\omega}(b). \]

**Proof** By induction on $M \in [1, \omega)$. The case $M = 1$ is clear. Let $M = M_0 + 1 > 1$.
Using the induction hypothesis, take $\varphi_0 : M_0 \to K$ such that
\[
\mu_\omega(\sum_{m < M_0} b_{\varphi_0(m)}^m) \geq \mu_\omega(b) - \left(\frac{K-1}{K}\right)^{M_0} \mu_\omega(b).\]

Put \(c = \sum_{m < M_0} b_{\varphi_0(m)}^m\). Since \(b - c = \sum_{i < K} (b_i^{M_0} - c)\), there exists \(j < K\) such that \(\mu_\omega(b_j^{M_0}) \geq \frac{1}{K} \mu_\omega(b - c)\). Then, \(\varphi = \varphi_0(\langle j \rangle)\) is as required. \(\square\)

For each \(n < \omega\), let
\[
M_n = \min\{ M < \omega \mid \left(\frac{n}{n+1}\right)^M < 2^{-n}\}. 
\]
Define \(f_1 \in \mathcal{F}\) by
\[
|\{ k < \omega \mid f_1(k) = n + 1 \}| = M_n, \text{ for all } n < \omega. 
\]
The next lemma implies that \(f_1\) satisfies the condition in Theorem 1.1.

**Lemma 1.4** \(\vdash_{\mathcal{B}(\omega)} \forall y \in \prod_{k < \omega} f_1(k) \exists x \in \prod_{k < \omega} f_1(k) \cap \mathcal{V} \exists^\infty k < \omega x(k) = y(k)\).

**Proof** For each \(n < \omega\), put \(J_n = \{ k < \omega \mid f_1(k) = n + 1 \}\). To show this lemma, let \(\dot{y}\) be a \(\mathcal{B}(\omega)\)-name such that \(\vdash^\ast \dot{y} \in \prod_{k < \omega} f_1(k)\). For each \(n < \omega\), using Lemma 1.3, take \(s_n \in \prod_{k \in J_n} f_1(k)\) such that
\[
\mu_\omega(\sum_{k \in J_n} |s_n(k) = \dot{y}(k)|) \geq 1 - \left(\frac{n}{n+1}\right)^{M_n}. 
\]
Put \(x = \bigcup_{n < \omega} s_n\). It is easy to check that
\[
\mu_\omega(\prod \forall^\infty n < \omega \exists k \in J_n x(k) = \dot{y}(k)) = 0. 
\]
So, it holds that \(\vdash^\ast \exists^\infty k < \omega x(k) = \dot{y}(k)\). \(\square\)

# 2 A forcing notion with the ccc which lifts up \(\theta^*\)

Define the forcing notion \((Q, \leq)\) by
\[
Q \subset 2^{<\omega} \times [2^\omega \times \mathcal{H}]^{<\omega}
\]
and, for any \((s, u) \in 2^{<\omega} \times [2^\omega \times \mathcal{H}]^{<\omega}\),
\[
(s, u) \in Q
\]
if and only if, for all \((x, h) \in u\) and all \(k < \omega\),
\[
\text{if } a_k^h \backslash \text{dom}(s) \neq \emptyset \text{ then } |a_k^h \backslash \text{dom}(s)| \geq |u| \text{ or } \exists i \in a_k^h \cap \text{dom}(s) x(i) \neq s(i), 
\]
and, for any \((s, u), (s', u') \in Q\),
\[
(s', u') \leq (s, u)
\]
if and only if
\[
s' \supset s \text{ and } u' \supset u \text{ and, for all } (x, h) \in u \text{ and all } k < \omega, \text{ if } a_k^h \cap \text{dom}(s') \backslash \text{dom}(s)) \neq \emptyset \text{ then } |a_k^h \backslash \text{dom}(s')| \geq |u'| \text{ or } \exists i \in a_k^h \cap \text{dom}(s') x(i) \neq s'(i). 
\]
We show that a finite support iteration by the above forcing notion lifts up the value \( \theta^* \). For this, we need several lemmas.

**Lemma 2.1** Let \( n < \omega \). Then, for every \((s, u) \in Q\), there is \( s' \in 2^{<\omega}\) such that \((s', u) \in Q\) and \((s', u) \leq (s, u)\) and \( n \in \text{dom}(s)\).

**Proof** For each \( j < \omega \), define \( \varphi_j : \mathcal{H} \rightarrow \omega \) by
\[
\varphi_j(h) = \text{the unique } k < \omega \text{ such that } j \in a_k^h.
\]
For each \( t \in 2^{<\omega} \), define \( \psi_t : 2^\omega \times \mathcal{H} \rightarrow \omega \) by
\[
\psi_t(x, h) = \begin{cases} 
|a_{\varphi_{\text{dom}(t)}(h)}^h \setminus \text{dom}(t)|, & \text{if } t \upharpoonright a_{\varphi_{\text{dom}(t)}(h)}^h \subset x, \\
|a_{\varphi_{\text{dom}(t)}(h)+1}^h|, & \text{otherwise}.
\end{cases}
\]
To show this lemma, let \( n < \omega \) and \((s, u) \in Q\). Put \( m = \text{dom}(s)\). Take \( M \in \omega \) such that \( n, m < M \) and \( |a_{\varphi_M(h)}^h \setminus M| \geq |u| \), for all \((x, h) \in u\).

By induction on \( j \in [m, M] \), take \( s_j : j \rightarrow 2 \) as follows:

- Put \( s_m = s \). Suppose that \( j \in [m, M) \) and \( s_j \) has been defined. Let \( l_j \) be the smallest element of \( \{ \psi_s(x, h) \mid (x, h) \in u \} \). Take \((x_j, h_j) \in u\) such that \( \psi_s(x_j, h_j) = l_j \). Set \( s_{j+1} = s_j \upharpoonright (1 - x_j(j)) \).

**Claim 2.2** \(|\{(x, h) \in u \mid \psi_s(x, h) < l\}| < l\), for all \( 0 < l < \omega \) and \( j \in [m, M] \).

\[ \because \] By induction on \( j \in [m, M] \). The case \( j = m \) is followed from the fact \((s, u) \in Q\). The case \( j = j_0 + 1 > m \) is followed from the fact \( \psi_s(x_{j_0}, h_{j_0}) \geq |u| \).

By Claim 2.2, it holds that \( l_j > 0 \), for all \( j \in [m, M] \). So, it holds that \((s_M, u) \in Q\) and \((s_M, u) \leq (s, u)\).

**Lemma 2.3** For each \((x, h) \in 2^\omega \times \mathcal{H}\),
\[
\{(s, u) \in Q \mid (x, h) \in u\} \text{ is dense in } Q.
\]

**Proof** Let \((x, h) \in 2^\omega \times \mathcal{H}\) and \((s, u) \in u\). Take \( M < \omega \) such that
1. \( |s| \leq M\),
2. if \( a_k^{h'} \setminus M \neq \phi \) then \( |a_k^{h'} \setminus M| > |u| \), for all \( k < \omega \) and \((x', h') \in u\).
3. if \( a_k^{h} \setminus M \neq \phi \) then \( |a_k^{h} \setminus M| > |u| \), for all \( k < \omega \).

Using Lemma 2.1, take \((t, u) \leq (s, u)\) such that \( \text{dom}(t) = M \). Then, it holds that \((t, u \cup \{(x, h)\}) \in Q\) and \((t, u \cup \{(x, h)\}) \leq (s, u)\).

**Lemma 2.4** \( Q \) satisfies the countable chain condition.
Proof. Let $W$ be an uncountable subset of $Q$. Using Lemma 2.1, replace $W$ by certain stronger conditions if necessary, we may assume that, for all $(s, u) ∈ W$

for all $(x, h) ∈ u$ and $k < ω$, if $a^h_k \setminus k \neq φ$ then $|a^h_k \setminus k| \geq 2|u|$.

Take $s_0 ∈ 2^{<ω}$ and $m < ω$ such that $W' = \{(s, u) ∈ W \mid s = s_0$ and $|u| = m\}$ is uncountable. Then, every elements in $W'$ are compatible. □

Let $\hat{G}$ be the canonical generic $Q$-name. Define $\hat{g}$ by

$\models Q \hat{g} = \bigcup \{s \mid (s, u) ∈ \hat{G}$, for some $u\}.$

Lemma 2.5 $\models Q \hat{g} ∈ 2^ω$ and $∀x ∈ 2^ω ∩ V \ ∀h ∈ H ∧ V \ ∀^∞ n < ω \ \hat{g} \upharpoonright a^h_n \neq x \upharpoonright a^h_n.$

Proof. This is directly followed from Lemmas 2.1 and 2.3. □

Let $\kappa$ be a regular uncountable cardinal and $P$ the $\kappa$-stage finite support iteration by the above forcing $Q$. Then, by the above arguments, it holds that $θ^* = κ$ in the generic model $V^P$. Since $P$ is finite support, it adds cofinally many Cohen reals. So, in $V^P$, the covering number $\text{cov}(M)$ of the meager ideal on the real line lifts up to $κ$. Furthermore, the next lemma shows that the unbounding number $b$ of $ω^ω$ lifts up to $κ$, too.

Lemma 2.6 There is a $Q$-name $\dot{d}$ such that $\models Q \dot{d} ∈ ω^ω$ dominates $ω^ω ∩ V$.

Proof. For each set $X$, denote by $0_X$ the constantly zero function from $X$ to 2.

Claim 2.7 For any $n < ω$,

$\{(s, u) ∈ Q \mid ∃m < ω$ ( $0_{[m, m+n]} ⊂ s$ ) $\}$ is dense in $Q$.

$\therefore$ Let $n < ω$ and $(s, u) ∈ Q$. Take $(t, u) ≤ (s, u)$ such that, for all $(x, h) ∈ u$ and $k < ω$,

if $a^h_k \setminus \text{dom}(t) \neq φ$ then $|a^h_k \setminus \text{dom}(t)| ≥ |u| + n$.

Define $t' : |t| + n → 2$ by $t ⊂ t'$ and $t'(|t| + j) = 0$, for $j < n$. It is easy to check that $(t', u) ∈ Q$ and $(t', u) ≤ (s, u)$. △

By Claim 2.7, it holds that $\models Q ∀n < ω \exists m < ω \ \hat{g} \upharpoonright [m, m + n] = 0_{[m, m+n]}.$

So, in $V^Q$, define $\hat{d} ∈ ω^ω$ by

$\hat{d}(n) =$ the smallest $m < ω$ such that $n ≤ m$ and $\hat{g} \upharpoonright [m, m + 2n] = 0_{[m, m+2n]}.$

To show $\hat{d}$ is a required one, let $f ∈ ω^ω$ and $(s, u) ∈ Q$. Without loss of generality, we may assume that $f$ is strictly increasing. Take $h ∈ H$ such that
\[ |a_k^h| \leq |a_{k+1}^h|, \text{ for all } k < \omega \text{ and } |\{ k < \omega \mid |a_k^h| = n \}| \geq f(n) + 1, \text{ for all } n < \omega. \]

By Lemma 2.3, take \((t, v) \leq (s, u)\) such that \((0_\omega, h) \in v\). Let \(k_0\) be the smallest \(k < \omega\) such that \(|t| \geq h(k)\) and set \(n_0 = |a_{k_0}^h| + |t|\). The next claim completes the proof of the lemma.

**Claim 2.8** \((t, v) \models Q \forall n > n_0 \ f(n) < \dot{d}(n)\).

.: To get a contradiction, assume that there are \((t', v') \leq (t, v)\) and \(n > n_0\) such that \((t', v') \models Q \dot{d}(n) \leq f(n)\). Replace \((t', v')\) by a stronger condition if necessary, we may assume that \((t', v')\) decides the value of \(\dot{d}(n)\). Let \(m < \omega\) be such that \((t', v') \models Q \dot{d}(n) = m\). Without loss of generality, we may assume that \(m + 2n \subset \text{dom}(t')\). Let \(k\) be the unique \(k < \omega\) such that \(m \in a_k^h\). By the choice of \(h\), it holds that \(|a_k^h|, |a_{k+1}^h| \leq n\).

\[ a_{k+1}^h \subset [m, m + 2n) \]

Since \((t', v') \models Q \dot{g} \uparrow [m, m + 2n) = 0_{[m, m+2n)}\), it holds that \(t' \upharpoonright a_{k+1}^h = 0_{a_{k+1}^h}\). This contradicts the facts that \((t', v') \leq (t, v)\) and \((0_\omega, h) \in v\) and \(\text{dom}(t) \cap [m, m + 2n) = \phi\). \(\square\)

In section 4, we give a generic model in which holds \(\theta^* = \omega_2\) and \(\text{cov}(M) = \omega_1\). But I do not known whether there is a model which satisfies \(b < \theta^*\).

**Question 2.1** Is \(b < \theta^*\) consistent with ZFC?

### 3 A forcing notion which lifts up \(\theta\)

In this section, we give a forcing notion which gives a generic model of \(\theta^* = \omega_1\) and \(\theta = \omega_2\). The forcing notion which we give here is constructed by the \(\omega_2\)-stage countable support iteration. We begin with the definition of a forcing notion \(BT_f\) for \(f \in \mathcal{F}\) which will be used each stage in the iteration.

Let \(f \in \mathcal{F}\). For each \(n < \omega\), denote \(\prod_{m<n} f(m)\) by \(S_n^f\). Put \(S^f = \bigcup_{n<\omega} S_n^f\). Note that \((S^f, \subset)\) is a tree. Define the forcing notion \((BT_f, \leq)\) by

\[ q \in BT_f \]

if and only if

1. \(q\) is a subtree of \(S^f\).
2. there is a function \(f' \in \mathcal{F}\) such that \(|\text{succ}_q(s)| \geq f'(|s|)\) for every \(s \in q\).

\[ q' \leq q \text{ if and only if } q' \subset q. \]

For each \(q \in BT_f\), define \(\pi_q \in \omega^\omega\) by

\[ \pi_q(n) = \max\{ k < \omega \mid \forall n' \geq n \ \forall s \in q \cap S_n^f \ |\text{succ}_q(s)| \geq k \}. \]
Note that $\pi_q \in \mathcal{F}$ for all $q \in \mathsf{BT}_f$. For each $k < \omega$, define the ordering $\leq_k$ on $\mathsf{BT}_f$ by

\[ q' \leq_k q \text{ if and only if } q' \leq q \text{ and } \pi_q \upharpoonright m_k = \pi_{q'} \upharpoonright m_k, \]

where $m_k$ denotes the smallest $m < \omega$ such that $\pi_q(m) > k$.

In [2], Bartoszyński, Judah and Shelah have used similar but more complicated forcing notions $Q_{f,g}$. The proof of the next lemma is similar to, but quite easier than the proof of Claim 2.6 in [2].

**Lemma 3.1** Let $\dot{e}$ be a $\mathsf{BT}_f$-name such that $\Vdash \dot{e} \in \mathbf{V}$. Then, for each $k < \omega$ and $q \in \mathsf{BT}_f$, there are $q' \leq_k q$ and a finite set $E$ such that $q' \Vdash \dot{e} \in E$.

**Proof** Let $\dot{e}$, $k < \omega$, $q \in \mathsf{BT}_f$ be as in the lemma. For each $s \in q$, denote by $q[s]$ the condition $\{ t \in q \mid s \subset t \text{ or } t \subset s \}$. Take $M < \omega$ such that $\pi_q(M) \geq 2k$. Set

\[ T = \{ s \in q \mid |s| \geq M \text{ and } \exists q' \leq_k q[s] \exists E (E \text{ is finite and } q' \Vdash \dot{e} \in E) \}. \]

Note that, whenever $s \in q \setminus T$ and $|s| \geq M$, $|\text{succ}_q(s) \cap T| < k$.

**Claim 3.2** $q \cap S^f_M \subset T$.

\[ \therefore \] To get a contradiction, assume that $s \in q \cap S^f_M \setminus T$. Let $U = \{ t \in q \setminus T \mid s \subset t \}$. Then, it holds that

\[ \forall t \in U \ (|\{ u \in U \mid t \subset u \text{ and } |u| = |t| + 1 \}| > \pi_q(|u|) - k). \]

This implies that $r = \{ s \mid j \mid j < |s| \} \cup U \in \mathsf{BT}_f$ and $r \leq_k q[s]$. Take $r' \leq r$ such that $r'$ decides $\dot{e}$. Take $t \in r'$ such that $\pi_{r'}(|t|) \geq k$. Since $r'[t] \leq_k q[t]$, we have that $t \in T$. This contradicts that $U \cap T = \phi$. $\triangle$

By Claim 3.2, for each $s \in q \cap S^f_M$, take $q_s \leq_k q[s]$ and a finite set $E_s$ such that $q_s \Vdash \dot{e} \in E_s$. Then $q' = \bigcup_{s \in q \cap S^f_M} q_s$ and $E = \bigcup_{s \in q \cap S^f_M} E_s$ satisfy this lemma. $\square$

**Corollary 3.3** $(\mathsf{BT}_f, (\leq_k)_{k < \omega})$ satisfies Axiom A and $\mathsf{BT}_f$ is $\omega^\omega$-bounding. $\square$

Let $\mathcal{G}$ be the canonical generic $\mathsf{BT}_f$-name. Define $\mathsf{BT}_f$-name $\dot{g}$ by

\[ \Vdash \dot{g} = \bigcup (\cap \mathcal{G}) \in \prod_{n < \omega} f(n). \]

Then, it is easy to check that

\[ \Vdash \forall x \in \prod_{n < \omega} f(n) \cap \mathbf{V} \forall^\infty n < \omega \ \dot{g}(n) \neq x(n). \]

Now we can describe how to construct a model which satisfies $\theta = \omega_2$ and $\theta^* = \omega_1$. Start with a ground model with $\text{CH}$. Let $\{ f_\alpha \mid \alpha < \omega_2 \} \subset \mathcal{F}$ be such that

\[ \{ \alpha < \omega_2 \mid f_\alpha = f \} \text{ is cofinal in } \omega_2 \text{ for each } f \in \mathcal{F}. \]
Define the $\omega_2$-stage countable support iteration $P_\alpha$ (for $\alpha \leq \omega_2$), $\dot{Q}_\alpha$ (for $\alpha < \omega_2$) by
\[ \forces \dot{Q}_\alpha = \text{BT}_f. \]
Let $P = P_{\omega_2}$. Then, by the above arguments, it holds that, in $V^P$, $\theta = \omega_2$ and $d = \omega_1$. Since $\text{cof}([\omega_1]^\omega, \subset) = \omega_1$ does always hold, it holds that, in $V^P$, $\theta^* \leq d = \omega_1$.

4 A generic model of $\theta = \omega_2$ and $\text{cov}(\mathcal{M}) = \omega_1$

In the previous section, we show that $\text{BT}_f$ does not lift up $\theta^*$. But, if we first add a dominating real then we get a certain function $f \in \mathcal{F}$ such that $\text{BT}_f$ lifts up $\theta^*$. In this section, we show that $\theta^*$ can be separated from $\text{cov}(\mathcal{M})$ by using it.

Lemma 4.1 Let $V$, $W$ be transitive models of ZFC such that $V \subset W$. Assume that $d \in W \cap \omega^\omega$ dominates $V \cap \omega^\omega$. In $W$, define $h \in H$ by
\[ |a^h_k| \leq |a^h_{k+1}|, \text{for all } k < \omega \text{ and } |\{ k < \omega \mid |a^h_k| = n \}| = d(n) + 1, \text{for all } n < \omega. \]
Then, it holds that $\forall^\infty m < \omega \exists k < \omega \ a^h_k \subset a^h_m$ for all $h' \in V \cap H$.

Proof Let $h' \in V \cap H$. In $V$, define $f_0, f_1 \in \omega^\omega$ by
\[ f_0(n) = \text{the smallest } m < \omega \text{ such that } \forall m' \geq m \ |a^h_{m'}| \geq 2n, \text{ and} \]
\[ f_1(n) = \text{max } a^h_{f_0(n+1)}. \]
Since $d$ dominates $f_0$, $f_1$, there is $n_0 < \omega$ such that $\forall n \geq n_0 \ f_0(n), f_1(n) < d(n)$. Put $k_0 = f_0(n_0)$. To show that $\forall k \geq k_0 \exists j < \omega \ a^h_j \subset a^h_k$, let $k \geq k_0$. Take $n < \omega$ such that $f_0(n) \leq k < f_0(n+1)$. Then, it holds that $|a^h_k| \geq 2n$ and max $a^h_k \leq \text{max } a^h_{f_0(n+1)} = f_1(n) \leq d(n)$. Since $[0, d(n))$ is covered by $\{ a^h_j \mid j < d(n) \}$ and $|a^h_j| \leq n$ for all $j < d(n)$, there is $j < d(n)$ such that $a^h_j \subset a^h_k$.

Lemma 4.2 Let $V$, $W$, $d$ and $h$ be as in Lemma 4.1. Working in $W$. Define $f \in \mathcal{F}$ by
\[ f(k) = 2|a^h_k|, \text{ for all } k < \omega. \]
Then, there is a $\text{BT}_f$-name $\dot{y}$ such that
\[ \forces \dot{y} \in 2^\omega \text{ and } \forall^\infty k < \omega \ \dot{y} \upharpoonright a^h_k \neq x \upharpoonright a^h_k, \text{ for all } x \in 2^\omega \cap V \text{ and } h' \in H \cap V. \]

Proof Working in $W$. Considering bijections from $f(k)$ to $a^h_k \forall k < \omega$, we may identify $\prod_{k<\omega} f(k)$ with $\prod_{k<\omega} a^h_k$. Let $\dot{G}$ be the canonical generic $\text{BT}_f$-name. Define $\text{BT}_f$-names $\dot{g}$ and $\dot{y}$ by
\[ \forces \dot{g} = \bigcup_{k<\omega} \dot{G}(k) \text{ and } \dot{y} = \bigcup_{k<\omega} \dot{g}(k). \]
Note that $\models \dot{y} \in \prod_{k<\omega} a_k^h 2$ and $\dot{y} \in 2^\omega$. It is easy to check that
\[ \models \forall x \in 2^\omega \cap W \forall^\infty k < \omega \dot{y} \upharpoonright a^h_k \neq x \upharpoonright a^h_k. \]
To show $\dot{y}$ is as required, let $x \in V \cap 2^\omega$ and $h' \in V \cap H$. Since it holds that $x \in W$ and $\forall^\infty m < \omega \exists k < \omega a^h_k \subset a^h_m$, we have that
\[ \models \forall^\infty m < \omega \dot{y} \upharpoonright a^h_m \neq x \upharpoonright a^h_m. \]

**Corollary 4.3**  Assume that CH holds. There are a forcing notion $R$ and $R$-name $\dot{y}$ such that

1. $R$ is proper and does not add a Cohen real and $|R| = \omega_1$.
2. $\models_R \dot{y} \in 2^\omega$ and $\forall x \in 2^\omega \cap V \forall h \in H \cap V \forall^\infty k < \omega \dot{y} \upharpoonright a^h_k \neq x \upharpoonright a^h_k$. \hfill \Box

Using Corollary 4.3, we can construct a generic model which satisfies $\text{cov}(M) = \omega_1$ and $\theta^* = \omega_2$. Start with a ground model with CH. Take an $\omega_2$-stage countable support iteration by the forcing notion as in Corollary 4.3. Since the iteration does not add a Cohen real, $\text{cov}(M)$ remains $\omega_1$. On the other hand, since functions $\dot{y} \in 2^\omega$ which satisfy (2) in the corollary is added cofinally, $\theta^*$ must be lifted up.

**References**

