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BSPFA Combined with One Measurable Cardinal

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Abstract

We consider consequences of BSPFA (Bounded Semi-Propre Forcing Axiom) combined with an existence of a measurable cardinal. The large cardinal assures existences of relevant semiproper preorders via Chang's Conjecture-type arguments.

Introduction

In [T], a new combinatorial principle \( \theta_{AC} \) is introduced. We recall its definition.

Definition. ([T]) \( \theta_{AC} \) holds, if for every one-to-one list \( r = (r_i \mid i < \omega_1) \) in \( \omega_2 \) and every \( S \subseteq \omega_1 \), there exist ordinals \( \gamma > \beta > \alpha \geq \omega_1 \) and an increasing continuous decomposition \( \gamma = \bigcup \{N_\nu \mid \nu < \omega_1 \} \) of \( \gamma \) into countable sets such that for all \( \nu < \omega_1, \) \( N_\nu \cap \omega_1 \subseteq S \) if and only if the following holds, where \( i = \text{o.t.}(N_\nu \cap \alpha), j = \text{o.t.}(N_\nu \cap \beta) \) and \( k = \text{o.t.}(N_\nu), \)

\[
\Delta(r_i, r_j) = \text{Max}\{\Delta(r_i, r_j), \Delta(r_i, r_k), \Delta(r_j, r_k)\}.
\]

The notation \( \Delta(r, r') \) stands for the least \( n < \omega \) such that \( r(n) \neq r'(n) \) for \( r, r' \in \omega_2 \) with \( r \neq r' \). We also recall.

Definition. BMM (Bounded Martin's Maximum) holds, if for any \( A \in H_{\omega_2}^V \) and any \( \Sigma_0 \)-formula \( \varphi \), if

\[
\text{\( \exists y \varphi(y, A) \in H_{\omega_2}^{V[\mathcal{G}]} \)}
\]

holds for some preorder \( P \) which preserves every stationary subset of \( \omega_1 \), then we already have \( \exists y \varphi(y, A) \in H_{\omega_2}^V \).

We may formulate a weaker forcing axiom by restricting the class of preorders to the semiproper ones.

Definition. BSPFA (Bounded Semi-Propre Forcing Axiom) holds, if for any \( A \in H_{\omega_2}^V \) and any \( \Sigma_0 \)-formula \( \varphi \), if

\[
\text{\( \exists y \varphi(y, A) \in H_{\omega_2}^{V[\mathcal{G}]} \)}
\]

holds for some preorder \( P \) which is semiproper, then we already have \( \exists y \varphi(y, A) \in H_{\omega_2}^V \).

In [T], it is shown

Theorem. ([T]) (1) BMM implies \( \theta_{AC} \),

(2) \( \theta_{AC} \) implies \( 2^\omega = 2^{\omega_1} = \omega_2 \).

In this note, we consider \( \theta_{AC}^* \) which is somewhat stronger than \( \theta_{AC} \) of [T] and show

(3) If BSFPA holds and there exists a measurable cardinal, then \( \theta_{AC}^* \) holds,

(4) \( \theta_{AC}^* \) implies both \( \theta_{AC} \) and CB (Complete Bounding).

While \( \theta_{AC} \) of [T] demands existences of \( \alpha, \beta \), and \( \gamma \) with \( \omega_1 < \alpha < \beta < \gamma \), our \( \theta_{AC}^* \) further demands \( \alpha = \omega_1 \). The consistency strength of the assumption in (1) is not well-known. A proper class of Woodin cardinals suffices (p. 867 in [W]). However they say it is unknown whether BMM implies \( 0^* \) or not.

On the other hand, if we have a type of reflecting cardinal (which itself is very much weaker than Mahlo) and a measurable cardinal above it (and so lots of measurable must exist below it), then we get the consistency of the assumption in (3) via a revised countable support iteration (say, see [M2]).
1. Basics with The One-to-one Lists in The Cantor Space

1.1 Definition. A one-to-one list \( r = \langle r_i \mid i < \omega_1 \rangle \) in \( \omega_2 \) means that for all \( i < \omega_1, r_i : \omega \rightarrow 2 \) and for all \( i, j < \omega_1, i \neq j \), then \( r_i \neq r_j \). In this case, we denote \( \Delta(r_i, r_j) = \min \{ n < \omega \mid r_i(n) \neq r_j(n) \} \). More generally, we consider a one-to-one list \( r = \langle r_i \mid i \in T \rangle \) on a stationary set \( T \subseteq \omega_1 \) in \( \omega_2 \). For a countable set \( X \) of ordinals, \( \text{o.t.}(X) \) denotes the order type of \( X \). Hence \( \text{o.t.}(X) < \omega_1 \). For any ordinals \( \alpha < \beta \), if \( \text{o.t.}(X \cap \alpha) < \text{o.t.}(X \cap \beta) < \omega_1 \), then we denote \( \Delta_X(\alpha, \beta) = \Delta(\text{r.o.t.}(X \cap \alpha), \text{r.o.t.}(X \cap \beta)) \). We usually simply write \( \Delta_X(\alpha, \beta) \) instead of \( \Delta_X(\alpha, \beta) \). For any ordinals \( \alpha, \beta \) and \( \gamma \), if \( \text{o.t.}(X \cap \alpha) < \text{o.t.}(X \cap \beta) < \text{o.t.}(X \cap \gamma) < \omega_1 \), then we denote \( \text{Max} \Delta_X(\alpha, \beta, \gamma) = \text{Max} \{ \Delta_X(\alpha, \beta), \Delta_X(\alpha, \gamma), \Delta_X(\beta, \gamma) \} \).

1.2 Lemma. Let \( r = \langle r_i \mid i < \omega_1 \rangle \) be a one-to-one list in \( \omega_2 \). Then there exists \( n < \omega \) such that both \( \{ i < \omega_1 \mid r_i(n) = 0 \} \) and \( \{ i < \omega_1 \mid r_i(n) = 1 \} \) are stationary.

Proof. Suppose not. For each \( n < \omega \), there is a club \( C_n \) and \( \epsilon_n \) such that for all \( i \in C_n \), \( r_i(n) = \epsilon_n \). Let \( C = \bigcap \{ C_n \mid n < \omega \} \). Then \( C \) is a club and for all \( i \in C \) and all \( n < \omega \), we have \( r_i(n) = \epsilon_n \). Hence \( \{ r_i \mid i \in C \} \) has one element. This is a contradiction.

1.3 Lemma. Let \( r = \langle r_i \mid i \in T \rangle \) be a one-to-one list on a stationary set \( T \) in \( \omega_2 \). Then there exist \( m < \omega \) and \( s \in \omega^2 \) such that both \( \{ i \in T \mid r_i[m = s \text{ and } r_i(m) = 0] \} \) and \( \{ i \in T \mid r_i[m = s \text{ and } r_i(m) = 1] \} \) are stationary.

Proof. Suppose not. For each \( m < \omega \) and \( s \in \omega^2 \), there exist a club \( C_{ms} \) and \( \epsilon_{ms} \) such that for all \( i \in C_{ms} \cap T \), we have if \( r_i[m = s] \), then \( r_i(m) = \epsilon_{ms} \). Let \( C = \{ C_{ms} \mid m < \omega, s \in \omega^2 \} \). Then \( C \) is a club and for all \( m < \omega \), all \( s \in \omega^2 \) and all \( i \in C \cap T \), we have if \( r_i[m = s] \), then \( r_i(m) = \epsilon_{ms} \). In particular, \( r_i(m) = \epsilon_{mr_i(n)} \). Hence for \( i, j \in C \cap T \), we may show \( r_i[m = r_j[m = m < \omega] \) by induction on \( m \). Hence \( \{ r_i \mid i \in C \cap T \} \) has one element. This is a contradiction.

1.4 Lemma. Let \( r = \langle r_i \mid i < \omega_1 \rangle \) be a one-to-one list in \( \omega_2 \). For any stationary \( S \) and any \( n < \omega_1 \), there exist \( m < \omega \) and \( s \in \omega^2 \) such that both \( \{ i \in S \mid r_i[m = s \text{ and } r_i(m) = 0] \} \) and \( \{ i \in S \mid r_i[m = s \text{ and } r_i(m) = 1] \} \) are stationary.

Proof. Let \( S \) and \( n \) be as given. Since \( \{ r_i[n \mid i \in S \} \) is finite, \( S \) gets partitioned into finitely many cells according to \( r_i[n \). But \( S \) is stationary. Hence one of them is stationary. So there is \( t \in \omega^2 \) such that \( T = \{ i \in S \mid r_i[n = t] \} \) is stationary. Now may apply lemma 1.3 to a one-to-one list \( \{ r_i[n, \omega \mid i \in T \} \) (somewhat abusive). Hence there exist \( m < \omega \) and \( u \in [n, m]^2 \) such that both \( \{ i \in S \mid r_i[n = t, r_i[m, m = u, r_i(m) = 0] \} \) and \( \{ i \in S \mid r_i[n = t, r_i[m, m = u, r_i(m) = 1] \} \) are stationary.

1.5 Lemma. Let \( r = \langle r_i \mid i < \omega_1 \rangle \) be a one-to-one list in \( \omega_2 \). For any \( n < \omega \), there exists a club \( C_n \) such that for any \( i \in C_n \) there is \( m \) with \( n \leq m < \omega \) such that both \( \{ j \in \omega_1 \mid r_j[m = r_i[m, r_j(m) = 0] \} \) and \( \{ j \in \omega_1 \mid r_j[m = r_i[m, r_j(m) = 1] \} \) are stationary.

Proof. Suppose not. For any club \( C \), there is \( i \in C \) such that for any \( m \) with \( n \leq m < \omega \), there is \( \eta \) such that \( \{ j \in \omega_1 \mid r_j[m = r_i[m, r_j(m) = \eta] \} \) is not stationary. Let \( S = \{ i < \omega_1 \mid \text{for all } m \text{ with } n \leq m < \omega, \text{there is } \eta \text{ such that } \{ j \in \omega_1 \mid r_j[m = r_i[m, r_j(m) = \eta] \} \text{ is not stationary} \} \). Then \( S \) is stationary. By lemma 1.4, we have \( m \) with \( n \leq m < \omega \) and \( s \in \omega^2 \) such that both \( S^0 = \{ i \in S \mid r_i[m = s, r_i(m) = 0] \} \) and \( S^1 = \{ i \in S \mid r_i[m = s, r_i(m) = 1] \} \) are stationary. Pick any \( i \in S^0(\neq \emptyset) \). Then \( r_i[m = s \text{ and } i \in S \). Hence there is \( \eta \) such that \( \{ j \in \omega_1 \mid r_j[m = s, r_j(m) = \eta] \} \) is not stationary. Since \( S^0 \) is stationary, we have \( \eta = 1 \). Similary, since \( S^1 \) is stationary, we have \( \eta = 0 \). This is a contradiction.

1.6 Lemma. Let \( r = \langle r_i \mid i < \omega_1 \rangle \) be a one-to-one list in \( \omega_2 \). Then there exists a club \( C_r \) such that for any \( i \in C_r \) and any \( n < \omega \), there is \( m \) with \( n \leq m < \omega \) such that both \( \{ j \in \omega_1 \mid r_j[m = r_i[m, r_j(m) = 0] \} \) and \( \{ j \in \omega_1 \mid r_j[m = r_i[m, r_j(m) = 1] \} \) are stationary.
Proof. Let $C_r = \bigcap \{C_r^n : n < \omega\}$. Then this $C_r$ works.

\[ \square \]

1.7 Lemma. Let $r = \langle r_i \mid i < \omega_1 \rangle$ be a one-to-one list in $\omega^2$. Then there exists a club $C_r$ such that for any $i < \infty$ and any $n < \omega$, we have $\{j < \omega_1 \mid \Delta(r_i, r_j) \geq n\}$ is stationary.

Proof. The $C_r$ above works.

\[ \square \]

1.8 Lemma. Let $r = \langle r_i \mid i < \omega_1 \rangle$ be a one-to-one list in $\omega^2$. Then there exist $n_r < \omega$ and a club $C_r$ such that

- Both $\{j < \omega_1 \mid r_j(n_r) = 0\}$ and $\{j < \omega_1 \mid r_j(n_r) = 1\}$ are stationary.

And so

- For any $i < \omega_1$, $\{j < \omega_1 \mid \Delta(r_i, r_j) \leq n_r\}$ is stationary.

While

- For any $i \in C_r$ and any $n < \omega$, $\{j < \omega_1 \mid \Delta(r_i, r_j) > n\}$ is stationary.

Proof. Let $n = n_r < \omega$ be any number such that both $\{j < \omega_1 \mid r_j(n) = 0\}$ and $\{j < \omega_1 \mid r_j(n) = 1\}$ are stationary. Let $C_r$ be as in above. These $n_r$ and $C_r$ work.

\[ \square \]

§ 2. Basics with Semiproper Preorders

2.1 Notation. Let $\lambda$ be a regular cardinal. We write $N \prec H_\lambda$, if the structure $(N, \in)$ is an elementary substructure of $(H_\lambda, \in)$. For $N$ and $M$, we denote $M \supseteq_{\text{end}} N$, if $M \supseteq N$ and $M \cap \omega_1 = N \cap \omega_1$. We denote 

\[ \langle X_i \mid i < \omega_1 \rangle \not\in X, \text{if } \langle X_i \mid i < \omega_1 \rangle \text{ is a sequence of continuously increasing countable subsets of } X \text{ and } \bigcup \{X_i \mid i < \omega_1\} = X. \]

2.2 Definition. Let $\kappa$ be a regular uncountable cardinal and $S \subseteq [\kappa]^{\omega}$. We say $S$ is semiproper, if there exists a club $C \subseteq [H_{(\kappa^+)^+}]^{\omega}$ such that for any $N \prec H_{(\kappa^+)^+}$ with $N \subseteq C$, there is a countable $M \prec H_{(\kappa^+)^+}$ such that $M \supseteq_{\text{end}} N$ and $M \cap \kappa \in S$.

2.3 Lemma. Let $\kappa$ be a regular uncountable cardinal, $S, T \subseteq [\kappa]^{\omega}$ be semiproper and disjoint. Then for any $B \subseteq \omega_1$, there is a semiproper p.o. set $P = P(S, T, B)$ such that in $V^P$, there is $\langle X_i \mid i < \omega_1 \rangle \not\in \kappa$ such that for all $i < \omega_1$,

- If $i \in B$, then $X_i \in S$,

- If $i \notin B$, then $X_i \in T$,

Hence

- $i \in B$ if and only if $X_i \in S$.

Proof. Let $p \in P$, if $p = \langle X_i^p \mid i \leq \alpha^p \rangle$ such that

- $p$ is continuously increasing and the $X_i^p$ are countable subsets of $\kappa$ with $\alpha^p < \omega_1$,

- For $i \leq \alpha^p$, we have

- If $i \in B$, then $X_i^p \in S$,

- If $i \notin B$, then $X_i^p \in T$.

For $p, q \in P$, let $q \leq p$, if $q \supseteq p$. 


We show that this $P$ works in a series of claims.

Claim 1. For any $p \in P$ and any $\xi \in \kappa$, there is $X$ such that $\xi \in X$, $q = p \cup \{(\alpha^p + 1, X)\} \in P$ and $q \leq p$.

Proof. According to $\alpha^p + 1 \in B$ or not, we have two cases.

Case 1. $\alpha^p + 1 \in B$: Since $S$ is semiproper, there is a countable $M \prec H_{(2^\kappa)^+}$ such that $p, \xi \in M$ and $M \cap \kappa \in S$. Let $X = M \cap \kappa$. Then this $X$ works.

Case 2. $\alpha^p + 1 \notin B$: Since $T$ is semiproper, there is a countable $M \prec H_{(2^\kappa)^+}$ such that $p, \xi \in M$ and $M \cap \kappa \in T$. Let $X = M \cap \kappa$. Then this $X$ works.

Claim 2. For $i < \omega_1$ and $\xi \in \kappa$, $D(i, \xi) = \{q \in P \mid i \leq \alpha^q, \xi \in X^q_{\alpha^q}\}$ is open dense in $P$.

Proof. By induction on $i$ for all $\xi$. By claim 1, it remains to deal with limit $i$. We show this by contradiction. Suppose for any $q \leq p$, $\alpha^q < i$. It suffices to derive a contradiction. Let $(i_n \mid n < \omega)$ be increasing such that $i_0 = \alpha^p$ and $\sup\{i_n \mid n < \omega\} = i$. According to $i \in B$ or not, we have two cases.

Case 1. $i \in B$: Let $M \prec H_{(2^\kappa)^+}$ be such that $i, p, \xi \in M$ and $M \cap \kappa \in S$. Let $(\xi_n \mid n < \omega)$ enumerate $M \cap \kappa$. By induction we have $(p_n \mid n < \omega)$ so that $p_0 = p, p_n \in P \cap M, i_n \leq \alpha^{p_{n+1}} < i$ and $\xi_n \in X^p_{\alpha^{p_{n+1}}}$. Let $q = \bigcup\{p_n \mid n < \omega\} \cup \{(i, M \cap \kappa)\}$. Then $q \in P$ and $q \leq p$ with $\alpha^q = i$. This is a contradiction.

Case 2. $i \notin B$: Similarly to case 1, let $M \prec H_{(2^\kappa)^+}$ be such that $i, p, \xi \in M$ and $M \cap \kappa \in T$. Let $(\xi_n \mid n < \omega)$ enumerate $M \cap \kappa$. By induction we have $(p_n \mid n < \omega)$ so that $p_0 = p, p_n \in P \cap M, i_n \leq \alpha^{p_{n+1}} < i$ and $\xi_n \in X^p_{\alpha^{p_{n+1}}}$. Let $q = \bigcup\{p_n \mid n < \omega\} \cup \{(i, M \cap \kappa)\}$. Then $q \in P$ and $q \leq p$ with $\alpha^q = i$. This is a contradiction.

Claim 3. $P$ is semiproper.

Proof. Let $P \in N \prec H_{(2^\kappa)^+}$ with $N \in C(S) \cap C(T)$, where $C(S)$ and $C(T)$ are clubs in $[H_{(2^\kappa)^+}]^\omega$ associated with semiproper $S$ and $T$ respectively. Let $p \in P \cap N$. We want to find $q \leq p$ which is $(P, N)$-semi-generic. According to $N \cap \omega_1 \notin B$ or not, we have two cases.

Case 1. $N \cap \omega_1 \in B$: Since $N \in C(S)$, we may take a countable $M \prec H_{(2^\kappa)^+}$ such that $M \supseteq_{\text{end}} N$ and $M \cap \kappa \in S$. Let $(p_n \mid n < \omega)$ be a $(P, M)$-generic sequence with $p_0 = p$. Let $q = \bigcup\{p_n \mid n < \omega\} \cup \{(M \cap \omega_1, M \cap \kappa)\}$. Then by claim 2, we know that $q \in P$ and so $q \leq p$. By construction, $q$ is $(P, M)$-generic and so $(P, N)$-semi-generic.

Case 2. $N \cap \omega_1 \notin B$: Similarly to case 1, take a countable $M \prec H_{(2^\kappa)^+}$ such that $M \supseteq_{\text{end}} N$ and $M \cap \kappa \in T$. Let $(p_n \mid n < \omega)$ be a $(P, M)$-generic sequence with $p_0 = p$. Let $q = \bigcup\{p_n \mid n < \omega\} \cup \{(M \cap \omega_1, M \cap \kappa)\}$. Then by claim 2, we know that $q \in P$ and so $q \leq p$. By construction, $q$ is $(P, M)$-generic and so $(P, N)$-semi-generic.

Claim 4. Let $G$ be any $P$-generic filter over $V$ and let $\langle X_i \mid i < \omega_1 \rangle = \bigcup G$. Then $\langle X_i \mid i < \omega_1 \rangle \cap \kappa$ and for $i < \omega_1$, we have

- If $i \in B$, then $X_i \in S$,
- If $i \notin B$, then $X_i \in T$.

Proof. By construction of $P$ and claim 2. Notice that $|\kappa| = \omega_1$ holds in the extension $V[G]$.

This completes the proof of lemma.
2.4 Lemma. Let κ be a regular uncountable cardinal and $S \subseteq [\kappa]^\omega$ be semiproper. Then there is a semiproper p.o. set $P = P(S)$ such that in $V^P$, there is $(X_i \mid i < \omega_1) \not\in \kappa$ such that for all $i < \omega_1$, $X_i \in S$.

Proof. The proof is entirely similar to and simpler than lemma 2.3.

\[ \square \]

§ 3. First Use of A Measurable Cardinal and BSPFA

We prepare a lemma with a measurable cardinal which is by now well-known with stronger statements.

3.1 Lemma. Let κ be a measurable cardinal with a normal measure D on κ. Let $N$ be a countable elementary substructure of $H_{(2^\kappa)^+}$ with $D \in N$.

(1) For any $\eta \in \kappa$ and any $s \in \bigcap (N \cap D)$ such that $\sup (N \cap \kappa)$, $\eta < s$, we may form a countable elementary substructure $M$ of $H_{(2^\kappa)^+}$ such that $N \cup \{s\} \subset M$ and $M \cap s = N \cap s = N \cap \kappa$.

(2) There is a continuously increasing countable elementary substructures $(N_i \mid i < \omega_1)$ of $H_{(2^\kappa)^+}$ such that $N_0 = N$ and $(\text{o.t.}(N_i \cap \kappa) \mid i < \omega_1)$ is a strictly increasing continuous sequence of countable ordinals.

(3) For any stationary $S \subseteq \omega_1$, there is a countable elementary substructure $M$ of $H_{(2^\kappa)^+}$ such that $N \subseteq \text{end} M$ and $\text{o.t.}(M \cap \kappa) \in S$.

Proof. For (1): Let $M = \{f(s) \mid f \in N\}$. Then this $M$ works.

For (2): Construct $(N_i \mid i < \omega_1)$ by recursion on $i$. At the successor stages, apply (1). At the limit stages, just take a union.

For (3): Immediate by (2).

\[ \square \]

3.2 Lemma. Let κ be a measurable cardinal and $r = \langle r_i \mid i < \omega_1 \rangle$ be a one-to-one list in $\omega^2$. For any countable $N < H_{(2^\kappa)^+}$ with $r, \kappa \in N$ and any $n < \omega$, there exists a countable $M < H_{(2^\kappa)^+}$ such that $M \supseteq \text{end} N$ and $\Delta_M(\omega_1, \kappa) \geq n$. Namely, $S(r, \kappa, n) = \{X \in [\kappa]^\omega \mid \Delta_X(\omega_1, \kappa) \geq n\}$ is semiproper.

Proof. Since $r \in N$, we may assume $C_r \in N$ and so $\delta = N \cap \omega_1 \in C_r$. Therefore $S = \{j < \omega_1 \mid \Delta(r_\delta, r_\kappa) \geq n\}$ is stationary. Since $\kappa$ is measurable and $\kappa \in N$, we may take a countable $M < H_{(2^\kappa)^+}$ such that $M \supseteq \text{end} N$ and $j = \text{o.t.}(M \cap \kappa) \in S$. Hence $\Delta_M(\omega_1, \kappa) = \Delta(r_{\text{o.t.}(M \cap \omega_1)}, r_{\text{o.t.}(M \cap \kappa)}) = \Delta(r_\delta, r_\kappa) \geq n$.

\[ \square \]

3.3 Lemma. Let κ be a measurable cardinal and $r = \langle r_i \mid i < \omega_1 \rangle$ be a one-to-one list in $\omega^2$. For any $n < \omega$, there exists a semiproper p.o. set $P$ such that in $V^P$, there exists $(X_i \mid i < \omega_1) \not\in \kappa$ such that for all $i < \omega_1$, $\Delta_{X_i}(\omega_1, \kappa) \geq n$.

Proof. Apply lemma 2.4 to $S(r, \kappa, n)$.

\[ \square \]

3.4 Lemma. (BSPFA) Let a measurable cardinal exist and $r = \langle r_i \mid i < \omega_1 \rangle$ be a one-to-one list in $\omega^2$. For any $n < \omega$, there exists $\beta$ with $\omega_1 < \beta < \omega_2$ and $(X_i \mid i < \omega_1) \not\in \beta$ such that for all $i < \omega_1$, $\Delta_{X_i}(\omega_1, \beta) \geq n$.

Proof. Apply BSPFA to lemma 3.3.

\[ \square \]

§ 4. Modifications and Summary
4.1 Lemma. Let \( n < \omega, \omega_1 < \beta < \omega_2 \) and \( \langle X_i | i < \omega_1 \rangle \not\sim \beta \) be such that for any \( i < \omega_1 \), \( \Delta_{X_i}(\omega_1, \beta) \geq n \). Then we have a continuously increasing \( \langle N_i | i < \omega_1 \rangle \) such that

- For all \( i < \omega_1, N_i \prec H_{\omega_2} \) and \( N_i \) is countable,
- \( \beta \in N_0, \bigcup \{ N_i | i < \omega_1 \} \supset \omega_1 \) and so \( \bigcup \{ N_i | i < \omega_1 \} \supset \beta \),
- For all \( i < \omega_1, \Delta_{N_i}(\omega_1, \beta) \geq n \).

**Proof.** Let \( \langle N_i | i < \omega_1 \rangle \) be any continuously increasing sequence of countable \( N_i \prec H_{\omega_2} \) such that \( \bigcup \{ N_i | i < \omega_1 \} \supset \omega_1 \) and \( N_i \in N_0 \). Then since \( \beta \prec \omega_2 \), we have \( \bigcup \{ N_i \cap \beta | i < \omega_1 \} = \beta \) and so \( C = \{ i < \omega_1 : X_i = N_i \cap \beta \} \) is a club. By reenumerating \( \{ N_i | i \in C \} \), we are done.

\[ \Box \]

4.2 Lemma. (BSPFA) Let a measurable cardinal exist and \( \kappa = \langle r_i | i < \omega_1 \rangle \) be a one-to-one list in \( \prec \omega_2 \). Then there exist \( n_\kappa < \omega \), a club \( C_\kappa, \beta_\kappa \) with \( \omega_1 < \beta_\kappa < \omega_2 \) and \( \langle N_i^\kappa | i < \omega_1 \rangle \) continuously increasing such that

- Both \( \{ j < \omega_1 | r_j(n_\kappa) = 0 \} \) and \( \{ j < \omega_1 | r_j(n_\kappa) = 1 \} \) are stationary,
- For any \( i < \omega_1, \{ j < \omega_1 | \Delta(r_i, r_j) \leq n_\kappa \} \) is stationary,
- For any \( i \in C_\kappa \) and any \( n < \omega, \{ j < \omega_1 | \Delta(r_i, r_j) > n \} \) is stationary,
- For any \( i < \omega_1, N_i^\kappa \prec H_{\omega_2} \) and \( N_i^\kappa \) is countable,
- \( \beta_\kappa \in N_0^\kappa, \bigcup \{ N_i^\kappa | i < \omega_1 \} \supset \omega_1 \) and so \( \bigcup \{ N_i^\kappa \cap \beta_\kappa | i < \omega_1 \} = \beta_\kappa \),
- For any \( i < \omega_1, \Delta_{N_i^\kappa}(\omega_1, \beta_\kappa) \geq n_\kappa + 1 \).

**Proof.** Combine lemma 1.8, lemma 3.4 and lemma 4.1.

\[ \Box \]

§ 5. Second Use of The Same Measurable Cardinal and BSPFA

5.1 Definition. Let \( \theta_{AC}^* \) denote the following statement. For any \( \kappa \) one-to-one list in \( \prec \omega_2 \) and any \( B \subseteq \omega_1 \), there exist \( \beta \) and \( \gamma \) with \( \omega_1 < \beta < \gamma < \omega_2 \) and \( \langle X_i | i < \omega_1 \rangle \not\sim \gamma \) such that for any \( i < \omega_1, i \in B \) if and only if \( \Delta_{X_i}(\omega_1, \beta) = \max \Delta_{X_i}(\omega_1, \beta, \gamma) \).

It is clear that \( \theta_{AC}^* \) implies \( \theta_{AC} \) of [T].

5.2 Theorem. (BSPFA) If there exists a measurable cardinal, then \( \theta_{AC}^* \) holds.

We show this in a series of lemmas.

5.3 Lemma. Let \( \kappa \) be a measurable cardinal and \( \kappa \) be a one-to-one list in \( \prec \omega_2 \). For any \( \beta \) with \( \omega_1 < \beta < \kappa \), any countable \( N \prec H_{\omega_2} \) with \( r, \beta, \kappa, N, \) exists there exists a countable \( M \prec H_{(2^\kappa)^+} \) such that \( M \supseteq \end N \) and \( \Delta_M(\omega_1, \beta) = \min \Delta_M(\omega_1, \beta, \kappa) \).

**Proof.** Since \( r \in N, \) we may assume \( C_r \in N \) and so \( N \cap \omega_1 \in C_r \). Hence for all \( n < \omega, \) we have \( \{ j < \omega_1 | \Delta(r \cap n \cap \omega_1, r_j) \geq n \} \) is stationary. Since \( \omega_1 < \beta \), we may calculate \( \Delta_N(\omega_1, \beta) = n \). Since \( \kappa \) is a measurable cardinal, we may choose a countable \( M \prec H_{(2^\kappa)^+} \) such that \( M \cap \beta = N \cap \beta \), if \( j = o.t.\langle M \cap n \rangle \), then \( \Delta(r \cap n \cap \omega_1, r_j) \geq n + 1 \). Since \( \Delta_M(\omega_1, \beta) = \Delta_N(\omega_1, \beta) = n < \Delta(r \cap n \cap \omega_1, r_j) = \Delta_M(\omega_1, \kappa) \), we have \( \Delta_M(\omega_1, \beta) = \min \Delta_M(\omega_1, \beta, \kappa) \).

\[ \Box \]

5.4 Lemma. (BSPFA) Let \( \kappa \) be a measurable cardinal and \( \kappa \) be a one-to-one list in \( \prec \omega_2 \). For any countable \( N \prec H_{(2^\kappa)^+} \) with \( r, \kappa, N, \) there exists a countable \( M \prec H_{(2^\kappa)^+} \) such that \( M \supseteq \end N \) and \( \Delta_M(\omega_1, \beta_r) = \max \Delta_M(\omega_1, \beta_r, \kappa) \).
Proof. Let \( \eta = r_{N\cap\omega_{1}}(n_{r}) \). Let \( \bar{\eta} \in \{0, 1\} \) and \( \eta \neq \bar{\eta} \). Since \( \{j < \omega_{1} \mid r_{j}(n_{r}) = \bar{\eta}\} \) is stationary, we may choose a countable \( M < H^{2e_{1}} \) such that \( M \equiv_{end} N \) and \( r_{o.t.}(M\cap\kappa)(n_{r}) = \bar{\eta} \). Hence \( \Delta_{M}(\omega_{1}, \kappa) \leq n_{r} \). On the other hand, since we may assume \( \langle N_{i}^{\delta} \mid i < \omega_{1}\rangle \in N, \) if \( \delta = N \cap \omega_{1}, \) then we have \( N_{i}^{\delta} \subseteq_{end} N \cap H_{\omega_{2}}. \) Since \( \beta_{r} \in N_{i}^{\delta}, \) we conclude \( N_{i}^{\delta} \cap \beta_{r} = N \cap \beta_{r} \) holds. So \( \Delta_{M}(\omega_{1}, \beta_{r}) = \Delta_{N}(\omega_{1}, \beta_{r}) = \Delta_{N_{i}^{\delta}}(\omega_{1}, \beta_{r}) \geq n_{r} + 1. \) Therefore, \( \Delta_{M}(\omega_{1}, \beta_{r}) = \Delta_{M}(\omega_{1}, \beta_{r}, \kappa). \)

5.5 Lemma. (BSPFA) Let \( \kappa \) be a measurable cardinal and \( r \) be a one-to-one list in \( \omega_{2}. \) Let \( S = \{X \in [\kappa]^{\omega} \mid \Delta_{X}(\omega_{1}, \beta_{r}) = \text{Max} \Delta_{X}(\omega_{1}, \beta_{r}, \kappa)\} \) and \( T = \{X \in [\kappa]^{\omega} \mid \Delta_{X}(\omega_{1}, \beta_{r}) = \text{Min} \Delta_{X}(\omega_{1}, \beta_{r}, \kappa)\}. \) Then both \( S \) and \( T \) are semiproper and disjoint.

Proof. By lemma 5.4 and lemma 5.3.

5.6 Lemma. (BSPFA) Let \( \kappa \) be a measurable cardinal. Let \( r = \langle r_{i} \mid i < \omega_{1}\rangle \) be a one-to-one list in \( \omega_{2} \) and \( B \subseteq \omega_{1}. \) Then there exists a semiproper p.o. set \( P \) such that in \( V^{P}, \) there is \( \langle Y_{i} \mid i < \omega_{1}\rangle \not\in \kappa \) such that for any \( i < \omega_{1}, i \in B \) if and only if \( \Delta_{Y_{i}}(\omega_{1}, \beta_{r}) = \text{Max} \Delta_{Y_{i}}(\omega_{1}, \beta_{r}, \kappa). \)

Proof. By lemma 5.5 and lemma 2.3.

Proof of theorem 5.2. Apply BSPFA to the p.o. set in lemma 5.6.

§ 6. \( \theta_{AC}^{*} \) implies CB

6.1 Definition. CB (complete bounding) stands for the following. For any \( f : \omega_{1} \rightarrow \omega_{1}, \) there exist \( \omega_{1} < \gamma < \omega_{2}, \) a club \( C \) and \( \langle X_{i} \mid i < \omega_{1}\rangle \not\in \gamma \) such that for all \( i \in C, f(i) < o.t. \langle X_{i}\rangle. \)

6.2 Theorem. \( \theta_{AC}^{*} \) implies CB.

Proof. We have two claims.

Claim 1. If for any one-to-one list \( r \) in \( \omega_{2}, \) there exist \( \omega_{1} < \beta < \omega_{2} \) and \( \langle X_{i} \mid i < \omega_{1}\rangle \not\in \beta \) such that \( \Delta_{X_{i}}(\omega_{1}, \beta) > 0, \) then CB holds.

Proof. Let \( f : \omega_{1} \rightarrow \omega_{1}. \) We may assume that for all \( i < \omega_{1}, i < f(i) \) and \( f \) is strictly increasing. Take a continuously increasing \( \langle N_{i} \mid i < \omega_{1}\rangle \) such that each \( N_{i} \) is countable, \( N_{i} < H_{\omega_{1}} \) and \( N_{i} \in \mathcal{N} \) and \( f \in \mathcal{N}. \) Notice that \( N_{i} \cap \omega_{1} < f(N_{i} \cap \omega_{1}) < N_{i+1} \cap \omega_{1}. \) It is easy to construct \( r \) so that \( r_{N \cap \omega_{1}}(0) = 1, \) for \( \xi \) with \( N_{i} \cap \omega_{1} < \xi \leq f(N_{i} \cap \omega_{1}), \) we have \( r_{\xi}(0) = 0. \) By assumption get \( \omega_{1} < \beta < \omega_{2} \) and \( \langle X_{i} \mid i < \omega_{1}\rangle \) such that for all \( i < \omega_{1}, \) we have \( \Delta_{X_{i}}(\omega_{1}, \beta) > 0. \) Let \( C = \{i < \omega_{1} \mid N_{i} \cap \omega_{1} = i = X_{i} \cap \omega_{1}, \omega_{1} \in X_{i}\}. \) Then for \( i \in C, \) since \( \Delta_{X_{i}}(\omega_{1}, \beta) = \Delta(r_{i}, r_{o.t.}(X_{i})) > 0, \) we have \( f(i) = f(N_{i} \cap \omega_{1}) < o.t. \langle X_{i}\rangle. \)

Claim 2. If \( \theta_{AC}^{*} \) holds, then for any one-to-one list \( r \) in \( \omega_{2}, \) there exist \( \omega_{1} < \beta < \omega_{2} \) and \( \langle X_{i} \mid i < \omega_{1}\rangle \not\in \beta \) such that \( \Delta_{X_{i}}(\omega_{1}, \beta) > 0. \)

Proof. By \( \theta_{AC}^{*} \) for \( \beta = \omega_{1}, \) there exist \( \omega_{1} < \beta < \gamma < \omega_{2} \) and \( \langle Y_{i} \mid i < \omega_{1}\rangle \not\in \gamma \) such that for all \( i < \omega_{1}, \) we have \( \Delta_{Y_{i}}(\omega_{1}, \beta) = \text{Max} \Delta_{Y_{i}}(\omega_{1}, \beta, \gamma). \) In particular, \( \Delta_{Y_{i}}(\omega_{1}, \beta) > 0. \) Let \( X_{i} = Y_{i} \cap \beta. \) Then these \( \beta \) and \( \langle X_{i} \mid i < \omega_{1}\rangle \) work.
§ 7. Additional Observations

Now we make a few observations. We may consider to directly force our \( \theta_{AC}^* \). Namely, we may add the following to [D].

**Theorem.** ([D], [M]) The following are equiconsistent.

- \( \text{Con(There exists a regular cardinal } \rho \text{ such that } \{ \kappa < \rho \mid \kappa \text{ is a measurable cardinal} \} \text{ is cofinal in } \rho \}) \)
- \( \text{Con}(\theta_{AC}^*) \)
- \( \text{Con}(\text{CB}) \)

Hence \( \theta_{AC} \) of [T] accordingly has a large cardinal upper-bound.

Next, similarly to \( \text{Con( PFA}^+ \leftrightarrow \text{CB}) \) (which we got from S. Todorcevic), we may show via \( \omega_1 \)-many Cohen reals \( r \),

**Theorem.** ([M]) \( \text{Con(PFA}^+ \text{ and } \neg \theta_{AC}^*) \) and so \( \text{Con(PFA}^+ \text{ and } \neg \text{BMM}) \).

Lastly, starting with a Souslin tree in the ground model and preserving it ([M1]), we have

**Theorem.** ([M]) \( \text{Con(There exists a Souslin tree and } \theta_{AC}^*) \) and so \( \text{Con( } \neg \text{MA and } \theta_{AC}^*) \)

Among others concerning the large cardinal strength of BMM, we may ask

**Question.** Does \( \theta_{AC} \) of [T] imply any large cardinal, say, CB?

More modestly,

**Question.** Does BMM imply the Weak Chang's Conjecture?

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