数理解析研究所講究録 1306 巻 2003 年 47-56

On Convex Fuzzy Games

Rodica Brânzei *
Faculty of Computer Science
"Alexandru Ioan Cuza" University, Iaşi, Romania
Stef Tijs
CentER and Department of Econometrics and Operations Research
Tilburg University, The Netherlands

Abstract

In this paper convex fuzzy games are defined, and their properties as well as properties of some solution concepts are presented.

1 Preliminaries

Let $N = \{1, 2, \ldots, n\}$ be a nonempty set of players considering possibilities of fuzzy cooperation, i.e. the players may be involved in cooperation with participation levels varying between 0 (non-cooperation) and 1 (full cooperation). Formally, a *fuzzy coalition* of players in N is a vector $s \in [0,1]^N$, whose the $i-$th coordinate s_i is called the *participation level* of player i. Instead of $[0,1]^N$ we will also write $\mathcal{F}^N$ for the set of fuzzy coalitions. Special cases of fuzzy coalitions are those corresponding to crisp coalitions $S \in 2^N$, which are denoted by e^S, where $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$ if $i \in N \setminus S$. Then $e^{\emptyset} = (0, \ldots, 0)$ stands for the empty coalition in a fuzzy setting, $e^N = (1, \ldots, 1)$ denotes the grand coalition, whereas e^i is the fuzzy coalition corresponding to the crisp coalition $S = \{i\}$ (and also the $i-$th standard basis vector in $\Re^N$). One can identify a fuzzy coalition with

*Corresponding author. E-mail address: branzeir@infoiasi.ro

a point in the hypercube $[0,1]^N$; the fuzzy coalitions e^S, $S \in 2^N$, are the $2^{|N|}$ extreme points (vertices) of this hypercube. A cooperative *fuzzy game* with player set N is a function $v : \mathcal{F}^N \to \Re$ with the property $v(e^{\emptyset}) = 0$, assigning to each fuzzy coalition the value achieved as the result of cooperation with participation levels s_i, $i \in N$. We denote the set of fuzzy games with player set N by FG^N. The set of non-empty fuzzy coalitions will be denoted by $\mathcal{F}_0^N$.

Many solution concepts for games with fuzzy coalitions have been developed: cores (Aubin (1974); Branzei et al. (2002c); Tijs et al. (2002b); Ishihara et al. (2003)); Shapley values (Aubin (1974), (1981); Butnariu (1978); Branzei et al. (2002a)) and Shapley functions (Tsurumi et al. (2001)); path solutions, path solution cover, hypercubes and compromise values (Branzei et al. (2002b)); monotonic allocation schemes such as FPMAS (Tsurumi et al. (2001)), pamas (Branzei et al. (2002a)), and bi-pamas (Tijs et al. (2002a)); the egalitarian solution (Branzei et al. (2002c)).

We briefly recall the definitions of those solution concepts that are of special interest for this paper.

Let $s \in \mathcal{F}^N$ and denote $car(s) = \{i \in N | s_i > 0\}$. Let $v \in FG^N$. The imputation set $I(v)$ of v is

$$I(v) = \left\{ x \in \Re^N \mid \sum_{i \in N} x_i = v(e^N), x_i \geq v(e^i) \text{ for each } i \in N \right\};$$

the Aubin *core* $C(v)$ of v (Aubin (1974)) is

$$C(v) = \left\{ x \in \Re^N \mid \sum_{i \in N} x_i = v(e^N), \sum_{i \in N} s_i x_i \geq v(s) \text{ for each } s \in \mathcal{F}^N \right\};$$

the proper core $C^P(v)$ of v (Tijs et al. (2002b)) is

$$C^P(v) = \left\{ x \in \Re^N \mid \sum_{i \in N} x_i = v(e^N), \sum_{i \in N} s_i x_i \geq v(s), s \in \mathcal{F}^N, car(s) \neq N \right\};$$

the crisp core $C^{cr}(v)$ of v (Tijs et al. (2002b)) is

$$C^{cr}(v) = \left\{ x \in \Re^N \mid \sum_{i \in N} x_i = v(e^N), \sum_{i \in car(e^S)} x_i \geq v(S) \text{ for each } S \in 2^N \right\}.$$

The dominance core (D-core) $DC(v)$ of v and stable sets K are based on dom_s and dom relations on $I(v)$. Let $x, y \in I(v)$ and $s \in \mathcal{F}^N$. Then $x\ dom_s\ y$ if $x_i > y_i$ for all $i \in car(s)$ and $\sum_{i\in N} s_i x_i \leq v(s)$; $x\ dom\ y$ if there is $s \in \mathcal{F}^N$ such that $x\ dom_s\ y$. The negation of $x\ dom\ y$ is denoted here by $\neg x\ dom\ y$.

$$DC(v) = \{x \in I(v) | \neg x\ dom\ y\ for\ all\ y \in I(v)\}$$

is the subset of $I(v)$ of undominated elements.
A stable set of v is a nonempty set K of imputations such that: for all $x, y \in K$, $\neg x\ dom\ y$, and for all $z \in I(v) \setminus K$, there is $x \in K$ with $x\ dom\ z$.

The *fuzzy Shapley value* $\phi(v)$ and the *fuzzy Weber set* $W(v)$ (Branzei et al. (2002a)) are given by:

$$\phi(v) = \frac{1}{|N|!} \sum_{\sigma \in \Pi(N)} m^\sigma(v); W(v) = conv\{m^\sigma(v) \mid \sigma \in \Pi(N)\},$$

where $\Pi(N)$ stands for the set of linear orderings of N, and $m^\sigma(v)$ for each $\sigma \in \Pi(N)$ is the *marginal vector* with for $i = \sigma(k)$, the $i-$th coordinate $m_i^\sigma(v)$ given by

$$m_i^\sigma(v) = v\left(\sum_{r=1}^{k} e^{\sigma(r)}\right) - v\left(\sum_{r=1}^{k-1} e^{\sigma(r)}\right).$$

One can identify a $\sigma \in \Pi(N)$ with an $n-$step walk along the edges of the hypercube of fuzzy coalitions starting in $e^\emptyset$ and ending in e^N by passing the vertices $e^{\sigma(1)}, e^{\sigma(1)} + e^{\sigma(2)}, \ldots, \sum_{r=1}^{n-1} e^{\sigma(r)}$. The vector $m^\sigma(v)$ records the changes in value from vertex to vertex.

A special class of fuzzy games with a non-empty Aubin core is the class of convex fuzzy games introduced in Branzei et al. (2002a). The purpose of this paper is on one hand to present the definition and (characterizing) properties for convex fuzzy games (Section 2), and on the other hand to offer an overview on special properties of solution concepts on the cone of convex fuzzy games, stressing on the solution concepts of participation monotonic allocation scheme, the egalitarian solution and the equal division core (Section 3). Section 4 concludes with some final remarks.

2 Definition and properties of convex fuzzy games.

Let N be a finite set and let $v : [0,1]^N \to \Re$ be a real-valued function on $[0,1]^N$. Then

(i) v is called a *supermodular function* on $[0,1]^N$ if

$$v(s \vee t) + v(s \wedge t) \geq v(s) + v(t) \text{ for all } s, t \in [0,1]^N,$$

where $s \vee t$ and $s \wedge t$ are those elements of $[0,1]^N$ with the i-th coordinate equal to $\max\{s_i, t_i\}$ and $\min\{s_i, t_i\}$, respectively;

(ii) v is called a *coordinate-wise convex function* if for each $i \in N$ and each $s^{-i} \in [0,1]^{N\setminus\{i\}}$ the function $g_{s^{-i}} : [0,1] \to \Re$ with $g_{s^{-i}}(t) = v(s^{-i} \parallel t)$ for each $t \in [0,1]$ is a convex function. Here $(s^{-i} \parallel t)$ is the element in $[0,1]^N$ with $(s^{-i} \parallel t)_j = s_j$ for each $j \in N \setminus \{i\}$ and $(s^{-i} \parallel t)_i = t$.

(iii) v is said to satisfy the *increasing average marginal return property (IAMR-property)* if for each $i \in N$, $s^1, s^2 \in \mathcal{F}^N$ with $s^1 \leq s^2$ and each $\varepsilon_1, \varepsilon_2 > 0$ with $s_i^1 + \varepsilon_1 \leq s_i^2 + \varepsilon_2 \leq 1$

$$\varepsilon_1^{-1}\left(v\left(s^1 + \varepsilon_1 e^i\right) - v\left(s^1\right)\right) \leq \varepsilon_2^{-1}\left(v\left(s^2 + \varepsilon_2 e^i\right) - v\left(s^2\right)\right).$$

The IAMR-property expresses the fact that an increase in participation level of any player in a smaller coalition yields per unit of participation level less than an increase in participation level in a bigger coalition under the condition that the reached level of participation in the first case is still not bigger than the reached participation level in the second case. The IAMR-property turns out to be crucial for convex fuzzy games as we see in Theorem 4.

Definition 1 *Let $v \in FG^N$. The fuzzy game v is called a convex fuzzy game if the function $v : [0,1]^N \to \Re$ is a supermodular and a coordinate-wise convex function on $[0,1]^N$.*

Remark 2 *A weaker definition of convexity, where only the supermodularity property is used, is given in Tsurumi et al. (2001).*

We denote the set of fuzzy games with player set N by CFG^N. Some properties of convex fuzzy games are given in the next proposition.

Proposition 3 *Let $v \in CFG^N$. Then the following properties hold:*
(i) (Increasing fuzzy marginal contribution for players). Let $i \in N$, $s^1, s^2 \in \mathcal{F}^N$ with $s^1 \leq s^2$ and let $\varepsilon \in \Re_+$ with $0 \leq \varepsilon \leq 1 - s_i^2$. Then

$$v\left(s^1 + \varepsilon e^i\right) - v\left(s^1\right) \leq v\left(s^2 + \varepsilon e^i\right) - v\left(s^2\right).$$

(ii)(Increasing fuzzy marginal contribution for coalitions). Let $s, t \in \mathcal{F}^N$ and $z \in \Re_+^N$ such that $s \leq t \leq t + z \leq e^N$. Then

$$v(s+z) - v(s) \leq v(t+z) - v(t).$$

(iii) (Stable marginal contribution property). For each $\sigma \in \Pi(N)$ the fuzzy marginal vector $m^\sigma(v)$ is a core element.

Proof. See Proposition 3 and 4, and Theorem 7 in Branzei et al. (2002a). ■

Theorem 4 *Let $v \in FG^N$. Then $v \in CFG^N$ iff the increasing average marginal return property (IAMR−property) holds.*

Proof. See Theorem 6 in Branzei et al. (2002a). ■

Remark 5 *Convex fuzzy games form a convex cone, that is for all $v, w \in CFG^N$ and all $a, b \geq 0$, $av + bw \in CFG^N$.*

For examples of convex fuzzy games the reader is referred to Branzei et al. (2002 a,b,c) and Tijs et al. (2002b).

3 Solution concepts for convex fuzzy games

First, we pay attention to the solution concepts for fuzzy games whose definitions are provided in Section 1 of this paper. As in the case of convex crisp games these solutions behave nicely on the class of convex fuzzy games. Let $v \in FG^N$; then the cooperative n-person game $cr(v)$ defined by $cr(v)(S) = v(e^S)$ for each $S \in 2^N$ is called the *crisp game corresponding to v*. For $v \in CFG^N$ the corresponding crisp game $cr(v)$ is also convex (see Proposition 2 in Branzei et al. (2002a)).

Theorem 6 *Let $v, w \in CFG^N$. Then*
(i) $C(v) = C(cr(v))$, $C(v) = W(v)$, and $C(v+w) = C(v) + C(w)$, $W(v) = W(cr(v))$;
(ii) $\phi(v) \in C(v)$ ($\phi(v)$ is the barycenter of the core), $\phi(v) = \phi(cr(v))$, and $\phi(v+w) = \phi(v) + \phi(w)$.

Proof. See Theorem 7 and Proposition 8 in Branzei et al. (2002a). ■

Remark 7 *The fact that $C(v) = W(v)$ does not necessarily imply that the fuzzy game v is convex (see Example 5 in Branzei et al. (2002a)).*

Theorem 8 *Let $v \in CFG^N$. Then*
(i) $C(v) = C^P(v) = C^{cr}(v)$;
(ii) $DC(v) = DC(cr(v))$;
(iii) $C(v) = DC(v)$;
(iv) $DC(v)$ is the unique stable set.

Proof. See Tijs et al. (2002b). ■

Interesting solution concepts for convex fuzzy games as those of participation monotonic allocation schemes (pamas) and the egalitarian solution introduced in Branzei et al. (2002a) and (2002c), respectively. We define these solutions and present briefly their properties in the rest of this section. In the definition of pamas the notion of t-restricted game plays a role.

Definition 9 *Let $v \in FG^N$ and $t \in \mathcal{F}^N$. The $t-$restricted game of v is the game $v_t : \mathcal{F}^N \to \Re$ given by $v_t(s) = v(t * s)$ for all $s \in \mathcal{F}^N$. Here $t * s$ is the coordinate-wise product of t and s, that is $(t * s)_i = t_i s_i$ for all $i \in N$.*

Remark 10 *If $v \in CFG^N$, then also $v_t \in CFG^N$ for each $t \in \mathcal{F}^N$. This is the fuzzy analogue of the fact that subgames of crisp convex games are convex.*

Definition 11 *A game $v \in FG^N$ is called totally balanced if $C(v) \neq \emptyset$ and $C(v_t) \neq \emptyset$ for all $t \in \mathcal{F}^N$.*

Definition 12 *Let $v \in FG^N$ be a totally balanced game. A scheme $[a_{t,i}]_{t \in \mathcal{F}^N, i \in N}$ is called a participation monotonic allocation scheme (pamas) if*
(i) $(a_{t,i})_{i \in N} \in C(v_t)$ for each $t \in \mathcal{F}^N$ (stability condition);
(ii) $t_i^{-1} a_{t,i} \geq s_i^{-1} a_{s,i}$ for each $s, t \in \mathcal{F}^N$ with $s \leq t$ and each $i \in car(s)$ (participation monotonicity condition).

Definition 13 *Let $v \in FG^N$ and $x \in C(v)$. Then we call x pamas–extendable if there exists a pamas $[a_{t,i}]_{t\in\mathcal{F}^N, i\in N}$ such that $a_{e^N,i} = x_i$ for each $i \in N$.*

Theorem 14 *Let $v \in CFG^N$. Then each $x \in C(v)$ is pamas–extendable.*

Proof. See Theorem 10 in Branzei et al. (2002a). ■

For each $v \in CFG^N$ the *total fuzzy Shapley value*, which is the scheme $[\phi_{t,i}]_{t\in\mathcal{F}^N, i\in N}$ with the fuzzy Shapley value of the restricted game v_t in each row t, is a pamas.

In the following we introduce the egalitarian solution for convex fuzzy games by adjusting the classical Dutta-Ray algorithm for finding the constrained egalitarian solution for convex crisp games. For each $s \in \mathcal{F}^N$, let $\lceil s \rfloor := \sum_{i=1}^n s_i$. Given $v \in CFG^N$ and $s \in \mathcal{F}_0^N$ we denote by $\alpha(s,v)$ the average worth of s with respect to the aggregated participation level of players in N, that is

$$\alpha(s,v) := \frac{v(s)}{\lceil s \rfloor}.$$

Note that $\alpha(s,v)$ can be viewed as a per participation-level-unit value of coalition s.

The next theorem (Theorem 6 in Branzei et al. (2002c)) guarantees that in each step k of the adjusted Dutta-Ray algorithm there is a unique maximal element in $\arg\sup_{s\in\mathcal{F}_0^N} \alpha(s, v_k)$ which corresponds to a crisp coalition, say S_k, implying that the adjusted Dutta-Ray algorithm is a finite algorithm.

Theorem 15 *Let $v \in CFG^N$. Then*
(i) $\sup_{s\in\mathcal{F}_0^N} \alpha(s,v) = \max_{T\in 2^N\setminus\{\emptyset\}} \alpha\left(e^T, v\right)$;
(ii) $T^* = \max\left(\arg\max_{T\in 2^N\setminus\{\emptyset\}} \alpha\left(e^T, v\right)\right)$ *generates the largest element in* $\arg\sup_{s\in\mathcal{F}_0^N} \alpha(s,v)$, *namely* e^{T^*}.

The egalitarian solution $E(v)$ of a convex fuzzy game v is obtained by the *adjusted Dutta-Ray algorithm* as follows:
Step 1. Let $N_1 := N$, $v_1 := v$. Let S_1 be the crisp coalition generating the largest element in $\arg\sup_{s\in\mathcal{F}_0^N} \alpha(s, v_1)$. Define $E_i(v) = \alpha\left(e^{S_1}, v_1\right)$ for each $i \in S_1$. If $S_1 = N$, then we stop else go to Step 2.
Step 2. Let $N_2 := N_1 \setminus S_1$ and v_2 defined for each $s \in [0,1]^{N\setminus S_1}$ by

$$v_2(s) = v_1\left(e^{S_1} \curvearrowright s\right) - v_1\left(e^{S_1}\right),$$

where $\left(e^{S_1} \curvearrowright s\right)$ is the element in $[0,1]^N$ with

$$\left(e^{S_1} \curvearrowright s\right)_i = \begin{cases} 1 & \textit{if } i \in S_1 \\ s_i & \textit{if } i \in N \setminus S_1 \end{cases}.$$

One can take the largest element e^{S_2} in $\arg\max_{S \in 2^{N_2} \setminus \{\emptyset\}} \alpha\left(e^S, v_2\right)$ and define $E_i(v) = \alpha\left(e^{S_2}, v_2\right)$ for all $i \in S_2$. If $S_1 \cup S_2 = N$ we stop; otherwise we continue by considering the convex fuzzy game v_3, etc.
After a finite number of steps the algorithm stops, and the obtained allocation $E(v)$ is called *the egalitarian solution of the convex fuzzy game v.*

Theorem 16 *Let $v \in CFG^N$. Then*
(i) $E(v) = E(cr(v))$;
(ii) $E(v) \in C(v)$;
(iii) $E(v)$ Lorenz dominates every other allocation $x \in C(v)$.

Proof. See Theorem 7 in Branzei et al. (2002c). ■

Remark 17 *Theorem 16 implies that we can calculate the egalitarian solution of a convex fuzzy game by considering the corresponding crisp game and applying on it the classical Dutta-Ray algorithm.*

Definition 18 *Given a cooperative fuzzy game v, we define the equal division core $EDC(v)$ as the set*

$$\left\{ x \in \Re^N \mid \sum_{i \in N} x_i = v(e^N),\ \nexists s \in \mathcal{F}_0^N \text{ s.t. } \alpha(s,v) > x_i \textit{ for all } i \in car(s) \right\}.$$

Each $x \in EDC(v)$ can be seen as a distribution of the value of the grand coalition e^N, where for each fuzzy coalition s, there is a player i with a positive participation level for which the pay-off x_i is at least as good as the equal division share $\alpha(s,v)$ of $v(s)$ in s.

Some interesting facts concerning the equal division core for convex fuzzy games are collected in

Theorem 19 *Let $v \in CFG^N$. Then*
(i) $C(v) \subseteq EDC(v)$;
(ii) $E(v) \in EDC(v)$;
(iii) $EDC(v) = EDC(cr(v))$.

Proof. See Theorem 8 in Branzei et al. (2002c). ■

Based on Theorems 18 and 19 and using Klijn et al. (2000) we have obtained in Branzei et al. (2002c) an axiomatic characterization of the egalitarian solution on the class of convex fuzzy games.

Theorem 20 *There is a unique solution on CFG^N satisfying the properties efficiency, equal division stability and max-consistency, and it is the egalitarian solution.*

Here equal division stability of a solution means that the solution assigns to any convex fuzzy game an element of the equal division core.

4 Final comments

First, note that for a convex fuzzy game all the presented solution concepts coincide with the corresponding solution concepts of the associated crisp game which is also convex. Therefore one can take the advantage of the available efficient algorithms for convex crisp games to compute solution concepts for convex fuzzy games. Note also the parallelism between the properties of the presented solutions for convex fuzzy games and those of convex crisp games. For other characterizing properties of convex fuzzy games we refer to Tijs and Branzei (2003).

References

[1] J.P. Aubin, Coeur et valeur des jeux flous à paiements latéraux, C.R. Acad. Sci. Paris 279 A (1974) 891-894.

[2] J.P. Aubin, Cooperative fuzzy games, Math. Oper. Res. 6 (1981) 1-13.

[3] R. Branzei, D. Dimitrov and S. Tijs, Convex fuzzy games and participation monotonic allocation schemes, CentER DP 2002-13, Tilburg University, Tilburg, The Netherlands, 2002a (to appear in Fuzzy Sets and Systems).

[4] R. Branzei, D. Dimitrov and S. Tijs, Hypercubes and compromise values for cooperative fuzzy games, CentER DP 2002-14, Tilburg University,

Tilburg, The Netherlands, 2002b (to appear in European Journal of Operational Research).

[5] R. Branzei, D. Dimitrov and S. Tijs, Egalitarianism in convex fuzzy games, CentER DP 2002-97, Tilburg University, Tilburg, The Netherlands, 2002c.

[6] D. Butnariu, E.P. Klement, Triangular Norm-Based Measures and Games with Fuzzy Coalitions, Kluwer Academic Publishers, Dordrecht, 1993.

[7] F. Klijn, M. Slikker, S. Tijs, J. Zarzuelo, The egalitarian solution for convex games: some characterizations, Mathematical Social Sciences 40 (2000) 111-121.

[8] S. Ishihara, S. Muto, S. Tijs and R. Branzei, Generalized cores and stable sets for fuzzy games, Mimeo, Tokyo Institute of Technology, Japan, 2003.

[9] E. Molina, J. Tejada, The equalizer and the lexicographical solutions for cooperative fuzzy games: characterizations and properties, Fuzzy Sets and Systems 125 (2002) 369-387.

[10] M. Sakawa, I. Nishizaki, A lexicographical concept in an $n-$person cooperative fuzzy game, Fuzzy Sets and Systems 61 (1994) 265-275.

[11] S. Tijs, R. Branzei, S. Muto, S. Ishihara and E.Fukuda, Fuzzy clan games and bi-monotonic allocation rules, CentER DP 2002-115, Tilburg University, Tilburg, The Netherlands, 2002a.

[12] S. Tijs, R. Branzei, S. Ishihara and S. Muto, On cores and stable sets for fuzzy games, CentER DP 2002-116, Tilburg University, Tilburg, The Netherlands, 2002b.

[13] S. Tijs and R. Branzei, Various characterizations of convex fuzzy games, Preprint, Department of Mathematics of the University of Genoa, Italy, 2003.

[14] S. Tsurumi, T. Tanino, M. Inuiguchi, A Shapley function on a class of cooperative fuzzy games, European Journal of Operational Research, 129 (2001) 569-618.