AN EXPLICIT FORMULA FOR THE LIMITING OPTIMAL GAIN IN THE FULL INFORMATION DURATION PROBLEM

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1 Introduction and summary

We consider here a full-information model for the duration problem [Ferguson, Hardwik, Tamaki 1992] with horizon $n$ tending to infinity. Our objective here is to determine the asymptotics for the optimal gain $V(n)$.

Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables, uniformly distributed on $[0, 1]$, where $X_n$ denotes the value of the object at the $n$-th stage from the end. We call an object relatively best if it possesses the largest value among those observed so far. The task is to select a relatively best object with the view of maximizing the duration it stays relatively best. Let $v(x, n)$ denote the optimal expected return when there are $n$ objects yet to be observed and the present maximum of past observations is $x$. Notice that $V(n) = v(0, n)$.

The Optimality Equation for $v(x, n)$ has form

$$v(x, n) = xv(x, n - 1) + \int_{x}^{1} \max\{w(t, n), v(t, n - 1)\} dt, \quad v(x, 0) = 0,$$  \hspace{1cm} (1)

where $w(x, n)$ denote the expected payoff given that the $n$th object from the last is a relatively best object of value $X_n = x$ and we select it.

$$w(x, n) = 1 + x + x^2 + \ldots + x^{n-1} = \frac{1 - x^n}{1 - x}.$$  \hspace{1cm} (2)

Denote the point of intersection of functions $v(x, n - 1)$ and $w(x, n)$ as $x_n$. It exists and unique because $w(x, n)$ are increasing in $x$ for every $n$ and $v(x, n)$ are nonincreasing in $x$ for every $n$, and $w(0, n) = 1 \leq \int_{0}^{1} w(t, n) dt \leq v(0, n)$, and $w(1, n) = n > 0 = v(1, n - 1)$.

So, we can rewrite (1) in the form

$$v(x, n) = xv(x, n - 1) + \int_{x}^{x_n} v(t, n - 1) dt + \int_{x_n}^{1} w(t, n) dt, \quad 0 \leq x \leq x_n$$  \hspace{1cm} (3)

and

$$v(x, n) = xv(x, n - 1) + \int_{x_n}^{1} w(t, n) dt, \quad x_n \leq x \leq 1.$$  \hspace{1cm} (4)
If we stop the selection with a relatively best object $X_n = x$, we receive $w(x, n)$. If we continue and select the next relatively best object, we expect to receive

$$u(x, n) = \sum_{k=1}^{n-1} x^{k-1} \int_x^1 w(t, n - k) dt$$

$$= \sum_{k=1}^{n-1} x^{k-1} \sum_{j=1}^{n-k} (1 - x^j)/j. \quad (5)$$

The problem is monotone [Ferguson et al, 1992], so the one-stage look-ahead rule (OLA) is optimal here and prescribes stopping if $w(x, n) \geq u(x, n)$; that is, if

$$\sum_{k=1}^{n} x^{k-1} \left( 1 - \sum_{j=1}^{n-k} (1 - x^j)/j \right) \geq 0. \quad (6)$$

It is equivalent that we stop selection on the step $n$ if the relatively best object has value $X_n \geq x_n$ (see the fig. 1) with $x_n$ as the solution of the equation (6).

![Graph](image)

**Fig. 1**

According to [Porosinski, 1987], $x_n$ written as $x_n = 1 - z_n/n$ satisfies the equation

$$\sum_{k=1}^{n} \left( 1 - \frac{z_n}{n} \right)^{k-1} \left( 1 - \sum_{j=1}^{n-k} \left( 1 - \frac{z_n}{n} \right)^{j}/j \right) = 0,$$

and from here $z_n$ must converge to a constant, $z_n \to z$, where $z \approx 2.11982$ satisfies the integral equation

$$\int_0^1 e^{-zv} \left[ 1 - \int_0^{1-v} \frac{1 - e^{-zu}}{u} du \right] dv = 0.$$
Lemma 1. \( V(n) \to \infty \) as \( n \to \infty \).

**Proof.** It follows immediately from the inequality \( v(x, n) \geq u(x, n + 1) \) for every \( x, n \) and \( u(0, n) = \sum_{j=1}^{n-1} 1/j \to \infty \) as \( n \to \infty \).

Lemma 2. There exists a constant \( C \) such that \( V(n) \leq Cn \) for all \( n \).

**Proof.** Suppose that it is true for some \( n-1 \). Then from (3)-(4) it follows that for every \( x \) \( v(x, n) \leq C(n-1) + \epsilon_n \) where \( \epsilon_n = \int_{x_n}^{1} w(t, n) dt \). Because \( z_n \to z \), the sequence
\[
\epsilon_n = \sum_{j=1}^{n} \frac{1-x_j^n}{j} = \sum_{j=1}^{n} \frac{1-(1-z_n^j)^j}{j} = \sum_{j=1}^{n} \frac{1-(1-z_1^n)^j}{j/n} \frac{1}{n}
\]
converges to \( \int_0^1 (1-e^{-zx})/xdx \approx 1.3700 \) and, consequently, it is bounded by some constant \( e \), i.e. \( \epsilon_n \leq e \). Hence, if we choose \( C \geq e \) we obtain \( v(x, n) \leq Cn \) for all \( x \).

Remark. So, the function \( V(n) \) tends to infinity not slower than \( \log n \) and not faster than \( Cn \).

Lemma 3. Function \( u(x, n) \) satisfies the equation
\[
u(x, n) = xu(x, n - 1) + \int_x^1 w(t, n - 1) dt, \quad u(x, 1) = 0.
\] (7)

**Proof.** It follows from direct calculations
\[
u(x, n) - xu(x, n - 1) = \sum_{k=1}^{n-1} x^{k-1} \int_x^1 w(t, n - k) dt - \sum_{k=1}^{n-2} x^k \int_x^1 w(t, n - 1 - k) dt
\]
\[
= \sum_{k=1}^{n-1} x^{k-1} \int_x^1 w(t, n - k) dx - \sum_{k=2}^{n-1} x^{k-1} \int_x^1 w(t, n - k) dx
\]
\[
= \int_x^1 w(t, n - k) dx.
\]

Let us introduce two new functions
\[
y(x, n) = v(x, n) - u(x, n + 1), \quad \Delta_n(x) = u(x, n) - w(x, n).
\]
In the interval \([0, x_n]\) both functions are non-negative and \(\Delta_n(x_n) = 0\). According to (3) and (7), \(y(x, n)\) satisfies the equation

\[
y(x, n) = xy(x, n - 1) + \int_{x}^{x_n} \left[ y(t, n - 1) + \Delta_n(t) \right] dt, \quad 0 \leq x \leq x_n,
\]

and \(y(x, n) = 0\), for \(x \geq x_n\). Also, notice that \(y(x, 1) = y(x, 2) = 0\) (because \(x_1 = x_2 = 0\)) and \(y(x, 3) = \int_{x}^{x_3} \Delta_3(t) dt\), where \(\Delta_3(x) = 1/2 - x - 5/2x^2\).

**Lemma 4.** Function \(\Delta_n(x)\) satisfies the equations

\[
\Delta_n(x) = x\Delta_{n-1}(x) + \int_{x}^{1} w(t, n-1) dt - 1, \quad \Delta_1(x) = -1.
\]

\[
\Delta_{n+1}(x) - \Delta_n(x) = \sum_{j=1}^{n} \frac{x^{n-j} - x^n}{j} - x^n.
\]

**Proof.** (9) follows from Lemma 3 and the simple identity

\[
w(x, n) = xw(x, n - 1) + 1.
\]

From (9) we obtain

\[
\Delta_{n+1}(x) - \Delta_n(x) = x \left[ \Delta_n(x) - \Delta_{n-1}(x) \right] + \frac{1-x^n}{n}.
\]

After \(n\) iterations we have

\[
\Delta_{n+1}(x) - \Delta_n(x) = \sum_{j=1}^{n} \frac{x^{n-j} - x^n}{j} + x^n[\Delta_1(x) - \Delta_0(x)].
\]

With \(\Delta_1(x) - \Delta_0(x) = -1\) it proves (10).

Differentiating (8) in \(x\) we obtain

\[
y'(x, n) = xy'(x, n - 1) - \Delta_n(x), \quad 0 \leq x \leq x_n,
\]

and, consequently, \(y'(x, 3) = -\Delta_3(x)\),

\[
y'(x, 4) = \begin{cases} -x\Delta_3(x) - \Delta_4(x), & \text{if } 0 \leq x \leq x_3 \\ -\Delta_4(x), & \text{if } x_3 < x \leq x_4 \end{cases}
\]

and

\[
y'(x, n) = -\sum_{j=i}^{n} x^{n-j}\Delta_j(x), \quad x_{i-1} \leq x \leq x_i, i = 3, 4, \ldots, n.
\]
In integral form (11) is

\[ y(x, n) = \sum_{j=i}^{n} \int_{x}^{x_j} t^{n-j} \Delta_j(t) dt, \quad x_{i-1} \leq x \leq x_i, i = 3, 4, ..., n. \] 

(12)

Letting in (12) \( x = 0 \) we obtain

Lemma 5.

\[ y_n = y(0, n) = \sum_{j=3}^{n} \int_{0}^{x_j} t^{n-j} \Delta_j(t) dt. \] 

(13)

Consider now the difference \( r_n = y_{n+1} - y_n \). Using (13) we can represent it as a sum of two expressions

\[ r_n = y_{n+1} - y_n = \sum_{j=3}^{n} \int_{0}^{x_j} t^{n-j} [\Delta_{j+1}(t) - \Delta_j(t)] dt + \sum_{j=3}^{n+1} \int_{x_{j-1}}^{x_j} t^{n-j+1} \Delta_j(t) dt \] 

(14)

Using (10) the first sum in (14) can be rewritten in the form

\[ \sum_{j=3}^{n} \int_{0}^{x_j} t^{n-j} \left[ \sum_{i=1}^{j} \frac{t^{j-i} - t^i}{i} \right] dt = \sum_{j=3}^{n} \left\{ \sum_{i=1}^{j} \left[ \frac{x_j^{n-i+1}}{n-i+1} - \frac{x_j^{n+1}}{n+1} \right] \right\} \frac{1}{i} \frac{x_j^{n+1}}{n+1} \] 

(15)

As \( n \to \infty \) letting \( x_n = 1 - z_n/n \) we obtain that (15) converges to the integral

\[ C_0 = \int_{0}^{1} e^{-z} \left[ \int_{0}^{u} \left( \frac{e^{\frac{zu}{u}} - 1}{v} + \frac{e^{\frac{zu}{u}}}{1-v} \right) dv - 1 \right] du. \] 

(16)

Let us prove that the second sum in (14) tends to zero. Firstly notice that \( \Delta_{j-1}(x) \leq 0 \) for \( x \geq x_{j-1} \), consequently, from Lemma 4 it follows that

\[ \Delta_j(x) \leq \int_{x}^{1} w(t, j-1) dt - 1 \leq \sum_{i=1}^{j-1} \frac{1}{i}, \quad x \in [x_{j-1}, x_j]. \]

Hence,

\[ 0 \leq \sum_{j=3}^{n} \int_{x_{j-1}}^{x_j} t^{n-j+1} \Delta_j(t) dt \]

\[ \leq \sum_{j=3}^{n+1} \sum_{i=1}^{j-1} \int_{x_{j-1}}^{x_j} \frac{t^{n-j+1}}{i} dt \]

\[ = \sum_{i=1}^{n} \frac{1}{i} \sum_{j=i+1}^{n+1} \left[ \frac{x_j^{n-j+2} - x_{j-1}^{n-j+2}}{n-j+2} \right] \]
\[
\frac{1}{n} \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} \left[ x_j^{n-j+1}(x_j - x_{j-1}) \right] 
\]

\[
\frac{1}{n} \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} \left[ (1 - \frac{z_j}{j})^{n-j+1} \left( \frac{z_{j-1}}{j-1} - \frac{z_j}{j} \right) \right].
\] (17)

Using the inequality \(1 - x \leq \exp(-x), \forall x\), (17) is not larger than

\[
\frac{1}{n} \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} \left[ \exp \left\{ -\frac{z_j}{j} (n - j + 1) \right\} \left( \frac{z_{j-1}}{j-1} - \frac{z_j}{j} \right) \right] 
\leq \exp(Z) \frac{1}{n} \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} \left[ \exp \left\{ -\frac{z_j(n+1)}{j} \right\} \left( \frac{z_{j-1}}{j-1} - \frac{z_j}{j} \right) \right],
\] (18)

where \(Z = \sup z_j\). The last sum (18) converges as \(n \to \infty\) to zero, because \(z_n \to z\) and corresponding for internal sum integral \(\int_{i+1}^{n+1} \exp\{-\frac{z(n+1)}{t}\} z/t^2 dt\) is not larger than \(\exp\{-z\}/(n+1)\).

Summarizing all, we obtain the next result.

Theorem 1. For large \(n\)

\[\frac{V_n}{n} \to C_0,\]

where \(C_0\) is the value of the integral (16).

Proof follows immediately from \(\lim_{n \to \infty} V_n/n = \lim_{n \to \infty} v(0, n)/n = \lim_{n \to \infty} [y_n + u(0, n + 1)]/n = \lim_{n \to \infty} [y_n + \sum_{j=1}^{n} 1/j]/n = C_0\). Package Mathematica calculates \(C_0 \approx 0.4351708\).

REFERENCES
