AN EXPLICIT FORMULA FOR THE LIMITING OPTIMAL GAIN IN THE FULL INFORMATION DURATION PROBLEM (Mathematics of Decision-making under uncertainty)

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AN EXPLICIT FORMULA FOR THE LIMITING OPTIMAL GAIN IN THE FULL INFORMATION DURATION PROBLEM

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1 Introduction and summary

We consider here a full-information model for the duration problem [Ferguson, Hardwik, Tamaki 1992] with horizon $n$ tending to infinity. Our objective here is to determine the asymptotics for the optimal gain $V(n)$.

Suppose that $X_1, X_2, ...$ are i.i.d. random variables, uniformly distributed on $[0, 1]$, where $X_n$ denotes the value of the object at the $n$-th stage from the end. We call an object relatively best if it possesses the largest value among those observed so far. The task is to select a relatively best object with the view of maximizing the duration it stays relatively best. Let $v(x, n)$ denote the optimal expected return when there are $n$ objects yet to be observed and the present maximum of past observations is $x$. Notice that $V(n) = v(0, n)$.

The Optimality Equation for $v(x, n)$ has form

$$v(x, n) = xv(x, n - 1) + \int_x^1 \max\{w(t, n), v(t, n - 1)\} dt, \quad v(x, 0) = 0,$$

where $w(x, n)$ denote the expected payoff given that the $n$th object from the last is a relatively best object of value $X_n = x$ and we select it.

$$w(x, n) = 1 + x + x^2 + ... + x^{n-1} = \frac{1 - x^n}{1 - x}. \quad (2)$$

Denote the point of intersection of functions $v(x, n - 1)$ and $w(x, n)$ as $x_n$. It exists and unique because $w(x, n)$ are increasing in $x$ for every $n$ and $v(x, n)$ are noncreasing in $x$ for every $n$, and $w(0, n) = 1 \leq \int_0^1 w(t, n) dt \leq v(0, n)$, and $w(1, n) = n > 0 = v(1, n - 1)$.

So, we can rewrite (1) in the form

$$v(x, n) = xv(x, n - 1) + \int_x^{x_n} v(t, n - 1) dt + \int_{x_n}^1 w(t, n) dt, \quad 0 \leq x \leq x_n \quad (3)$$

and

$$v(x, n) = xv(x, n - 1) + \int_x^1 w(t, n) dt, \quad x_n \leq x \leq 1. \quad (4)$$
If we stop the selection with a relatively best object \( X_n = x \), we receive \( w(x, n) \). If we continue and select the next relatively best object, we expect to receive

\[
 u(x, n) = \sum_{k=1}^{n-1} x^{k-1} \int_x^1 w(t, n - k) dt 
 = \sum_{k=1}^{n-1} x^{k-1} \sum_{j=1}^{n-k} (1 - x^j)/j. 
\] (5)

The problem is monotone [Ferguson et al, 1992], so the one-stage look-ahead rule (OLA) is optimal here and prescribes stopping if \( w(x, n) \geq u(x, n) \); that is, if

\[
 \sum_{k=1}^{n} x^{k-1} \left( 1 - \sum_{j=1}^{n-k} (1 - x^j)/j \right) \geq 0. 
\] (6)

It is equivalent that we stop selection on the step \( n \) if the relatively best object has value \( X_n \geq x_n \) (see the fig. 1) with \( x_n \) as the solution of the equation (6).

![Graph](image)

According to [Porosinski, 1987], \( x_n \) written as \( x_n = 1 - z_n/n \) satisfies the equation

\[
 \sum_{k=1}^{n} \left( 1 - \frac{z_n}{n} \right)^{k-1} \left( 1 - \sum_{j=1}^{n-k} \left( 1 - \frac{z_n}{n} \right)^j / j \right) = 0,
\]

and from here \( z_n \) must converge to a constant, \( z_n \to z \), where \( z \approx 2.11982 \) satisfies the integral equation

\[
 \int_0^1 e^{-zu} \left[ 1 - \int_0^{1-u} \frac{1 - e^{-zu}}{u} du \right] dv = 0.
\]
Lemma 1. $V(n) \to \infty$ as $n \to \infty$.

Proof. It follows immediately from the inequality $v(x, n) \geq u(x, n+1)$ for every $x, n$ and $u(0, n) = \sum_{j=1}^{n-1} \frac{1}{j} \to \infty$ as $n \to \infty$.

Lemma 2. There exists a constant $C$ such that $V(n) \leq Cn$ for all $n$.

Proof. Suppose that it is true for some $n - 1$. Then from (3)-(4) it follows that for every $x$ $v(x, n) \leq C(n - 1) + \epsilon_n$ where $\epsilon_n = \sum_{j=1}^{n} \frac{1 - x_j}{j}$. Because $z_n \to z$, the sequence

$$\epsilon_n = \sum_{j=1}^{n} \frac{1 - x_n^j}{j}$$

converges to $\int_1^1 (1 - e^{-zx})/xdx \approx 1.3700$ and, consequently, it is bounded by some constant $\epsilon$, i.e. $\epsilon_n \leq \epsilon$. Hence, if we choose $C \geq \epsilon$ we obtain $v(x, n) \leq Cn$ for all $x$.

Remark. So, the function $V(n)$ tends to infinity not slower than $\log n$ and not faster than $Cn$.

Lemma 3. Function $u(x, n)$ satisfies the equation

$$u(x, n) = xu(x, n - 1) + \int_{x}^{1} w(t, n - 1)dt, \quad u(x, 1) = 0. \quad (7)$$

Proof. It follows from direct calculations

$$u(x, n) - xu(x, n - 1) = \sum_{k=1}^{n-1} x^{k-1} \int_{x}^{1} w(t, n - k)dt - \sum_{k=1}^{n-2} x^{k} \int_{x}^{1} w(t, n - 1 - k)dt$$

$$= \sum_{k=1}^{n-1} x^{k-1} \int_{x}^{1} w(t, n - k)dx - \sum_{k=2}^{n-1} x^{k-1} \int_{x}^{1} w(t, n - k)dx$$

$$= \int_{x}^{1} w(t, n - k)dx.$$
In the interval \([0, x_n]\) both functions are non-negative and \(\Delta_n(x_n) = 0\). According to (3) and (7), \(y(x, n)\) satisfies the equation

\[
y(x, n) = xy(x, n - 1) + \int_x^{x_n} y(t, n - 1) + \Delta_n(t) \, dt, \quad 0 \leq x \leq x_n, \tag{8}
\]

and \(y(x, n) = 0\), for \(x \geq x_n\). Also, notice that \(y(x, 1) = y(x, 2) = 0\) (because \(x_1 = x_2 = 0\)) and \(y(x, 3) = \int_x^{x_3} \Delta_3(t) \, dt\), where \(\Delta_3(x) = 1/2 - x - 5/2x^2\).

**Lemma 4.** Function \(\Delta_n(x)\) satisfies the equations

\[
\Delta_n(x) = x\Delta_{n-1}(x) + \int_x^1 w(t, n-1) \, dt - 1, \quad \Delta_1(x) = -1. \tag{9}
\]

\[
\Delta_{n+1}(x) - \Delta_n(x) = \sum_{j=1}^{n} \frac{x^{n-j} - x^n}{j} - x^n. \tag{10}
\]

**Proof.** (9) follows from Lemma 3 and the simple identity

\[
w(x, n) = xw(x, n - 1) + 1.
\]

From (9) we obtain

\[
\Delta_{n+1}(x) - \Delta_n(x) = x \left[ \Delta_n(x) - \Delta_{n-1}(x) \right] + \frac{1 - x^n}{n}.
\]

After \(n\) iterations we have

\[
\Delta_{n+1}(x) - \Delta_n(x) = \sum_{j=1}^{n} \frac{x^{n-j} - x^n}{j} + x^n[\Delta_1(x) - \Delta_0(x)].
\]

With \(\Delta_1(x) - \Delta_0(x) = -1\) it proves (10).

Differentiating (8) in \(x\) we obtain

\[
y'(x, n) = xy'(x, n - 1) - \Delta_n(x), \quad 0 \leq x \leq x_n,
\]

and, consequently, \(y'(x, 3) = -\Delta_3(x)\),

\[
y'(x, 4) = \begin{cases} -x\Delta_3(x) - \Delta_4(x), & \text{if } 0 \leq x \leq x_3 \\ -\Delta_4(x), & \text{if } x_3 < x \leq x_4 \end{cases}
\]

and

\[
y'(x, n) = -\sum_{j=i}^{n} x^{n-j}\Delta_j(x), \quad x_{i-1} \leq x \leq x_i, i = 3, 4, \ldots, n. \tag{11}
\]
In integral form (11) is

$$y(x, n) = \sum_{j=i}^{n} \int_{x}^{x_{j}} t^{n-j} \Delta_{j}(t) dt, \quad x_{i-1} \leq x \leq x_{i}, \ i = 3, 4, ..., n. \quad (12)$$

Letting in (12) $x = 0$ we obtain

Lemma 5.

$$y_{n} = y(0, n) = \sum_{j=3}^{n} \int_{0}^{x} j \ t^{n-j} \Delta_{j}(t) dt. \quad (13)$$

Consider now the difference $r_{n} = y_{n+1} - y_{n}$. Using (13) we can represent it as a sum of two expressions

$$r_{n} = y_{n+1} - y_{n} = \sum_{j=3}^{n} \int_{0}^{x_{j}} (\Delta_{j+1}(t) - \Delta_{j}(t)) dt + \sum_{j=3}^{n+1} \int_{x_{j-1}}^{x_{j}} t^{n-j+1} \Delta_{j}(t) dt \quad (14)$$

Using (10) the first sum in (14) can be rewritten in the form

$$\sum_{j=3}^{n} \int_{0}^{x_{j}} t^{n-j} \left[ \sum_{i=1}^{j} \left( \frac{t^{j-i+1}}{i} - \frac{x_{j}^{n-i+2}}{i} \right) - \frac{x_{j}^{n+1}}{n+1} \right] dt = \sum_{j=3}^{n} \left\{ \sum_{i=1}^{j} \left( \frac{x_{j}^{n-i+1}}{n-i+1} - \frac{x_{j+1}^{n+1}}{n+1} \right) \frac{1}{i} - \frac{x_{j}^{n+1}}{n+1} \right\}. \quad (15)$$

As $n \to \infty$ letting $x_{n} = 1 - z_{n}/n$ we obtain that (15) converges to the integral

$$C_{0} = \int_{0}^{1} e^{-\frac{z}{u}} \left[ \int_{0}^{u} \left( \frac{e^{\frac{zv}{u}} - 1}{v} + \frac{e^{\frac{zv}{u}}}{1-v} \right) dv - 1 \right] du. \quad (16)$$

Let us prove that the second sum in (14) tends to zero. Firstly notice that $\Delta_{j-1}(x) \leq 0$ for $x \geq x_{j-1}$, consequently, from Lemma 4 it follows that

$$\Delta_{j}(x) \leq \int_{x}^{1} w(t, j-1) dt - 1 \leq \sum_{i=1}^{j-1} \frac{1}{i}, \quad x \in [x_{j-1}, x_{j}].$$

Hence,

$$0 \leq \sum_{j=3}^{n+1} \int_{x_{j-1}}^{x_{j}} t^{n-j+1} \Delta_{j}(t) dt \leq \sum_{j=3}^{n+1} \sum_{i=1}^{j-1} \int_{x_{j-1}}^{x_{j}} \left( \frac{t^{n-j+1}}{i} \right) dt$$

$$= \sum_{i=1}^{n+1} \frac{1}{i} \sum_{j=i+1}^{n} \left[ \frac{x_{j}^{n-j+2} - x_{j-1}^{n-j+2}}{n-j+2} \right]$$
\[
\leq \sum_{i=1}^{n} \frac{1}{i} \sum_{j=i+1}^{n+1} x_j^{n-j+1}(x_j - x_{j-1})
\]
\[
= \sum_{i=1}^{n} \frac{1}{i} \sum_{j=i+1}^{n+1} (1 - \frac{z_j}{j})^{n-j+1}\left(\frac{z_{j-1}}{j-1} - \frac{z_j}{j}\right).
\tag{17}
\]

Using the inequality $1 - x \leq \exp(-x), \forall x$, (17) is not larger than
\[
\leq \exp\{Z\} \sum_{i=1}^{n} \frac{1}{i} \sum_{j=i+1}^{n+1} \left(\frac{z_j}{j}\right)^{n+1}\left(\frac{z_{j-1}}{j-1} - \frac{z_j}{j}\right),
\tag{18}
\]
where $Z = \sup z_j$. The last sum (18) converges as $n \to \infty$ to zero, because $z_n \to z$ and corresponding for internal sum integral $\int_{i+1}^{n+1} \exp\{-\frac{z(n+1)}{t}\}z/t^2dt$ is not larger than $\exp\{-z\}/(n+1)$.

Summarizing all, we obtain the next result.

**Theorem 1.** For large $n$

\[
\frac{V_n}{n} \to C_0,
\]
where $C_0$ is the value of the integral (16).

Proof follows immediately from $\lim_{n \to \infty} V_n/n = \lim_{n \to \infty} v(0, n)/n = \lim_{n \to \infty}[y_n + u(0, n + 1)]/n = \lim_{n \to \infty}[y_n + \sum_{j=1}^{n} 1/j]/n = C_0$. Package Mathematica calculates $C_0 \approx 0.4351708$.

**REFERENCES**
