<table>
<thead>
<tr>
<th>Title</th>
<th>PRECISE ASYMPTOTIC FORMULAS FOR NONLINEAR EIGENVALUE PROBLEMS (Variational Problems and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Shibata, Tetsutaro</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 2003年, 1307: 1-12</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2003-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/42828">http://hdl.handle.net/2433/42828</a></td>
</tr>
<tr>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
1. Introduction. We consider the following nonlinear two-parameter problem

\[-u''(x) + \lambda u(x)^q = \mu u(x)^p, \ x \in I = (0, 1),\]

\[u(x) > 0, \ x \in I,\]

\[u(0) = u(1) = 0,\]  

where \(1 < q < p\) and \(\lambda, \mu > 0\) are parameters.

The purpose of this paper is to establish the asymptotic formulas for the eigencurve \(\mu = \mu(\lambda)\) with the exact second term as \(\lambda \to \infty\) by using a variational method. We also establish the critical relationship between \(p\) and \(q\) from a point of view of the decaying rate of the second term of \(\mu(\lambda)\).

In Shibata [8], by using a standard variational framework (see Section 2), the variational eigencurve \(\mu = \mu(\lambda)\) was defined to analyze \(S_{\lambda, \mu}\) and the following asymptotic formula for \(\mu(\lambda)\) as \(\lambda \to \infty\) was established:

\[\mu(\lambda) = C_1 \lambda^{(p+3)/(2p-q+3)} + o(\lambda^{(p+3)/(2p-q+3)}),\]  

(1.2)
\[ C_1 = \left( \frac{(p + 1)(q + 3)}{(p + 3)(q + 1)} \frac{1}{\gamma^{p+1}} \frac{2}{p-q} \sqrt{\frac{\pi(q+1)}{2}} \left( \frac{p+1}{q+1} \right) \frac{q+3}{2(p-q)} \frac{\Gamma\left(\frac{q+3}{2(p-q)}\right)}{\Gamma\left(\frac{p+3}{2(p-q)}\right)} \right)^{\frac{q+3-p}{2p-q+3}}, \]

\[ \Gamma(r) = \int_0^\infty y^{r-1}e^{-y}dy \quad (r > 0). \]

(1.3)

By this formula, we understood the first term of \( \mu(\lambda) \) as \( \lambda \to \infty \). However, the remainder estimate of \( \mu(\lambda) \) has not been obtained. The purpose here is to obtain the exact second term of \( \mu(\lambda) \) as \( \lambda \to \infty \). We emphasize that the second term depends deeply on the relationship between \( p \) and \( q \), and the critical case is \( p = (3q - 1)/2 \). As far as the author knows, this kind of criticality is new for two-parameter problems and great interest by itself. Finally, it should be mentioned that the asymptotic behavior of such eigencurve is also effected by the variational framework (cf. [6, 7]).

2. Main Results. Let \( H_0^1(I) \) be the usual real Sobolev space. \( \|u\|_r \) denotes the usual \( L^r \)-norm. For \( u \in H_0^1(I) \)

\[ E_\lambda(u) := \frac{1}{2}\|u'\|_2^2 + \frac{1}{q+1} \lambda \|u\|_{q+1}^{q+1}, \]

\[ M_\gamma := \{u \in H_0^1(I) : \|u\|_{p+1} = \gamma\}, \]

where \( \gamma > 0 \) is a fixed constant. For a given \( \lambda > 0 \), we call \( \mu(\lambda) \) the variational eigenvalue when the following conditions (2.1)-(2.2) are satisfied:

\[ (\lambda, \mu(\lambda), u_\lambda) \in \mathbb{R}_+ \times \mathbb{R}_+ \times M_\gamma \ satisfies \ (1.1). \]

(2.1)

\[ E_\lambda(u_\lambda) = \inf_{u \in M_\gamma} E_\lambda(u). \]

(2.2)
Then $\mu(\lambda)$ is obtained as a Lagrange multiplier and is represented explicitly as follows:

$$
\mu(\lambda) = \frac{\|u'_\lambda\|_2^2 + \lambda\|u_\lambda\|_{q+1}^{q+1}}{\gamma^{p+1}}.
$$

(2.3)

The existence of $\mu(\lambda)$ for a given $\lambda > 0$ is ensured in [8, Theorem 2.1] and $\mu(\lambda)$ is continuous for $\lambda > 0$ (cf. [8, Theorem 2.2]). Finally, let

$$
K_1 := \left( 2 \left( \frac{q+1}{p+1} \right)^{(q-1)/(2(p-q))} \Gamma \left( \frac{1}{q+1} \right) \Gamma \left( \frac{q-1}{2(q+1)} \right) C_1^{(q-1)/(2(p-q))} \right)^{2(q+1)/(q-1)},
$$

$$
K_2 := \frac{1}{2} \int_0^1 \frac{s^{(2p-3q-1)/2}(1-s^{p+1})}{(1-s^{p-q})^{3/2}} ds,
$$

$$
K_3 := \frac{2^{2(p+2)/(q+1)}}{q+1} \int_0^1 \frac{y^{(2p-2q+2)/(q+1)}}{(1+y)^{2(p+2)/(q+1)}(1-y)^{(2p-2q+2)/(q+1)}} dy,
$$

$$
J_0 = \frac{\sqrt{\pi}}{p-q} \frac{q+3}{p+3} \frac{\Gamma \left( \frac{q+3}{2(p-q)} \right)}{\Gamma \left( \frac{p+3}{2(p-q)} \right)}.
$$

**Theorem 2.1.** (1) Assume $p > (3q-1)/2$. Then the following asymptotic formula holds as $\lambda \to \infty$:

$$
\mu(\lambda) = C_1 \lambda^{(p+3)/(2p-q+3)} \left\{ 1 + C_2 (1 + o(1)) \lambda^{-2(p+1)(q+1)/((2p-q+3)(q-1))} \right\},
$$

(2.4)

where

$$
C_2 = K_1 \left( 1 - \frac{2(p-q)K_2}{(2p-q+3)J_0} \right).
$$

(2) Assume $p < (3q-1)/2$. Then as $\lambda \to \infty$:

$$
\mu(\lambda) = C_1 \lambda^{(p+3)/(2p-q+3)} \left\{ 1 - C_3 (1 + o(1)) \lambda^{-(p+1)/(q-1)} \right\},
$$

(2.5)

where

$$
C_3 = \frac{2(p-q)}{(2p-q+3)J_0 K_1^{(2p-q+3)/(2(q+1))}}.
$$
(3) Assume $p = (3q - 1)/2$. Then as $\lambda \to \infty$:

$$\mu(\lambda) = C_1 \lambda^{(p+3)/(2p-q+3)} \left\{ 1 - C_4 (1 + o(1)) \lambda^{-2(p+1)(q+1)/((2p-q+3)(q-1))} \log \lambda \right\},$$

(2.6)

where

$$C_4 = \frac{2(p-q)(p+1)}{(q-1)(2p-q+3)^2 J_0} K_1.$$

The basic idea of the proof is as follows. Put

$$\nu(\lambda) = \lambda^{\frac{p-q}{2(p-q)}} \mu(\lambda)^{\frac{1-q}{2(p-q)}},$$

$$w_\lambda(t) = \left( \frac{\mu(\lambda)}{\lambda} \right)^{\frac{1}{p-q}} u_\lambda(x), \quad t = \nu(\lambda) \left( x - \frac{1}{2} \right).$$

(2.7)

Then it follows from (1.1) that $w_\lambda$ satisfies

$$-w_\lambda''(t) = w_\lambda(t)^p - w_\lambda(t)^q, \quad t \in I_{\nu(\lambda)} := \left(-\frac{1}{2} \nu(\lambda), \frac{1}{2} \nu(\lambda)\right),$$

$$w_\lambda(t) > 0, \quad t \in I_{\nu(\lambda)},$$

$$w_\lambda \left( \pm \frac{1}{2} \nu(\lambda) \right) = 0.$$

(2.8)

Then by [8, Lemma 5.1],

$$\nu(\lambda) \to \infty$$

(2.9)

as $\lambda \to \infty$. Put $z_\lambda = w_\lambda / \|w_\lambda\|_\infty$. Then it is easy to see from (2.3) that

$$\mu(\lambda) = \frac{\lambda^{(p+3)/(2(p-q))} \mu(\lambda)^{-(q+3)/(2(p-q))} (\|w_\lambda''\|_2^2 + \|w_\lambda\|_{q+1}^{q+1})}{\gamma^{p+1}} = \frac{\lambda^{(p+3)/(2(p-q))} \mu(\lambda)^{-(q+3)/(2(p-q))} \|w_\lambda\|_{p+1}^{p+1}}{\gamma^{p+1}} = \frac{\lambda^{(p+3)/(2(p-q))} \mu(\lambda)^{-(q+3)/(2(p-q))} \|w_\lambda\|_{\infty}^{p+1} \|z_\lambda\|_{p+1}^{p+1}}{\gamma^{p+1}}.$$

(2.10)

Therefore, it is crucial to study the asymptotic behavior of $\|w_\lambda\|_\infty$ and $\|z_\lambda\|_{p+1}$ as
3. Asymptotic behavior of $\|w_\lambda\|_\infty$. We put

$$\|w_\lambda\|_\infty = \left( \frac{p+1}{q+1}(1 + \epsilon(\lambda)) \right)^{1/(p-q)}.$$  \hfill (3.1)

Then by [8, (5.10), Lemma 5.2], we know that $\epsilon(\lambda) > 0$ and $\epsilon(\lambda) \to 0$ as $\lambda \to \infty$.

**Lemma 3.1.** The following equality holds for $\lambda > 0$:

$$\nu(\lambda) = \sqrt{2(q+1)} \left( \frac{p+1}{q+1}(1 + \epsilon(\lambda)) \right)^{-(q-1)/(2(p-q))} L(\epsilon(\lambda)),$$  \hfill (3.2)

where

$$L(\epsilon) = \int_0^1 \frac{1}{m(\epsilon, s)} \, ds,$$  \hfill (3.3)

$$m(\epsilon, s) = \sqrt{sq^{q+1} - sp^{p+1} + \epsilon(1 - s^{p+1})} \quad (\epsilon > 0).$$

**Proof.** Multiply the equation in (2.8) by $w'_\lambda$. Then for $t \in I_{\nu(\lambda)}$

$$\frac{d}{dt} \left( \frac{1}{2}(w'_\lambda(t))^2 + \frac{1}{p+1}w_\lambda(t)^{p+1} - \frac{1}{q+1}w_\lambda(t)^{q+1} \right) = 0.$$  

We know that $w_\lambda(0) = \|w_\lambda\|_\infty$ and $w'_\lambda(0) = 0$, since $u_\lambda(1/2) = \|u_\lambda\|_\infty$ and $u'_\lambda(1/2) = 0$. Then put $t = 0$ to obtain

$$\frac{1}{2}w'_\lambda(t)^2 + \frac{1}{p+1}w_\lambda(t)^{p+1} - \frac{1}{q+1}w_\lambda(t)^{q+1} \equiv \frac{1}{p+1}\|w_\lambda\|_{p+1}^p - \frac{1}{q+1}\|w_\lambda\|_{q+1}^q.$$  

Note that $w'_\lambda(t) < 0$ for $t \in (0, \nu(\lambda)/2)$, since $u'_\lambda(x) < 0$ for $x \in (1/2, 1)$. Then it follows from this and (3.1) that for $t \in (0, \nu(\lambda)/2)$

$$-z'_\lambda(t) = \|w_\lambda\|_{\infty}^{(q-1)/2} \sqrt{\frac{2}{q+1}} \sqrt{z_\lambda(t)^{q+1} - z_\lambda(t)^{p+1} + \epsilon(\lambda)(1 - z_\lambda(t)^{p+1})}$$  \hfill (3.4)

$$= \|w_\lambda\|_{\infty}^{(q-1)/2} \sqrt{\frac{2}{q+1}m(\epsilon(\lambda), z_\lambda(t))}.$$  

Put $s = z_\lambda$. Then (3.1) and (3.4) yield

$$\frac{\nu(\lambda)}{2} = \int_0^{\nu(\lambda)/2} \frac{-z'_\lambda(t)}{\sqrt{\frac{2}{q+1}} \|w_\lambda\|_{\infty}^{(q-1)/2} m(\epsilon(\lambda), z_\lambda(t))} \, dt$$

$$= \frac{\sqrt{q+1}}{2} \left( \frac{p+1}{q+1}(1 + \epsilon(\lambda)) \right)^{-(q-1)/(2(p-q))} \int_0^1 \frac{1}{m(\epsilon(\lambda), s)} \, ds.$$  

This implies (3.2). \hfill \square
Lemma 3.2. For $0 < \epsilon \ll 1$

\[
L(\epsilon) = \frac{\Gamma\left(\frac{1}{q+1}\right) \Gamma\left(\frac{q-1}{2(q+1)}\right)}{(q+1)\sqrt{\pi}} \epsilon^{-(q-1)/(2(q+1))} + o(\epsilon^{-(q-1)/(2(q+1))}).
\] (3.5)

Proof. Put

\[
L_1(\epsilon) := L(\epsilon) - \int_0^1 \frac{1}{\sqrt{s^{q+1}+\epsilon}} ds. 
\] (3.6)

Put $s = \epsilon^{1/(q+1)} \tan^{2/(q+1)} \theta$. Then

\[
\int_0^1 \frac{1}{\sqrt{s^{q+1}+\epsilon}} ds = \frac{2}{q+1} \epsilon^{-(q-1)/(2(q+1))} \int_0^{\tan^{-1}(1/\sqrt{\epsilon})} \sin^{-(q-1)/(q+1)} \theta \cos^{-2/(q+1)} \theta d\theta
\]

\[
= \frac{2}{q+1} (1+o(1)) \epsilon^{-(q-1)/(2(q+1))} \int_0^{\pi/2} \sin^{-(q-1)/(q+1)} \theta \cos^{-2/(q+1)} \theta d\theta 
= \frac{1}{q+1} (1+o(1)) \epsilon^{-(q-1)/(2(q+1))} \frac{\Gamma\left(\frac{1}{q+1}\right) \Gamma\left(\frac{q-1}{2(q+1)}\right)}{\sqrt{\pi}}.
\] (3.7)

Next, we calculate $L_1(\epsilon)$. Note that for $0 \leq s \leq 1$

\[
m(\epsilon, s) = \sqrt{s^{q+1}(1-s^{p-q}) + \epsilon(1-s^{p+1})} \geq \sqrt{(s^{q+1}+\epsilon)(1-s^{p-q})}. \]

(3.8)

By this, we obtain

\[|L_1(\epsilon)| = \int_0^1 \frac{(1+\epsilon)s^{p+1}}{m(\epsilon, s)\sqrt{s^{q+1}+\epsilon}(m(\epsilon, s)+\sqrt{s^{q+1}+\epsilon})} ds \leq (1+\epsilon) \int_0^1 \frac{s^{p+1}}{(s^{q+1}+\epsilon)^{3/2} \sqrt{1-s^{p-q}}(1+\sqrt{1-s^{p-q}})} ds \leq 2 \int_0^1 \frac{s^{p+1}}{(s^{q+1}+\epsilon)^{3/2} \sqrt{1-s^{p-q}}} ds \leq 2 \int_0^1 \frac{s^{p+1}}{(s^{q+1}+\epsilon)^{3/2} \sqrt{1-s^{p-q}}} ds + 2 \int_{\delta}^1 \frac{s^{p+1}}{(s^{q+1}+\epsilon)^{3/2} \sqrt{1-s^{p-q}}} ds := I + II,\]
where $0 < \delta \ll 1$ is a fixed constant. Let $C_{j,\delta} > 0$ ($j = 1, 2, \cdots$) be constants depending only on $\delta$. Put $s = \sin^{2/(p-q)} \theta$. Then

$$II \leq \frac{2}{\delta^{3(q+1)/2}} \int_{\delta}^{1} \frac{1}{\sqrt{1 - s^{p-q}}} ds$$

$$= \frac{2}{\delta^{3(q+1)/2}} \int_{\sin^{-1} \delta^{(p-q)/2}}^{1} \sin^{(2+q-p)/(p-q)} \theta d\theta$$

$$\leq C_{1,\delta}.$$ (3.10)

Moreover, put $s = \epsilon^{1/(q+1)} t$. Then for $0 < \epsilon \ll 1$

$$I \leq \frac{2}{\sqrt{1 - \delta^{p-q}}} \int_{0}^{\delta} \frac{\epsilon^{(p+1)/(q+1)} t^{p+1}}{t^{q+1} + 1} \frac{\epsilon^{1/(q+1)} dt}{\epsilon^{3/2}}$$

$$\leq 2 \frac{\delta^{p+1}}{\sqrt{1 - \delta^{p-q}}} \epsilon^{(2p-3q+1)/(2(q+1))} = o(\epsilon^{-(q-1)/(2(q+1))}).$$ (3.11)

By (3.9)–(3.11), we have

$$|L_1(\epsilon)| = o(\epsilon^{-(q-1)/(2(q+1))}).$$

By this, (3.6) and (3.7), we obtain (3.5). $\square$

**Lemma 3.3.** As $\lambda \to \infty$

$$\epsilon(\lambda) = K_1(1 + o(1)) \lambda^{-(2p+1)/(q+1)} \lambda^{-(q-1)/(2(q+1)(2p-q+1))}. \quad (3.12)$$

**Proof.** By (1.2) and (2.7), we have

$$\nu(\lambda) = \lambda^{(p-1)/(2(p-q))} \mu(\lambda)^{(1-q)/(2(p-q))}$$

$$= C_1(1 + o(1)) \lambda^{(p+1)/(2p-q+3)}.$$ (3.13)

On the other hand, by Lemmas 3.1–3.2 and Taylor expansion, we have

$$\nu(\lambda) = \sqrt{2(q+1)} \left(\frac{p+1}{q+1}\right)^{-(q-1)/(2(p-q))} (1 + \epsilon(\lambda))^{-(q-1)/(2(p-q))} L(\epsilon(\lambda))$$

$$= \sqrt{2} \left(\frac{p+1}{q+1}\right)^{-(q-1)/(2(p-q))} \frac{\Gamma \left(\frac{1}{q+1}\right) \Gamma \left(\frac{q-1}{2(q+1)}\right)}{\sqrt{\pi(q+1)}} \epsilon(\lambda)^{-(q-1)/(2(q+1))} (1 + o(1)).$$
By this and (3.13), we obtain (3.12).

4. **Asymptotic behavior of **$\|z_\lambda\|_{p+1}$**. By (3.4) and putting $s = z_\lambda(t)$, we have

$$
\|z_\lambda\|_{p+1}^{p+1} = 2 \int_0^{\nu(\lambda)/2} z_\lambda(t)^{p+1} dt
$$

$$
= 2 \int_0^{\nu(\lambda)/2} z_\lambda(t)^{p+1} \frac{-z_\lambda'(t)}{\|w_\lambda\|_{\infty}(q-1)/2 \sqrt{2/q+1} m(\epsilon(\lambda),z_\lambda(t))} dt
$$

$$
= \frac{\sqrt{2(q+1)}}{\|w_\lambda\|_{\infty}(q-1)/2} J(\epsilon(\lambda)),
$$

(4.1)

where

$$
J(\epsilon) := \int_0^1 \frac{s^{p+1}}{m(\epsilon,s)} ds \quad (\epsilon > 0).
$$

(4.2)

Therefore, we study the precise asymptotics of $J(\epsilon)$ as $\epsilon \to 0$. Put $s = \sin^{2/(p-q)} \theta$.

Then as $\epsilon \to 0$

$$
J(\epsilon) \to J(0) = \int_0^{\pi/2} \frac{s^{(2p-q+1)/2}}{\sqrt{1-s^{p-q}}} ds
$$

$$
= \frac{2}{p-q} \int_0^{\pi/2} \sin^{(p+3)/(p-q)} \theta d\theta
$$

$$
= \sqrt{\pi} \frac{q+3}{p-q} \frac{\Gamma\left(\frac{q+3}{2(p-q)}\right)}{\Gamma\left(\frac{p+3}{2(p-q)}\right)}
$$

$$
= J_0.
$$

(4.3)

We use here the formulas

$$
\int_0^{\pi/2} \sin^r \theta d\theta = \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{r}{2}+1\right)} \quad (r > -1),
$$

(4.4)

$$
\Gamma(r+1) = r\Gamma(r).
$$

Therefore, put

$$
J_1(\epsilon) := J(\epsilon) - J_0 := -\epsilon J_2(\epsilon),
$$

$$
J_2(\epsilon) := \int_0^1 \frac{s^{p+1}(1-s^{p+1})}{m(\epsilon,s)m(0,s)(m(\epsilon,s) + m(0,s))} ds.
$$

(4.5)

We study the asymptotic behavior of $J_2(\epsilon)$ as $\epsilon \to 0$. 
Lemma 4.1. (1) If \( p > (3q - 1)/2 \), then \( J_2(\epsilon) \rightarrow K_2 \) as \( \epsilon \rightarrow 0 \).

(2) If \( p < (3q - 1)/2 \), then as \( \epsilon \rightarrow 0 \)

\[
J_2(\epsilon) = K_3(1 + o(1))\epsilon^{(2p-3q+1)/(2(q+1))}.
\] (4.6)

(3) If \( p = (3q - 1)/2 \), then as \( \epsilon \rightarrow 0 \)

\[
J_2(\epsilon) = -\frac{1}{2(q+1)}(1 + o(1))\log \epsilon.
\] (4.7)

Proof. (1) Since \( p > (3q - 1)/2 \), we have \( (2p - 3q - 1)/2 > -1 \). Therefore, by Lebesgue's convergence theorem, as \( \epsilon \rightarrow 0 \)

\[
J_2(\epsilon) \rightarrow \frac{1}{2} \int_0^1 \frac{s^{(2p-3q-1)/2}(1-s^{p+1})}{(1-s^{p-q})^{3/2}} ds = K_2.
\]
This completes the proof.

(2) Step 1. Assume that \( p < (3q - 1)/2 \). We introduce \( J_3(\epsilon) \) to approximate \( J_2(\epsilon) \):

\[
J_3(\epsilon) := \int_0^1 \frac{s^{(2p-q+1)/2}}{\sqrt{s^{q+1} + \epsilon}(s^{(q+1)/2} + \sqrt{s^{q+1} + \epsilon})} ds
\]

\[= J_4(\epsilon, \delta) + J_5(\epsilon, \delta)
\]

\[
:= \int_0^\delta \frac{s^{(2p-q+1)/2}}{\sqrt{s^{q+1} + \epsilon}(s^{(q+1)/2} + \sqrt{s^{q+1} + \epsilon})} ds
+ \int_\delta^1 \frac{s^{(2p-q+1)/2}}{\sqrt{s^{q+1} + \epsilon}(s^{(q+1)/2} + \sqrt{s^{q+1} + \epsilon})} ds,
\] (4.8)

where \( 0 < \delta \ll 1 \) is a fixed small constant. We study the asymptotic behavior of \( J_3, J_4 \)
and \( J_5 \) as \( \epsilon \rightarrow 0 \). Note that \( 0 < (2p - 2q + 2)/(q+1) < 1 \), since \( p < (3q - 1)/2 \). Then put \( s = \epsilon^{1/(q+1)} \tan^{2/(q+1)} \theta \) and \( y = \tan(\theta/2) \) to obtain

\[
J_3(\epsilon) = 2^{q+1}\epsilon^{(2p-3q+1)/(2(q+1))} \int_0^{\tan^{-1}\sqrt{\epsilon}} \frac{\tan^{(2p-2q+2)/(q+1)} \theta}{1 + \sin \theta} d\theta
\]

\[= K_3(1 + o(1))\epsilon^{(2p-3q+1)/(2(q+1))}.
\] (4.9)
Similarly, we obtain
\[ J_4(\epsilon, \delta) = K_3(1 + o(1))\epsilon^{(2p-3q+1)/(2(q+1))}, \quad J_5(\epsilon, \delta) \leq \frac{1}{\delta^{q+1}}. \tag{4.10} \]
Since \( p < (3q - 1)/2 \), this along with (4.9) implies that \( J_3(\epsilon)/J_4(\epsilon, \delta) \to 1 \) as \( \epsilon \to 0 \) for a fixed \( \delta \).

**Step 2.** We show that as \( \epsilon \to 0 \)
\[ \frac{J_2(\epsilon)}{J_3(\epsilon)} \to 1. \tag{4.11} \]
Let an arbitrary \( 0 < \delta \ll 1 \) be fixed. Put
\[ J_2(\epsilon) = J_6(\epsilon, \delta) + J_7(\epsilon, \delta) \]
\[ := \int_0^\delta \frac{s^{p+1}(1-s^{p+1})}{m(\epsilon,s)m(0,s)(m(\epsilon,s)+m(0,s))}ds \tag{4.12} \]
\[ + \int_\delta^1 \frac{s^{p+1}(1-s^{p+1})}{m(\epsilon,s)m(0,s)(m(\epsilon,s)+m(0,s))}ds. \]
Then for \( 0 < \epsilon \ll 1 \)
\[ |J_7(\epsilon, \delta)| \leq C_{2,\delta}\int_\delta^1 \frac{1-s^{p+1}}{(1-s^{p-q})^{3/2}}ds \leq C_{3,\delta}. \tag{4.13} \]
Moreover, by (3.8), we obtain
\[ (1 - \delta^{p+1})\int_0^\delta \frac{s^{(2p-q+1)/2}}{\sqrt{s^{q+1}} + \epsilon(s^{(q+1)/2} + \sqrt{s^{q+1}} + \epsilon)}ds \leq J_6(\epsilon, \delta) \]
\[ \leq \frac{1}{(1 - \delta^{p-q})^{3/2}}\int_0^\delta \frac{s^{(2p-q+1)/2}}{\sqrt{s^{q+1}} + \epsilon(s^{(q+1)/2} + \sqrt{s^{q+1}} + \epsilon)}ds. \]
This implies
\[ (1 - \delta^{p+1})J_4(\epsilon, \delta) \leq J_6(\epsilon, \delta) \leq \frac{1}{(1 - \delta^{p-q})^{3/2}}J_4(\epsilon, \delta). \tag{4.14} \]
By (4.10), (4.13) and (4.14), we see that \( J_7(\epsilon, \delta) = o(J_6(\epsilon, \delta)) \) as \( \epsilon \to 0 \) for a fixed \( \delta \), since \( p < (3q - 1)/2 \). Then by (4.9), (4.10) and (4.12)–(4.14),
\[ (1 - \delta^{p+1}) \leq \liminf_{\epsilon \to 0} \frac{J_6(\epsilon, \delta)}{J_4(\epsilon, \delta)} = \liminf_{\epsilon \to 0} \frac{J_2(\epsilon)}{J_3(\epsilon)} \leq \limsup_{\epsilon \to 0} \frac{J_2(\epsilon)}{J_3(\epsilon)} \]
\[ = \limsup_{\epsilon \to 0} \frac{J_6(\epsilon, \delta)}{J_4(\epsilon, \delta)} \leq \frac{1}{(1 - \delta^{p-q})^{3/2}}. \tag{4.15} \]
By letting $\delta \to 0$, we obtain (4.11). Then by (4.9) and (4.11), we obtain (4.6).

(3) If $p = (3q - 1)/2$, then by the asymptotic formula

$$\tan^{-1}x = \frac{\pi}{2} - \frac{1}{x} + O\left(\frac{1}{x^3}\right) \quad (x \gg 1),$$

and Taylor expansion of $\tan x$ at $x = \pi/4$ and (4.9), we obtain (4.7) by direct calculation. $\square$

5. Proof of Theorem 2.1. By (2.10), (3.1), (4.1) and (4.5), we have

$$\mu(\lambda)^{(2p-q+3)/(2(p-q))} = \frac{\sqrt{2(q+1)}}{\gamma^{p+1}}\lambda^{(p+3)/(2(p-q))}\|w_{\lambda}\|_{\infty}^{(2p-q+3)/2}J(\epsilon(\lambda))$$

$$= \frac{\sqrt{2(q+1)}}{\gamma^{p+1}}\lambda^{(p+3)/(2(p-q))}\left(\frac{p+1}{q+1}\right)^{(2p-q+3)/(2(p-q))}$$

$$\times (1 + \epsilon(\lambda))^{(2p-q+3)/(2(p-q))}(J_0 - \epsilon(\lambda)J_2(\epsilon(\lambda))).$$

Moreover, it is easy to check that

$$\left(\frac{\sqrt{2(q+1)}}{\gamma^{p+1}}\right)^{2(p-q)/(2p-q+3)}\frac{p+1}{q+1}J_0^{2(p-q)/(2p-q+3)} = C_1.$$

By this, (5.1) and Taylor expansion, we obtain

$$\mu(\lambda) = C_1\lambda^{(p+3)/(2(p-q)+3)}$$

$$\times \left(1 + \epsilon(\lambda) - \frac{2(p-q)}{(2p-q+3)J_0}(1 + o(1))\epsilon(\lambda)J_2(\epsilon(\lambda))\right).$$

Then by Lemma 3.3, Lemma 4.1 and direct calculation, we obtain Theorem 2.1. Thus the proof is complete. $\square$

REFERENCES