LOCATION OF BLOW UP POINTS
OF LEAST ENERGY SOLUTIONS
TO THE BREZIS-NIRENBERG EQUATION

FUTOSHI TAKAHASHI (高橋太)
Tokyo National College of Technology
（東京工業高等専門学校 非常勤）

1. Introduction. Let Ω be a smooth bounded domain in \( \mathbb{R}^N \), \( N \geq 4 \) and \( p = \frac{N+2}{N-2} \). In this article, we return to the well-studied problem \((P_\epsilon)\):

\[
\begin{aligned}
-\Delta u &= u^p + \epsilon u & \text{in } \Omega, \\
u &> 0 & \text{in } \Omega, \\
u|_{\partial \Omega} &= 0,
\end{aligned}
\]

where \( \epsilon > 0 \) is a parameter.

The exponent \( p \) is called the critical Sobolev exponent in the sense that the Sobolev embedding \( H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega) \) is continuous but not compact. So from the variational viewpoint, this problem belongs to the limit case of the Palais-Smale compactness condition, and the classical arguments do not apply to the questions related to the existence or nonexistence and multiplicity of solutions of this problem.

In pioneering work [3], Brezis and Nirenberg proved that, in spite of possible failure of the Palais-Smale compactness condition, \((P_\epsilon)\) has at least one non-trivial solution on a general bounded domain \( \Omega \) when \( \epsilon \in (0, \lambda_1) \), where \( \lambda_1 \) denotes the first eigenvalue of \(-\Delta\) with Dirichlet boundary condition.

On the other hand when \( \epsilon = 0 \), it is known that problem \((P_0)\) reflects the topology and the geometry of the domain \( \Omega \). Pohozaev showed that if \( \Omega \) is star-shaped, then \((P_0)\) has no non-trivial solutions [7]. In other cases Bahri and Coron [1] proved that \((P_0)\) has a solution when \( \Omega \) has non-trivial topology in the sense that \( H_d(\Omega, \mathbb{Z}_2) \neq \{0\} \) for some positive integer \( d \), where \( H_d(\Omega, \mathbb{Z}_2) \) denotes the \( d \)-th homology group of \( \Omega \) with \( \mathbb{Z}_2 \) coefficients. Furthermore Ding [5] and Passaseo [8] proved that even if \( \Omega \) is contractible, \((P_0)\) can still have a solution if the geometry of \( \Omega \) is non-trivial in some sense.

Because of the different nature of the problem when \( \epsilon > 0 \) and \( \epsilon = 0 \), it is interesting to study the asymptotic behavior of solutions \( u_\epsilon \) of \((P_\epsilon)\) as \( \epsilon \to 0 \). In this direction, Han [9] and Rey [12][13] proved independently the following result, which had been conjectured previously by Brezis and Peletier [4].
Theorem 0. (Han [9], Rey [12]) Let $u_{\epsilon}$ be a solution of problem $(P_{\epsilon})$ and assume
\[
\frac{\int_{\Omega} |\nabla u_{\epsilon}|^2 dx}{(\int_{\Omega} |u_{\epsilon}|^{p+1} dx)^{\frac{2}{p+1}}} = S + o(1) \quad \text{as } \epsilon \to 0,
\]
where $S$ is the best Sobolev constant in $\mathbb{R}^N$:
\[
S = \pi N(N-2) \left( \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)} \right)^{\frac{2}{N}}.
\]
Then we have (after passing to a subsequence):

1. There exists $a_{\infty} \in \Omega$ (interior point) such that
   \[
   |\nabla u_{\epsilon}|^2 \rightharpoonup S^\frac{N}{2} \delta_{a_{\infty}} \quad \text{as } \epsilon \to 0
   \]
   in the sense of Radon measures of the compact space $\overline{\Omega}$, where $\delta_a$ is the Dirac measure supported by $a \in \mathbb{R}^N$.

2. The $a_{\infty}$ above is a critical point of the (positive) Robin function $H(a,a)$ on $\Omega$:
   \[
   \nabla_a H(a_{\infty}, a_{\infty}) = 0,
   \]
   where $H(x,a)$ is the regular part of the Green's function $G(x,a)$:
   \[
   H(x,a) := \frac{1}{(N-2)\omega_N} |x-a|^{2-N} - G(x,a),
   \]
   in which $\omega_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}$ is the $(N-1)$ dimensional volume of $S^{N-1}$ and
   \[
   \begin{cases}
   -\Delta_x G(x,a) = \delta_a(x), & x \in \Omega, \\
   G(x,a)|_{x \in \partial\Omega} = 0.
   \end{cases}
   \]

3. We have an exact blow up rate of the $L^\infty$-norm of $u_{\epsilon}$ as $\epsilon \to 0$:
   \[
   \lim_{\epsilon \to 0} \epsilon \|u_{\epsilon}\|_{L^\infty(\Omega)} = \left( N(N-2) \right)^{\frac{N-4}{2}} \frac{(N-2)^3 \omega_N}{2C_N} H(a_{\infty}, a_{\infty}), \quad \text{if } N \geq 5,
   \]
   \[
   \lim_{\epsilon \to 0} \epsilon \log \|u_{\epsilon}\|_{L^\infty(\Omega)} = 4\omega_4 H(a_{\infty}, a_{\infty}), \quad \text{if } N = 4,
   \]
   where
   \[
   C_N = \int_0^\infty \frac{s^{N-1}}{(1+s^2)^{N-2}} ds = \frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N-4}{2}\right)}{2\Gamma(N-2)}.
   \]
In this article, we restrict our attention to a particular family of solutions to \((P_\epsilon)\), namely the solutions \((\overline{u}_\epsilon)_{\epsilon \in (0, \lambda_1)}\) obtained by the method of Brezis and Nirenberg. We call \((\overline{u}_\epsilon)\) the least energy solutions to the problem \((P_\epsilon)\).

Before stating our main result, we recall the construction of least energy solutions by Brezis and Nirenberg.

For \(\epsilon \in (0, \lambda_1)\), define

\[
S_\epsilon := \inf_{u \in H^1_0(\Omega)} \left\{ \int_\Omega |\nabla u|^2 dx - \epsilon \int_\Omega u^2 dx \right\}. \tag{1.1}
\]

Since the constraint on \(\|u\|_{L^{p+1}(\Omega)}\) is not preserved under weak convergence in \(H^1_0(\Omega)\), it is not obvious that \(S_\epsilon\) is achieved or not. By using the fact that \(S_\epsilon < S\) if \(\epsilon > 0\), Brezis-Nirenberg proved that any minimizing sequence for (1.1) is compact in \(H^1_0(\Omega)\) and (1.1) is achieved by some positive function \(u^0_\epsilon \in H^1_0(\Omega)\). Furthermore if \(\epsilon < \lambda_1\), then it follows \(S_\epsilon > 0\) and

\[
\overline{u}_\epsilon := S_\epsilon^{-\frac{N-2}{4}} u^0_\epsilon \tag{1.2}
\]
is a solution to \((P_\epsilon)\).

By Global Compactness Theorem of Struwe [14], we know that the least energy solutions \(\overline{u}_\epsilon\) blow up at exactly one point in \(\Omega\) as \(\epsilon \to 0\). That is, there exist \(\lambda_\epsilon > 0\) with \(\lambda_\epsilon \to 0\) \((\epsilon \to 0)\) and \(a_\epsilon \in \Omega\) with \(\lambda_\epsilon / \text{dist}(a_\epsilon, \partial\Omega) \to 0\) \((\epsilon \to 0)\) such that

\[
\|\nabla (\overline{u}_\epsilon - \alpha_N PU_{\lambda_\epsilon, a_\epsilon})\|_{L^2(\Omega)} \to 0 \quad \text{as} \quad \epsilon \to 0, \tag{1.3}
\]

where \(\alpha_N = (N(N-2))^{-\frac{N-2}{4}}\).

Here for \(\lambda > 0\) and \(a \in \Omega\), we define

\[
U_{\lambda, a}(x) := \left( \frac{\lambda}{\lambda^2 + |x-a|^2} \right)^{\frac{N-2}{2}}, \quad x \in \mathbb{R}^N \tag{1.4}
\]

and \(PU_{\lambda, a} := U_{\lambda, a} - \varphi_{\lambda, a} \in H^1_0(\Omega)\), where \(\varphi_{\lambda, a}\) is the harmonic extension of \(U_{\lambda, a}|_{\partial\Omega}\) to \(\Omega\):

\[
\left\{ \begin{array}{l}
-\Delta \varphi_{\lambda, a} = 0 \text{ in } \Omega, \\
\varphi_{\lambda, a}|_{\partial\Omega} = U_{\lambda, a}|_{\partial\Omega}.
\end{array} \right. \tag{1.5}
\]

We call any accumulation point of \((a_\epsilon)_{\epsilon > 0}\) a blow up point of \((\overline{u}_\epsilon)\). Note that if \(a_\infty \in \overline{\Omega}\) is a blow up point of \((\overline{u}_\epsilon)_{\epsilon > 0}\), then by passing to a subsequence,
we see $|\nabla u_{\varepsilon}|^{2} \rightarrow S^{N}_{\delta_{a_{\infty}}} \delta_{a_{\infty}}$ as $\varepsilon \to 0$, and by construction, $(\bar{u}_{\varepsilon})$ is a minimizing sequence for the best Sobolev constant. So from the result of Han and Rey, we know that $a_{\infty} \in \Omega$ (interior point) and $a_{\infty}$ is a critical point of the Robin function on $\Omega$.

Our main result is to further locate the blow up point $a_{\infty}$ of the least energy solutions on a general bounded domain $\Omega$ in $\mathbb{R}^N$, $N \geq 4$.

**Theorem 1.** Let $a_{\infty}$ be a blow up point of the least energy solutions $(\bar{u}_{\varepsilon})$ obtained by the method of Brezis and Nirenberg. Then $a_{\infty}$ is a minimum point of the Robin function of $\Omega$:

$$H(a_{\infty}, a_{\infty}) = \inf_{a \in \Omega} H(a, a).$$

To prove Theorem 1, we will make a precise asymptotic expansion of the value $S_{\varepsilon}$ as $\varepsilon \to 0$. For this purpose, we combine the method developed by Isobe [10] [11] and technical calculations in Rey [12] [13]. As a by-product of our method, we prove that the blow up point is the interior point of $\Omega$ by using only an energy comparison argument. Also we can give another explanation of the exact blow up rate of $L^{\infty}$-norm of $\bar{u}_{\varepsilon}$ along the line of our context.

Wei [15] treated the subcritical problem:

$$\begin{cases}
-\Delta u = u^{p_{\varepsilon}} & \text{in } \Omega, \\
u > 0 & \text{in } \Omega, \\
u|_{\partial \Omega} = 0
\end{cases}$$

where $\varepsilon > 0$, and he proved that as $\varepsilon \to 0$, the least energy solutions to this problem blow up at exactly one point, and the blow up point is a minimum point of the Robin function. His method is the usual blow-up (rescaling) technique and he obtained a second order expansion of the rescaled function, which leads to an asymptotic expansion as $\varepsilon \to 0$ of the value

$$\inf_{u \in H^{1}_{0}(\Omega)} \left\{ \int_{\Omega} |\nabla u|^{2} dx \right\} \quad \text{subject to } \|u\|_{L^{p+1-\varepsilon}(\Omega)} = 1.$$

In the course of the proof, he used the result of Han and Rey, and a crucial pointwise estimate obtained by Han for the rescaled function.

We might follow the method of Wei to study the problem $(P_{\varepsilon})$ when $N \geq 5$, but even in this case, I believe that our method is more consistent and somewhat simpler because we do not need any use of Pohozaev identity, Kelvin transformation and Gidas-Ni-Nirenberg theory. See also [6].
2. Asymptotic behavior of $S_{\epsilon}$. In this section, we obtain an asymptotic formula of the value $S_{\epsilon}$ as $\epsilon \to 0$ and derive the suitable upper bound for $S_{\epsilon}$. See Lemma 2.5 and Lemma 2.7.

For $\epsilon \in (0, \lambda_1)$, let $v_{\epsilon}^0 \in H_0^1(\Omega)$ be a solution to the minimization problem (1.1).

Define
\[ v_{\epsilon} := S^\frac{N-2}{4} v_{\epsilon}^0. \] (2.1)

Then (1.2), (1.3) and $S_{\epsilon} = S + o(1)$ as $\epsilon \to 0$ imply
\begin{align*}
\| \nabla (v_{\epsilon} - \alpha PU_{\lambda_1 a}) \|_{L^2(\Omega)} & \to 0 \text{ as } \epsilon \to 0, \quad (2.2) \\
\int_{\Omega} v_{\epsilon}^{p+1} \, dx & = S^\frac{N}{2}. \quad (2.3)
\end{align*}

Define for $\eta > 0$,
\[ M(\eta) := \left\{ v \in H_0^1(\Omega) : \begin{array}{llll} 
\exists \alpha > 0, |\alpha - \alpha_N| < \eta, \exists a \in \Omega, \exists \lambda > 0 \\
\text{with } \lambda/d(a, \partial\Omega) < \eta \\
\text{such that } \| \nabla (v - \alpha PU_{\lambda_1 a}) \|_{L^2(\Omega)} < \eta.
\end{array} \right\} \]
where $d(a, \partial\Omega) = \text{dist}(a, \partial\Omega)$.

It is proved in [1]: Proposition 7, that for $v \in M(\eta)$ and $\eta > 0$ small enough, the minimization problem:

Minimize
\[ \left\{ \| \nabla (v - \alpha PU_{\lambda a}) \|_{L^2(\Omega)} : \begin{array}{llll} 
\alpha \in (\alpha_N - 2\eta, \alpha_N + 2\eta), \\
\lambda > 0, a \in \Omega, \\
\lambda/d(a, \partial\Omega) < 2\eta.
\end{array} \right\} \] (2.4)

has a unique solution $(\alpha^0, \lambda^0, a^0) \in (\alpha_N - 2\eta, \alpha_N + 2\eta) \times \mathbb{R}_+ \times \Omega$.

Let $a_{\infty} \in \overline{\Omega}$ be a blow up point of $(\bar{u}_\epsilon)_{\epsilon > 0}$. By definition of the blow up point, there exist $\epsilon_n \to 0, \lambda_n \to 0, \Omega \ni a_n \to a_{\infty}$ such that $(v_n := v_{\epsilon_n}, d_n := d(a_n, \partial\Omega))$

\[ \| \nabla (v_n - \alpha_N PU_{\lambda_n a_n}) \|_{L^2(\Omega)} \to 0, \quad \lambda_n/d_n \to 0 (n \to \infty). \] (2.5)

(2.5) implies there exists $\eta_n \to 0$ such that $v_n \in M(\eta_n)$. We denote the unique solution $(\alpha^0_n, \lambda^0_n, a^0_n)$ to (2.4) for $v = v_n, \eta = \eta_n$ again by $(\alpha_n, \lambda_n, a_n)$.

Then by our choice of $(\alpha_n, \lambda_n, a_n)$, if we write
\[ v_n = \alpha_n PU_{\lambda_n a_n} + w_n, \quad w_n \in H_0^1(\Omega), \] (2.6)
it follows that
\[ \alpha_n \to \alpha_N = (N(N-2))^{N-2}, \quad a_n \to a_\infty, \]
\[ \frac{\lambda_n}{d_n} \to 0 \quad \text{where} \quad d_n = \text{dist}(a_n, \partial \Omega), \]
\[ w_n \in E_{\lambda_n, a_n}, \quad w_n \to 0 \quad \text{in} \quad H_0^1(\Omega) \]
(2.7)
as \( n \to \infty \). Here for \( \lambda > 0 \) and \( a \in \Omega \),
\[ E_{\lambda, a} := \{ w \in H_0^1(\Omega) : 0 = \int_\Omega \nabla w \cdot \nabla PU_{\lambda, a} \, dx \} \]
(2.8)
In the following, we estimate
\[ J_n := \int_\Omega |\nabla v_n|^2 \, dx - \epsilon_n \int_\Omega v_n^2 \, dx \]
(2.9)
by using the expression (2.6).

**Lemma 2.1.** (Asymptotic behavior of \( H_0^1 \) norm of the main part)
As \( n \to \infty \), we have
\[ \int_\Omega |\nabla PU_{\lambda_n, a_n}|^2 \, dx = N(N-2)A - (N-2)^2 \omega_N^2 H(a_n, a_n) \lambda_n^{N-2} + O\left(\frac{\lambda_n^N}{d_n^N} \log\left(\frac{\lambda_n}{d_n}\right)\right), \]
where
\[ A = \int_{\mathbb{R}^N} U_{\lambda_n, a_n}^{p+1} \, dx = \frac{\Gamma(N/2)}{\Gamma(N)} \pi^{N/2}. \]
**Proof.** We have
\[ \int_\Omega |\nabla PU_{\lambda_n, a_n}|^2 \, dx = \int_\Omega -\Delta PU_{\lambda_n, a_n} \cdot PU_{\lambda_n, a_n} \, dx \]
\[ = N(N-2) \int_\Omega U_{\lambda_n, a_n}^p \cdot (U_{\lambda_n, a_n} - \varphi_{\lambda_n, a_n}) \, dx \]
\[ = N(N-2) \int_\Omega U_{\lambda_n, a_n}^{p+1} \, dx - N(N-2) \int_\Omega U_{\lambda_n, a_n}^p \varphi_{\lambda_n, a_n} \, dx \]
\[ =: N(N-2)I_1 - N(N-2)I_2. \]
(2.10)
Here we have used the fact that $PU_{\lambda_n, a_n} \in H^1_0(\Omega)$ satisfies the equation

$$-\Delta PU_{\lambda_n, a_n} = N(N-2)U^p_{\lambda_n, a_n} \quad \text{in} \ \Omega. \quad (2.11)$$

Now,

$$I_1 = \int_{\Omega} U^{p+1}_{\lambda_n, a_n} dx = \int_{\mathbb{R}^N} U^{p+1}_{\lambda_n, a_n} dx - \int_{\mathbb{R}^N \setminus \Omega} U^{p+1}_{\lambda_n, a_n} dx$$

$$= A + O \left( \int_{\mathbb{R}^N \setminus B_{d_n}(a_n)} U^{p+1}_{\lambda_n, a_n} dx \right)$$

$$= A + O \left( \lambda_n^N \int_{r=d_n}^{r=\infty} \frac{r^{N-1}}{(\lambda_n^2 + r^2)^N} dr \right) \quad (r = |x - a_n|)$$

$$= A + O \left( \frac{\lambda_n^N}{d_n^N} \right). \quad (2.12)$$

We divide $I_2$ in the second term of (2.10) as

$$I_2 = \int_{\Omega} U^p_{\lambda_n, a_n} \varphi_{\lambda_n, a_n} dx$$

$$= \int_{\Omega \setminus B_{d_n/2}(a_n)} U^p_{\lambda_n, a_n} \varphi_{\lambda_n, a_n} dx + \int_{B_{d_n/2}(a_n)} U^p_{\lambda_n, a_n} \varphi_{\lambda_n, a_n} dx$$

$$=: I_2^1 + I_2^2. \quad (2.13)$$

Now,

$$I_2^1 = \int_{\Omega \setminus B_{d_n/2}(a_n)} U^p_{\lambda_n, a_n} \varphi_{\lambda_n, a_n} dx$$

$$= O \left( \| \varphi_{\lambda_n, a_n} \|_{L^\infty(\Omega)} \int_{\Omega \setminus B_{d_n/2}(a_n)} U^p_{\lambda_n, a_n} dx \right)$$

$$= O \left( \lambda_n^{N-2} \cdot \frac{N+2}{d_n^{N-2}} \cdot \frac{N-1}{d_n^{N-2}} \int_{r=d_n/2}^{r=\infty} \frac{r^{N-1}}{(\lambda_n^2 + r^2)^{N+2}} dr \right)$$

$$= O \left( \frac{\lambda_n^N}{d_n^N} \right). \quad (2.14)$$

Here, we have used the estimate

$$\| \varphi_{\lambda_n, a_n} \|_{L^\infty(\Omega)} = O \left( \frac{\lambda_n^{N-2}}{d_n^{N-2}} \right), \quad (2.15)$$
which is a consequence of (1.5) and the maximum principle of harmonic functions.

In calculating $I_2^2$, we make a Taylor expansion of $\varphi_{\lambda_n,a_n}$ on $B_{d_n/2}(a_n)$:

$$
\varphi_{\lambda_n,a_n} = \varphi_{\lambda_n,a_n}(a_n) + \nabla \varphi_{\lambda_n,a_n}(a_n) \cdot (x - a_n) + O \left( \| \nabla^2 \varphi_{\lambda_n,a_n} \|_{L^\infty(B_{d_n/2}(a_n))} |x - a_n|^2 \right).
$$

Note that we have

$$
\varphi_{\lambda_n,a_n}(a_n) = (N - 2)\omega_N \lambda_{n}^{\frac{N-2}{2}} H(a_n, a_n) + O \left( \frac{\lambda_n^{N+2}}{d_n^N} \right) \quad (2.16)
$$

by [13]; Proposition 1, and

$$
\| \nabla^2 \varphi_{\lambda_n,a_n} \|_{L^\infty(B_{d_n/2}(a_n))} = O \left( \frac{\lambda_n^{N-2}}{d_n^N} \right) \quad (2.17)
$$

by the elliptic estimate $d_n^k \| \nabla^k \varphi_{\lambda_n,a_n} \|_{L^\infty(B_{d_n/2}(a_n))} \leq \| \varphi_{\lambda_n,a_n} \|_{L^\infty(\Omega)}$ ($k \in \mathbb{N}$) for a harmonic function $\varphi_{\lambda_n,a_n}$.

Then by (2.16), (2.17) and the oddness of the integral, we calculate:

$$
I_2^2 = \int_{B_{d_n/2}(a_n)} U_{\lambda_n,a_n}^p \varphi_{\lambda_n,a_n} dx
$$

$$
= \int_{B_{d_n/2}(a_n)} U_{\lambda_n,a_n}^p \varphi_{\lambda_n,a_n}(a_n) dx
$$

$$
+ \int_{B_{d_n/2}(a_n)} U_{\lambda_n,a_n}^p \nabla \varphi_{\lambda_n,a_n}(a_n) \cdot (x - a_n) dx
$$

$$
+ \int_{B_{d_n/2}(a_n)} U_{\lambda_n,a_n}^p \cdot O(\| \nabla^2 \varphi_{\lambda_n,a_n} \|_{L^\infty(B_{d_n/2}(a_n))} |x - a_n|^2) dx
$$

$$
= \{(N - 2)\omega_N \lambda_{n}^{\frac{N-2}{2}} H(a_n, a_n) + O \left( \frac{\lambda_n^{N+2}}{d_n^N} \right)\} \int_{B_{d_n/2}(a_n)} U_{\lambda_n,a_n}^p dx + 0
$$

$$
+ O \left( \frac{\lambda_n^{N-2}}{d_n^N} \int_{B_{d_n/2}(a_n)} U_{\lambda_n,a_n}^p |x - a_n|^2 dx \right)
$$

$$
= \left( \frac{N-2}{N} \right)\omega_N \lambda_{n}^{N-2} H(a_n, a_n) + O \left( \frac{\lambda_n^N}{d_n^N} \right) + O \left( \frac{\lambda_n^N}{d_n^N} |\log(\frac{\lambda_n}{d_n})| \right). \quad (2.18)
$$

Here in the last equality, we have used the estimates

$$
\int_{B_{d_n/2}(a_n)} U_{\lambda_n,a_n}^p dx = \omega_N \int_0^{d_n/2} \left( \frac{\lambda_n}{\lambda_n^2 + r^2} \right)^{N+2} r^{N-1} dr
$$
\[
\omega_N \lambda_{n}^{\frac{N-2}{n^2}} \int_{0}^{d_n/2\lambda_{n}} \frac{s^{N-1}}{(1+s^2)^{\frac{N+2}{2}}}ds = \omega_N \lambda_{n}^{\frac{N-2}{n^2}} \left( \int_{0}^{\infty} - \int_{d_n/2\lambda_{n}}^{\infty} \right)
\]
\[
= \frac{\omega_N}{N} \lambda_{n}^{\frac{N+2}{n^2}} + O\left( \frac{\lambda_{n}^{\frac{N}{n} \log \lambda_{n}}}{d_n^2} \right)
\] (2.19)

\[
\int_{B_{d_n/2}(a_n)} U_{\lambda_{n},a_{n}}^{2} O(|x-a_{n}|^2)dx = O\left( \lambda_{n}^{\frac{N}{n} \log \lambda_{n}} \int_{0}^{d_n/2\lambda_{n}} \frac{s^{N+1}}{(1+s^2)^{N-2}}ds \right)
\]
\[
= o\left( \lambda_{n}^{\frac{N}{n} \log \lambda_{n}} \int_{0}^{d_n/2\lambda_{n}} \frac{s^{N+1}}{(1+s^2)^{N-2}}ds \right)
\] (2.20)

and the estimate of the Robin function:
\[
H(a_{n}, a_{n}) = \frac{1}{(N-2)\omega_N} \left( \frac{1}{2d_n} \right)^{N-2} + o\left( \frac{1}{d_n^{N-2}} \right) \text{ as } d_n \to 0
\] (2.21)

(see [13]: (2.8)).

(2.19) is a consequence of
\[
\int_{0}^{\infty} \frac{s^{N-1}}{(1+s^2)^{N-2}}ds = \frac{\Gamma\left( \frac{N}{2} \right) \Gamma\left( \frac{N-4}{2} \right)}{2 \Gamma\left( \frac{N+2}{2} \right)} = \frac{1}{N},
\]
where we used the formula
\[
\int_{0}^{\infty} \frac{s^{\alpha}}{(1+s^2)^{\beta}}ds = \frac{\Gamma\left( \frac{1+\alpha}{2} \right) \Gamma\left( \frac{2\beta-\alpha-1}{2} \right)}{2 \Gamma(\beta)}
\] (2.22)

for \( \alpha > 0, \beta > 0 \) and \( 2\beta - \alpha - 1 > 0 \).

From (2.10)-(2.18), we obtain the conclusion of Lemma 2.1. \( \square \)

**Lemma 2.2. (Asymptotic behavior of \( L^2 \) norm of the main part)**

When \( N \geq 5 \), we have
\[
\int_{\Omega} PU_{\lambda_{n},a_{n}}^{2}dx = \omega_N C_N \lambda_{n}^2 + o(\lambda_{n}^2) \text{ as } n \to \infty,
\]
where
\[
C_N = \int_{0}^{\infty} \frac{s^{N-1}}{(1+s^2)^{N-2}}ds = \frac{\Gamma\left( \frac{N}{2} \right) \Gamma\left( \frac{N-4}{2} \right)}{2 \Gamma(\frac{N}{2})}.
\]

When \( N = 4 \), we have
\[
\int_{\Omega} PU_{\lambda_{n},a_{n}}^{2}dx = \omega_N \lambda_{n}^2 |\log \lambda_{n}| + o(\lambda_{n}^2 |\log \lambda_{n}|)
\]
\[
+ O\left( \frac{\lambda_{n}^2}{d_n} \right)^{1/2} + O\left( \frac{\lambda_{n}^2}{d_n^2} \right) \text{ as } n \to \infty,
\]
**Proof** \((N \geq 5)\). We extend \(PU_{\lambda_{n},a_{n}}\) and \(\varphi_{\lambda_{n},a_{n}}\) to \(\mathbb{R}^{N}\) by setting \(PU_{\lambda_{n},a_{n}} = 0\) in \(\mathbb{R}^{N} \setminus \Omega\) and \(\varphi_{\lambda_{n},a_{n}} = U_{\lambda_{n},a_{n}}\) in \(\mathbb{R}^{N} \setminus \Omega\). We denote them again by \(PU_{\lambda_{n},a_{n}}\) and \(\varphi_{\lambda_{n},a_{n}}\) respectively.

Since \(PU_{\lambda_{n},a_{n}} = U_{\lambda_{n},a_{n}} - \varphi_{\lambda_{n},a_{n}}\), we have

\[
\int_{\Omega} PU_{\lambda_{n},a_{n}}^{2} \, dx = \int_{\Omega} U_{\lambda_{n},a_{n}}^{2} \, dx + \int_{\Omega} \varphi_{\lambda_{n},a_{n}}^{2} \, dx \quad + \quad O\left(\left(\int_{\Omega} U_{\lambda_{n},a_{n}}^{2} \, dx\right)^{1/2}\left(\int_{\Omega} \varphi_{\lambda_{n},a_{n}}^{2} \, dx\right)^{1/2}\right). \tag{2.23}
\]

We estimate the first term in (2.23) as follows: By monotonicity of the integral, we have

\[
\int_{B_{d_{n}}(a_{n})} U_{\lambda_{n},a_{n}}^{2} \, dx \leq \int_{\Omega} U_{\lambda_{n},a_{n}}^{2} \, dx \leq \int_{B_{R}(a_{n})} U_{\lambda_{n},a_{n}}^{2} \, dx, \tag{2.24}
\]

where \(R = \text{diam}(\Omega)\).

Calculation shows

\[
\int_{B_{d_{n}}(a_{n})} U_{\lambda_{n},a_{n}}^{2} \, dx = \omega_{N} \int_{0}^{d_{n}} \left(\frac{\lambda_{n}^{2}}{\lambda_{n}^{2} + r^{2}}\right)^{N-2} r^{N-1} \, dr
\]

\[
= \omega_{N} \lambda_{n}^{2} \int_{0}^{d_{n}/\lambda_{n}} s^{N-1} \left(1 + s^{2}\right)^{-N+1} \, ds
\]

\[
= \omega_{N} \lambda_{n}^{2} \left(C_{N} + O\left(\int_{d_{n}/\lambda_{n}}^{\infty} \frac{s^{N-1}}{\left(1 + s^{2}\right)^{N-2}} \, ds\right)\right)
\]

\[
= \omega_{NC_{N}} \lambda_{n}^{2} + O\left(\frac{\lambda_{n}^{N-2}}{d_{n}^{N-4}}\right),
\]

here we have used the assumption \(N \geq 5\).

The same calculation shows

\[
\int_{B_{R}(a_{n})} U_{\lambda_{n},a_{n}}^{2} \, dx = \omega_{N} C_{N} \lambda_{n}^{2} + O(\lambda_{n}^{N-2}).
\]

So dividing both the integrals of (2.24) by \(\omega_{N} C_{N} \lambda_{n}^{2}\) and noting \((\lambda_{n}/d_{n}) = o(1)\) (see (2.7)), we obtain

\[
\lim_{n \to \infty} \frac{\int_{\Omega} U_{\lambda_{n},a_{n}}^{2} \, dx}{\omega_{N} C_{N} \lambda_{n}^{2}} = 1,
\]
\[ \int_{\Omega} U_{\lambda_{n},a_{n}}^{2} \, dx = \omega_{N} C_{N} \lambda_{n}^{2} + o(\lambda_{n}^{2}) \quad (n \to \infty). \tag{2.25} \]

To estimate the second term in (2.23), we divide the integral in two parts:

\[ \int_{\Omega} \varphi_{\lambda_{n},a_{n}}^{2} \, dx = \int_{B_{d_{n}}(a_{n})} \varphi_{\lambda_{n},a_{n}}^{2} \, dx + \int_{\Omega \setminus B_{d_{n}}(a_{n})} \varphi_{\lambda_{n},a_{n}}^{2} \, dx. \]

Then:

\[ \int_{B_{d_{n}}(a_{n})} \varphi_{\lambda_{n},a_{n}}^{2} \, dx = O \left( \| \varphi_{\lambda_{n},a_{n}} \|_{L^\infty(\Omega)}^{2} \cdot \text{vol}(B_{d_{n}}(a_{n})) \right) = O \left( \left( \frac{\lambda_{n}^{N-2}}{d_{n}^{N-2}} \right)^{2} \cdot d_{n}^{N} \right) = O\left( \frac{\lambda_{n}^{N-2}}{d_{n}^{N-4}} \right) \]

by (2.15), and

\[ \int_{\Omega \setminus B_{d_{n}}(a_{n})} \varphi_{\lambda_{n},a_{n}}^{2} \, dx = O \left( \int_{\mathbb{R}^{N} \setminus B_{d_{n}}(a_{n})} U_{\lambda_{n},a_{n}}^{2} \, dx \right) = O \left( \int_{d_{n}}^{\infty} \left( \frac{\lambda_{n}}{\lambda_{n}^{2} + r^{2}} \right)^{N-2} r^{N-1} \, dr \right) = O\left( \frac{\lambda_{n}^{N-2}}{d_{n}^{N-4}} \right), \]

since \( 0 < \varphi_{\lambda_{n},a_{n}} < U_{\lambda_{n},a_{n}} \) in \( \Omega \) and \( \varphi_{\lambda_{n},a_{n}} = U_{\lambda_{n},a_{n}} \) on \( \mathbb{R}^{N} \setminus \Omega \).

In conclusion, we have

\[ \int_{\Omega} \varphi_{\lambda_{n},a_{n}}^{2} \, dx = O\left( \frac{\lambda_{n}^{N-2}}{d_{n}^{N-4}} \right) = o(\lambda_{n}^{2}) \quad \text{as} \ n \to \infty. \tag{2.26} \]

By (2.23), (2.25) and (2.26), we have the conclusion of Lemma 2.2. \( \square \)

From Lemma 2.1, Lemma 2.2 and the fact that

\[ \int_{\Omega} |\nabla v_{n}|^{2} \, dx = \alpha_{n}^{2} \int_{\Omega} |\nabla P U_{\lambda_{n},a_{n}}|^{2} \, dx + \int_{\Omega} |\nabla w_{n}|^{2} \, dx \]

(which follows since \( w_{n} \in E_{\lambda_{n},a_{n}}; \) see (2.8)), we have the following lemma, for example when \( N \geq 5 \).
Lemma 2.3. (Asymptotic behavior of \( J_n \)) When \( N \geq 5 \), we have

\[
J_n := \int_{\Omega} |\nabla v_n|^2 \, dx - \varepsilon_n \int_{\Omega} v_n^2 \, dx
\]
\[
= \alpha_n^2 \left\{ N(N-2)A - (N-2)^2 \omega_N^2 H(a_n, a_n) \lambda_n^{N-2} \right\} - \varepsilon_n \alpha_n^2 \omega_N C_N \lambda_n^2
+ \| \nabla w_n \|_{L^2(\Omega)}^2 - \varepsilon_n \| w_n \|_{L^2(\Omega)}^2 + O\left( \frac{\lambda_n^{N}}{d_n^{N}} |\log(\frac{\lambda_n}{d_n})| \right) + o(\varepsilon_n \lambda_n^2)
+ O(\varepsilon_n \lambda_n \| w_n \|_{L^2(\Omega)})
\] as \( n \to \infty \).

To proceed further, we need the precise asymptotic behavior of \( \alpha_n \) as \( n \to \infty \). This is given by the next lemma.

Lemma 2.4. (Asymptotic behavior of \( \alpha_n \))

When \( N \geq 4 \), we have

\[
\alpha_n^2 = \alpha_N^2 + \alpha_N^2 \left( \frac{N-2}{N} \right) \left( \frac{2 \omega_N^2}{A} \right) H(a_n, a_n) \lambda_n^{N-2} + O(\| \nabla w_n \|_{L^2(\Omega)}^2) + o\left( \frac{\lambda_n^{N-2}}{d_n^{N-2}} \right)
\] as \( n \to \infty \), where \( \alpha_N = (N(N-2))^{\frac{N-2}{4}} \).

Proof. After extending \( v_n, PU_{\lambda_n, a_n} \), and \( w_n \) by 0 outside \( \Omega \), we have

\[
S^{N/2} = \int_{\Omega} v_n^{p+1} \, dx = \int_{\mathbb{R}^N} |\alpha_n PU_{\lambda_n, a_n} + w_n|^{p+1} \, dx
\] (2.27)

by (2.3). We set \( W_n := -\alpha_n \varphi_{\lambda_n, a_n} + w_n \), here as before, \( \varphi_{\lambda_n, a_n} \) is extended to \( \mathbb{R}^N \) by \( U_{\lambda_n, a_n} \) on \( \mathbb{R}^N \setminus \Omega \).

By expanding the right hand side of (2.27), we have

\[
S^{N/2} = \int_{\mathbb{R}^N} (\alpha_n U_{\lambda_n, a_n} + W_n)^{p+1} \, dx
\]
\[
= \alpha_n^{p+1} \int_{\mathbb{R}^N} U_{\lambda_n, a_n}^{p+1} \, dx + (p+1)\alpha_n^p \int_{\mathbb{R}^N} U_{\lambda_n, a_n}^p W_n \, dx
+ O \left( \int_{\mathbb{R}^N} U_{\lambda_n, a_n}^{p-1} W_n^2 \, dx + \int_{\mathbb{R}^N} |W_n|^{p+1} \, dx \right).
\] (2.28)

First, we know

\[
\alpha_n^{p+1} \int_{\mathbb{R}^N} U_{\lambda_n, a_n}^{p+1} \, dx = \alpha_n^{p+1} A.
\] (2.29)
Next, by using the equation $-\Delta U_{\lambda, a_n} = N(N-2)U_{\lambda, a_n}^p$ in $\mathbb{R}^N$, we calculate

$$(p + 1)\alpha_n^p \int_{\mathbb{R}^N} U_{\lambda, a_n}^p W_n dx = \frac{2\alpha_n^p}{(N-2)^2} \int_{\mathbb{R}^N} (-\Delta U_{\lambda, a_n}) W_n dx$$

$$= \frac{2\alpha_n^p}{(N-2)^2} \int_{\mathbb{R}^N} \nabla U_{\lambda, a_n} \cdot \nabla W_n dx$$

$$= \frac{2\alpha_n^p}{(N-2)^2} \int_{\mathbb{R}^N} (\nabla P U_{\lambda, a_n} + \nabla \varphi_{\lambda, a_n}) \cdot (-\alpha_n \nabla \varphi_{\lambda, a_n} + \nabla w_n) dx$$

$$= -\frac{2\alpha_n^{p+1}}{(N-2)^2} \int_{\mathbb{R}^N} |\nabla \varphi_{\lambda, a_n}|^2 dx$$

$$= -\frac{2\alpha_n^{p+1}}{(N-2)^2} \left\{ (N-2)^2 \omega_N^2 H(a_n, a_n) \lambda_n^{N-2} + O\left( \frac{\lambda_n^N}{d_n^N} |\log(\frac{\lambda_n}{d_n})| \right) \right\}$$

$$= -2\alpha_n^{p+1} \omega_N^2 H(a_n, a_n) \lambda_n^{N-2} + O\left( \frac{\lambda_n^N}{d_n^N} |\log(\frac{\lambda_n}{d_n})| \right). \quad (2.30)$$

Here we have used the fact that $\varphi_{\lambda, a_n}$ is a harmonic function on $\Omega$, $w_n \in E_{\lambda, a_n}$ and

$$\int_{\mathbb{R}^N} |\nabla \varphi_{\lambda, a_n}|^2 dx = \int_{\mathbb{R}^N} |\nabla U_{\lambda, a_n}|^2 dx - \int_{\mathbb{R}^N} |\nabla P U_{\lambda, a_n}|^2 dx$$

$$= (N - 2)^2 \omega_N^2 H(a_n, a_n) \lambda_n^{N-2} + O\left( \frac{\lambda_n^N}{d_n^N} |\log(\frac{\lambda_n}{d_n})| \right) \quad (2.31)$$

by Lemma 2.1.

Now, we claim that the error term in (2.28) can be estimated as

$$O\left( \int_{\mathbb{R}^N} U_{\lambda, a_n}^{p-1} W_n^2 dx + \int_{\mathbb{R}^N} |W_n|^{p+1} dx \right) = O\left( \|\nabla w_n\|_{L^2(\Omega)}^2 \right) + O\left( \frac{\lambda_n^N}{d_n^N} \right). \quad (2.32)$$

Indeed, we divide the integral as

$$\int_{\mathbb{R}^N} U_{\lambda, a_n}^{p-1} W_n^2 dx = \int_{\mathbb{R}^N \setminus \Omega} U_{\lambda, a_n}^{p-1} W_n^2 dx + \int_{\Omega} U_{\lambda, a_n}^{p-1} W_n^2 dx. \quad (2.33)$$

Since $W_n = -\alpha_n U_{\lambda, a_n}$ on $\mathbb{R}^N \setminus \Omega$, the first term in (2.33) is estimated as

$$\int_{\mathbb{R}^N \setminus \Omega} U_{\lambda, a_n}^{p-1} W_n^2 dx = \alpha_n^2 \int_{\mathbb{R}^N \setminus \Omega} U_{\lambda, a_n}^{p+1} dx = O\left( \int_{\mathbb{R}^N \setminus B_{d_n}(a_n)} U_{\lambda, a_n}^{p+1} dx \right).$$

Now we compute

$$\int_{\mathbb{R}^N \setminus B_{d_n}(a_n)} U_{\lambda, a_n}^{p+1} dx = \omega_N \int_{d_n}^{\infty} \left( \frac{\lambda_n}{\lambda_n^2 + r^2} \right)^N r^{N-1} dr = O\left( \frac{\lambda_n^N}{d_n^N} \right),$$
so we have
\[
\int_{\mathbb{R}^N \setminus \Omega} U_{\lambda_n,a_n}^{p-1} W_n^2 dx = O\left(\frac{\lambda_n^N}{d_n^N}\right). \tag{2.34}
\]
Substituting \(W_n\) by \(-\alpha_n \varphi_{\lambda_n,a_n} + w_n\) in the second term in (2.33), we have
\[
\int_{\Omega} U_{\lambda_n,a_n}^{p-1} W_n^2 dx = \alpha_n^2 \int_{\Omega} U_{\lambda_n,a_n}^{p-1} \varphi_{\lambda_n,a_n}^2 dx + \int_{\Omega} U_{\lambda_n,a_n}^{p-1} w_n^2 dx 
+ O\left((\int_{\Omega} U_{\lambda_n,a_n}^{p-1} w_n^2 dx)^{1/2}(\int_{\Omega} U_{\lambda_n,a_n}^{p-1} \varphi_{\lambda_n,a_n}^2 dx)^{1/2}\right). \tag{2.35}
\]
Now by Hölder and Sobolev inequality, we find
\[
\int_{\Omega} U_{\lambda_n,a_n}^{p-1} w_n^2 dx = O\left(\int_{\Omega} U_{\lambda_n,a_n}^{p+1} dx\right)^\frac{p-1}{p+1} \left(\int_{\Omega} w_n^{p+1} dx\right)^\frac{2}{p+1}.
\]
On the other hand, when we estimate the first term in (2.35), we divide the integral as
\[
\int_{\Omega} U_{\lambda_n,a_n}^{p-1} \varphi_{\lambda_n,a_n}^2 dx = \int_{B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p-1} \varphi_{\lambda_n,a_n}^2 dx + \int_{\Omega \setminus B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p-1} \varphi_{\lambda_n,a_n}^2 dx. \tag{2.37}
\]
First term in (2.37) is estimated as
\[
\int_{B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p-1} \varphi_{\lambda_n,a_n}^2 dx = O\left(\left\|\varphi_{\lambda_n,a_n}\right\|_{L^\infty(\Omega)}^2 \cdot \int_{B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p-1} dx\right)
= O\left(\left(\frac{\lambda_n^{N-2} - \Pi}{d_n^{N-2}}\right)^2 \cdot \lambda_n^2 d_n^{N-4}\right) = O\left(\frac{\lambda_n^N}{d_n^N}\right). \tag{2.38}
\]
Here we have used the fact
\[
\int_{B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p-1} dx = \omega_N \int_0^{d_n} \left(\frac{\lambda_n}{\lambda_n^2 + r^2}\right)^2 r^{N-1} dr = O\left(\lambda_n^2 d_n^{N-4}\right),
\]
since \(N \geq 5\).
Second term in (2.37) is estimated as before:
\[
\int_{\Omega \setminus B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p-1} \varphi_{\lambda_n,a_n}^2 dx = O\left(\int_{\Omega \setminus B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p+1} dx\right) = O\left(\frac{\lambda_n^N}{d_n^N}\right). \tag{2.39}
\]
By (2.37)-(2.39), we have
\[
\int_{\Omega} U_{\lambda_n,a_n}^{p-1} \varphi_{\lambda_n,a_n}^2 dx = O\left(\frac{\lambda_n^N}{d_n^N}\right). \tag{2.40}
\]
Combining (2.35), (2.36) and (2.40), we obtain

\[ \int_{\Omega} U_{\lambda_{n},a_{n}}^{p-1} W_{n}^{2} dx = O \left( \|
abla w_{n}\|_{L^{2}(\Omega)}^{2} \right) + O \left( \frac{\lambda_{n}^{N}}{d_{n}^{N}} \right). \]  

(2.41)

Finally, by Sobolev inequality and convex inequality \((a + b)^{t} \leq C(a^{t} + b^{t})\) for some \(C > 0\) \((a, b > 0, t > 1)\), we have

\[
\int_{\mathbb{R}^{N}} |W_{n}|^{p+1} dx = O \left( \left( \int_{\mathbb{R}^{N}} |\nabla W_{n}|^{2} dx \right)^{\frac{p+1}{2}} \right)
= O \left( \left( \int_{\mathbb{R}^{N}} |\nabla \varphi_{\lambda_{n},a_{n}}|^{2} dx + \int_{\mathbb{R}^{N}} |\nabla w_{n}|^{2} dx \right)^{\frac{p+1}{2}} \right)
= O \left( \left( \int_{\mathbb{R}^{N}} |\nabla \varphi_{\lambda_{n},a_{n}}|^{2} dx \right)^{\frac{p+1}{2}} \right) + O \left( \left( \int_{\mathbb{R}^{N}} |\nabla w_{n}|^{2} dx \right)^{\frac{p+1}{2}} \right). \]

(2.42)

(Recall we extend \(\varphi_{\lambda_{n},a_{n}}\) to \(\mathbb{R}^{N} \setminus \Omega\) by \(U_{\lambda_{n},a_{n}}\)). So by (2.42), (2.31) and the estimate \(H(a_{n}, a_{n}) = O \left( \frac{1}{d_{n}^{N-2}} \right)\) (see (2.21)), we obtain

\[
\int_{\mathbb{R}^{N}} |W_{n}|^{p+1} dx = O \left( \int_{\mathbb{R}^{N}} \left( \frac{\lambda_{n}^{N-2}}{d_{n}^{N-2}} \right)^{\frac{N}{2}} dx \right) + O \left( \|
abla w_{n}\|_{L^{2}(\Omega)}^{2} \right)
= O \left( \frac{\lambda_{n}^{N}}{d_{n}^{N}} \right) + o \left( \frac{\lambda_{n}^{N-2}}{d_{n}^{N-2}} \right). \]

(2.43)

Combining (2.33), (2.34), (2.41) and (2.43), we conclude the claim (2.32).

Returning to (2.28) and using (2.29), (2.30) and (2.32), we obtain

\[ S^{N/2} = \alpha_{n}^{p+1} A - 2\alpha_{n}^{p+1} \cdot \omega_{N}^{2} H(a_{n}, a_{n}) \lambda_{n}^{N-2} + O \left( \|
abla w_{n}\|_{L^{2}(\Omega)}^{2} \right) + o \left( \frac{\lambda_{n}^{N-2}}{d_{n}^{N-2}} \right). \]

Dividing the both sides by \(A\) and noting that \(\frac{S^{N/2}}{A} = \alpha_{n}^{p+1}\), we have

\[ \alpha_{n}^{p+1} = \alpha_{n}^{p+1} - \alpha_{n}^{p+1} \left( \frac{2\omega_{N}^{2}}{A} \right) H(a_{n}, a_{n}) \lambda_{n}^{N-2} + O \left( \|
abla w_{n}\|_{L^{2}(\Omega)}^{2} \right) + o \left( \frac{\lambda_{n}^{N-2}}{d_{n}^{N-2}} \right). \]

From this we can derive the conclusion. \(\square\)

Combining Lemma 2.3 and Lemma 2.4, we obtain:

**Lemma 2.5.** (Asymptotic behavior of \(S_{\epsilon_{n}}\))
As $n \to \infty$,
\begin{align*}
S_{\epsilon_n} &:= \inf_{v \in H_0^1(\Omega) \atop \|v\|_{L^{p+1}(\Omega)} = 1} \left\{ \int_{\Omega} |\nabla v|^2 dx - \epsilon_n \int_{\Omega} v^2 dx \right\} \\
&= S \cdot S^{-\frac{N}{2}} J_n \\
&= S + S \left( \frac{N-2}{N} \right) \left( \frac{\omega_N^2}{A} \right) H(a_n, a_n) \lambda_n^{N-2} - \epsilon_n \left( \frac{S \omega_N C_N}{N(N-2)A} \right) \lambda_n^2 \\
&\quad + O \left( \| \nabla w_n \|_{L^2(\Omega)}^2 \right) + o \left( \frac{\lambda_n^{N-2}}{d_n^{N-2}} \right) + o(\epsilon_n \lambda_n^2). \quad (N \geq 5)
\end{align*}

\begin{align*}
S_{\epsilon_n} &= S + S \left( \frac{\omega_4^2}{A} \right) H(a_n, a_n) \lambda_n^2 - \epsilon_n \left( \frac{S \omega_4}{8A} \right) \lambda_n^2 |\log \lambda_n| \\
&\quad + O \left( \| \nabla w_n \|_{L^2(\Omega)}^2 \right) + o \left( \frac{\lambda_n^2}{d_n^2} \right) + o(\epsilon_n \lambda_n^2 |\log \lambda_n|). \quad (N = 4)
\end{align*}

As for the "w-part" of $v_n$, we have the following estimate due to Rey [13](Appendix C:(C.1)).

**Lemma 2.6.** As $n \to \infty$, we have
\[ \| \nabla w_n \|_{L^2(\Omega)}^2 = o\left( \frac{\lambda_n^{N-2}}{d_n^{N-2}} \right) + o(\epsilon_n \lambda_n^2). \]

Now, we need the appropriate bound of the value $S_{\epsilon_n}$ from the above. The restriction that we consider only least energy solutions is essential in the next lemma.

**Lemma 2.7. (Upper bound of $S_{\epsilon}$)**
For any $a \in \Omega$ and $\rho > 0$, there exists $\epsilon_0 = \epsilon_0(a, \rho)$ such that if $\epsilon \in (0, \epsilon_0)$, then the following holds:
\begin{align*}
S_\epsilon &= \inf_{v \in H_0^1(\Omega) \atop \|v\|_{L^{p+1}(\Omega)} = 1} \left\{ \int_{\Omega} |\nabla v|^2 dx - \epsilon \int_{\Omega} v^2 dx \right\} \\
&\leq S - \left( \frac{N-4}{N-2} \right) \epsilon \left\{ \frac{S \omega_N C_N}{N(N-2)A} - \rho \right\} \left[ \frac{2C_N \epsilon}{(N-2)^3 \omega_N H(a, a)} \right]^{\frac{2}{N-4}}
\end{align*}
when $N \geq 5$. 

\[ \boxed{\text{128}} \]
\[ S_{\epsilon} \leq S - \frac{S\epsilon\omega_{4}}{16Ae}\exp\left(\frac{-8\omega_{4}H(a,a) + \epsilon/\epsilon + 2\rho}{\epsilon}\right) \]

when \( N = 4 \).

**Proof** \((N \geq 5)\). For \( a \in \Omega \) and \( \epsilon > 0 \), define \( \psi_{\epsilon,a} \in H_{0}^{1}(\Omega) \) as

\[ \psi_{\epsilon,a} := S^{-\frac{(N-2)}{4}}\alpha_{N}PU_{\lambda_{a}(\epsilon),a}, \quad (2.44) \]

where

\[ \lambda_{a}(\epsilon) := \left[ \frac{2CN\epsilon}{(N-2)^{3}\omega_{N}H(a,a)} \right]^{\frac{N-2}{4}}. \quad (2.45) \]

Note that \( \lambda_{a}(\epsilon) \) is the unique minimum point of the function

\[ f(\lambda) = K_{1}H(a,a)\lambda^{N-2} - K_{2}\epsilon\lambda^{2} \]

for \( \lambda > 0 \), and it gives the minimum value

\[ \min_{\lambda > 0} f(\lambda) = f(\lambda_{a}(\epsilon)) = -\left( \frac{N-4}{N-2} \right)K_{2}\epsilon K_{1}H(a,a) \]

\[ = -\left( \frac{N-4}{N-2} \right) \epsilon \left( \frac{S\omega_{N}C_{N}}{N(N-2)A} \right) \left( \frac{2C_{N}\epsilon}{(N-2)^{3}\omega_{N}H(a,a)} \right)^{\frac{2}{N-4}}. \quad (2.46) \]

Here, we denote

\[ K_{1} = S \left( \frac{N-2}{N} \right) \left( \frac{\omega_{N}^{2}}{A} \right), \quad K_{2} = \frac{S\omega_{N}C_{N}}{N(N-2)A}. \quad (2.47) \]

Define

\[ J_{\epsilon}(\psi) := \frac{\int_{\Omega} |\nabla \psi|^{2} dx - \epsilon \int_{\Omega} \psi^{2} dx}{(\int_{\Omega} |\psi|^{p+1} dx)^{\frac{2}{p+1}}} \quad (2.48) \]

for \( \psi \in H_{0}^{1}(\Omega) \setminus \{0\} \).

Now, we claim that:

\[ J_{\epsilon}(\psi_{\epsilon,a}) = S - \left( \frac{N-4}{N-2} \right) \epsilon \left\{ \frac{S\omega_{N}C_{N}}{N(N-2)A} \right\} \left[ \frac{2C_{N}\epsilon}{(N-2)^{3}\omega_{N}H(a,a)} \right]^{\frac{2}{N-4}} \]

\[ + o(\epsilon^{\frac{N-2}{N-4}}) \quad (2.49) \]
Indeed, as in the calculation in the proof of Lemma 2.1, Lemma 2.2 (note now $d(a, \partial\Omega)$ is a constant independent of $\epsilon$), we have

\[
\int_{\Omega} |\nabla \psi_{\epsilon,a}|^2 dx = S \cdot S^{-\frac{N}{2}} \alpha_{N}^2 \int_{\Omega} |\nabla PU_{\lambda_{a}(\epsilon),a}|^2 dx
\]

\[
= S - S \left( \frac{N-2}{N} \right) \left( \frac{\omega_{N}^2}{A} \right) H(a,a)\lambda_{a}^{N-2}(\epsilon) + o(\lambda_{a}^{N-2}(\epsilon)), \quad (2.50)
\]

\[
\int_{\Omega} \psi_{\epsilon,a}^2 dx = S \cdot S^{-\frac{N}{2}} \alpha_{N}^2 \int_{\Omega} PU_{\lambda_{a}(\epsilon),a}^2 dx
\]

\[
= \frac{S\omega_{N}C_{N}}{N(N-2)A} \lambda_{a}^{2}(\epsilon) + o(\lambda_{a}^{2}(\epsilon)) \quad (2.51)
\]

as $\epsilon \to 0$.

Also by an argument similar to the one in the proof of Lemma 2.4, we have

\[
\int_{\Omega} |\psi_{\epsilon,a}|^{p+1} dx = S^{-\frac{N}{2}} \alpha_{N}^{p+1} \int_{\Omega} |PU_{\lambda_{a}(\epsilon),a}|^{p+1} dx
\]

\[
= \frac{1}{A} \left\{ \int_{\Omega} U_{\lambda_{a}(\epsilon),a}^{p+1} dx + (p+1) \int_{\Omega} U_{\lambda_{a}(\epsilon),a}^{p} \varphi_{\lambda_{a}(\epsilon),a} dx \\
+ O \left( \int_{\Omega} U_{\lambda_{a}(\epsilon),a}^{p-1} \varphi_{\lambda_{a}(\epsilon),a}^{2} dx + \int_{\Omega} |\varphi_{\lambda_{a}(\epsilon),a}|^{p+1} dx \right) \right\}
\]

\[
= \frac{1}{A} \left\{ A - 2\omega_{N}^{2} \lambda_{a}^{N-2}(\epsilon)H(a,a) + o(\lambda_{a}^{N-2}(\epsilon)) \right\}
\]

\[
= 1 - \left( \frac{2\omega_{N}^{2}}{A} \right) \lambda_{a}^{N-2}(\epsilon)H(a,a) + o(\lambda_{a}^{N-2}(\epsilon)). \quad (2.52)
\]

Note that $S^{N/2} = \alpha_{N}^{2}N(N-2)A = \alpha_{N}^{p+1}A$.

So, by (2.50)-(2.52) and $(1+x)^{-\frac{2}{p+1}} = 1 - \frac{2}{p+1}x + o(x)$ as $x \to 0$, we obtain

\[
J_{\epsilon}(\psi_{\epsilon,a})
\]

\[
= \left\{ S - S \left( \frac{N-2}{N} \right) \left( \frac{\omega_{N}^2}{A} \right) H(a,a)\lambda_{a}^{N-2}(\epsilon) + o(\lambda_{a}^{N-2}(\epsilon)) \right. \\
\left. - \epsilon \left( \frac{S\omega_{N}C_{N}}{N(N-2)A} \right) \lambda_{a}^{2}(\epsilon) + o(\epsilon \lambda_{a}^{2}(\epsilon)) \right\}
\]

\[
\times \left\{ 1 + \frac{2}{p+1} \left( \frac{2\omega_{N}^{2}}{A} \right) H(a,a)\lambda_{a}^{N-2}(\epsilon) + o(\lambda_{a}^{N-2}(\epsilon)) \right\}
\]

\[
= S + S \left( \frac{N-2}{N} \right) \left( \frac{\omega_{N}^2}{A} \right) H(a,a)\lambda_{a}^{N-2}(\epsilon) - \epsilon \left( \frac{S\omega_{N}C_{N}}{N(N-2)A} \right) \lambda_{a}^{2}(\epsilon)
\]
\[ + o(\epsilon^2(\epsilon)) + o(\lambda_a^{N-2}(\epsilon)) \]
\[ = S - \left( \frac{N - 4}{N - 2} \right) \epsilon \left\{ \frac{S\omega_NC_N}{N(N - 2)A} \right\} \left[ \frac{2C_N\epsilon}{(N - 2)^3\omega_NH(a,a)} \right]^{2} \]
\[ + o(\epsilon^{\frac{N - 2}{N - 4}}) \]
\[ \text{(2.53)} \]
as \( \epsilon \to 0 \).

This proves the claim. The last equality in (2.53) follows from our choice of \( \lambda_a(\epsilon) \) (see (2.46)) and the fact
\[ \epsilon\lambda_a^2(\epsilon) = C_1\lambda_a^{N-2}(\epsilon) = C_2\epsilon^{\frac{N - 2}{N - 4}} \]
by the definition of \( \lambda_a(\epsilon) \) (see (2.45)), where \( C_1, C_2 \) are constants independent of \( \epsilon \).

From (2.49) and the definition of \( S_\epsilon \), we obtain the conclusion of Lemma 2.7.

\[ \square \]

3. Proof of Theorem. In this section, we prove Theorem 1 by using lemmas we prepared in the previous section.

First, we will show that the blow up point \( a_\infty \) is in the interior of \( \Omega \).

Indeed, suppose the contrary. Then \( a_\infty \in \partial\Omega \) and \( d_n = d(a_n, \partial\Omega) \to 0 \) as \( n \to \infty \). Then by Lemma 2.5, Lemma 2.6 and the estimate (2.21), we can find constants \( C_1, C_2, C_3 > 0 \) such that
\[ S_\epsilon_n = S + S \left( \frac{N - 2}{N} \right) \left( \frac{\omega_N^2}{A} \right) \lambda_n \]
\[ + O(\|\nabla w_n\|_{L^2(\Omega)}^2) + o\left( \frac{\lambda_n^{N-2}}{d_n^{N-2}} \right) + o(\epsilon_n^2) \]
\[ \geq S + C_1 \left( \frac{\lambda_n^{N-2}}{d_n^{N-2}} \right) - C_2\epsilon_n \lambda_n \]
\[ \geq S - \left( \frac{\lambda_n^{N-2}}{d_n^{N-2}} \right) C_2\epsilon_n \left\{ \frac{2C_2\epsilon_n}{(N - 2)C_1(d_n^{-1})} \right\}^{\frac{N - 2}{N - 4}} \]
\[ = S - C_3\epsilon_n^{\frac{N - 2}{N - 4}}d_n^{\frac{2(N - 2)}{N - 4}} = S + o(\epsilon_n^{\frac{N - 2}{N - 4}}), \]
\[ \text{(3.1)} \]
since we assume \( d_n \to 0 \).

Here as in the proof of Lemma 2.7, we have used the fact that \( f(\lambda) = C_4\lambda^{N-2} - C_5\lambda^2 \) has the unique global minimum value \(- \left( \frac{N - 4}{N - 2} \right) C_5 \left( \frac{2C_2}{(N - 2)C_4} \right)^{\frac{N - 2}{N - 4}} \)
for \( \lambda > 0 \), where \( C_4 = C_1(d_n^{-1}), C_5 = C_2\epsilon_n \).
On the other hand, we know that $S_{\epsilon_n} \leq S - C \epsilon_n^{N-4} + o(\epsilon_n^{N-4})$ for some $C > 0$ (see Lemma 2.7 (2.49)). This contradicts (3.1), so we conclude that $a_{\infty}$ is in the interior of $\Omega$.

Now, since we have proved that $d_n \geq C$ for some constant $C > 0$ uniformly in $n$, we may drop $d_n$ in the asymptotic formulas Lemma 2.5 and Lemma 2.6.

Therefore, we can find $p_n > 0, p_n \to 0$ and $q_n > 0, q_n \to 0$ such that

\[
S_{\epsilon_n} = S + S \left( \frac{N - 2}{N} \right) \left( \frac{\omega_n^2}{A} \right) H(a_n, a_n) \lambda_n^{N-2} - \epsilon_n \left( \frac{S \omega_n C_n}{N(N - 2)A} \right) \lambda_n^2
+ o(\lambda_n^{N-2}) + o(\epsilon_n \lambda_n^2)
\geq S + (K_1 H(a_n, a_n) - p_n) \lambda_n^{N-2} - (K_2 + q_n) \epsilon_n \lambda_n^2
\geq S - \left( \frac{N - 4}{N - 2} \right) (K_2 + q_n) \epsilon_n \left[ \frac{2(K_2 + q_n) \epsilon_n}{(N - 2)(K_1 H(a_n, a_n) - p_n)} \right]^{\frac{2}{N-4}}
\] (3.2)

where $K_1, K_2$ are defined in (2.47). The last inequality of (3.2) follows again by the property of the function $f(\lambda) = C_4 \lambda^{N-2} - C_5 \lambda^2$.

Combine (3.2) with Lemma 2.7, we have

\[
S - \left( \frac{N - 4}{N - 2} \right) (K_2 + q_n) \epsilon_n \left[ \frac{2(K_2 + q_n) \epsilon_n}{(N - 2)(K_1 H(a_n, a_n) - p_n)} \right]^{\frac{2}{N-4}}
\leq S_{\epsilon_n} \leq S - \left( \frac{N - 4}{N - 2} \right) (K_2 - \rho) \epsilon_n \left[ \frac{2K_2 \epsilon_n}{(N - 2)K_1 H(a, a)} \right]^{\frac{2}{N-4}}
\]

for any $a \in \Omega$ and $\rho > 0$, if $n$ sufficiently large.

From this we obtain

\[
(K_2 + q_n) \epsilon_n \left[ \frac{2(K_2 + q_n) \epsilon_n}{(N - 2)(K_1 H(a_n, a_n) - p_n)} \right]^{\frac{2}{N-4}} \geq (K_2 - \rho) \epsilon_n \left[ \frac{2K_2 \epsilon_n}{(N - 2)K_1 H(a, a)} \right]^{\frac{2}{N-4}}.
\]

Dividing both sides by $\epsilon_n^{\frac{N-2}{4}}$ and letting $n \to \infty$, we have

\[
K_2 \left[ \frac{2K_2}{(N - 2)K_1 H(a_\infty, a_\infty)} \right]^{\frac{2}{N-4}} \geq (K_2 - \rho) \left[ \frac{2K_2}{(N - 2)K_1 H(a, a)} \right]^{\frac{2}{N-4}}.
\] (3.3)

For $\rho > 0$ can be arbitrary small, (3.3) implies

\[
H(a_\infty, a_\infty) \leq H(a, a)
\]
for any $a \in \Omega$.

Therefore we conclude that $a_\infty$ minimizes the Robin function $H(a, a)$. This completes the proof of Theorem. 

\[\square\]

References


