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Kyoto University
An existence result for some semi-linear elliptic equation in bent strip-like unbounded domains

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1 Introduction and Main Result

Let $N \geq 2$ and $\Omega$ be an unbounded domain in $\mathbb{R}^N$. We consider the following equation

$$
\begin{cases}
\Delta u + \lambda u = u^p_+ & \text{in } \Omega, \\
u \in H^1_0(\Omega),
\end{cases}
$$

(1)

where $\lambda \geq 0$ and $1 < p < \infty$ if $N = 2$, $1 < p < (N + 2)/(N - 2)$ if $N \geq 3$ are given constants. It is well-known that (1) has a positive solution if $\Omega$ is bounded. In general, the existence of a positive solution of (1) is unknown if $\Omega$ is unbounded. Esteban and Lions showed in [4] that if $\Omega$ satisfies following condition (EL) then there is no nontrivial solution.

(EL) There exists a vector $X \in \mathbb{R}^N$ such that $\nu(x) \cdot X \geq 0$ and $\nu(x) \cdot X \not\equiv 0$ on $x \in \partial \Omega$, where $\nu(x)$ is the outer unit normal vector of $\Omega$.

On the other hand, many authors showed existence result. (cf. [1, 3, 5, 7] and references therein). In this paper, we will give an existence result in bent strip-like unbounded domains. We use following notations.

$$
S_d := \{x = (x', x_N) \in \mathbb{R}^N; |x'| < d\},
\hat{S}_d := \{x = (x', x_N) \in S_d; x_N > 0\}.
$$

In [6], we conjectured that if $\lambda \geq 0$ and $\Omega$ satisfying the following condition (\Omega1) then there is a nontrivial solution.

(\Omega1) $\Omega$ is a domain in $\mathbb{R}^N$ and $\partial \Omega$ is Lipschitz continuous. There are $K \in \mathbb{N}\backslash\{1\}$, a bounded set $A$ and congruent transformations $\Lambda_j$ ($1 \leq j \leq K$) such that $\Omega = A \cup \Lambda_1(\hat{S}_d) \cup \cdots \cup \Lambda_K(\hat{S}_d)$ and $\Lambda_i(\hat{S}_d) \cap \Lambda_j(\hat{S}_d) = \emptyset$ if $i \neq j$.

This conjecture is still open. In this paper, we consider the following stronger conditions (\Omega2), (\Omega3) in two dimensional case. Here after, we assume $N = 2$. 

There are $d > 0$, a smooth curve $\{c(s)\}_{s \in \mathbb{R}}$ parameterized by arc length with the curvature $\kappa(s)$ such that $\text{supp}\{\kappa\}$ is compact and $\Phi : S_d \to \Omega$ is bijective, where $\Phi$ is defined by $\Phi(y) := c(y_2) + y_1 e(y_2)$ and $e(s)$ is the unit normal vector of $c(s)$.

$\Omega$ satisfies $(\Omega 1)$, $\exists \Omega_0 \subset \Omega$ s.t. $\Omega_0$ satisfies $(\Omega 2)$.

Remark. If $\Omega$ satisfies $(\Omega 2)$ then
$$\Omega = \{x \in \mathbb{R}^2; \text{dist}(x, \{c(s)\}) < d\}.$$ So $\Omega$ is a bent strip-like domain.

Remark. $\Omega$ satisfies $(\Omega 2)$ then $\Omega$ satisfies $(\Omega 3)$ with $\Omega = \Omega_0$. $\Omega$ satisfies $(\Omega 3)$ then $\Omega$ satisfies $(\Omega 1)$.

Now we state our main theorem.

**Theorem A.** Suppose that $N = 2$, $\lambda \geq 0$ and the following equation has unique nontrivial solution up to $x_2$ transformation.
$$\begin{cases}
-\Delta v + \lambda v = v_+^p & \text{in } S, \\
v \in H_0^1(S).
\end{cases}$$ (2)

If $(\|\kappa\|_{L^\infty} d)^2 < 1 - 2^{(1-p)/(1+p)}$ then (1) has a nontrivial solution.

Remark. If $\lambda = 0$, (2) has unique nontrivial solution up to $x_2$ transformation by [2].

## 2 Preliminaries

At first, we state notations. For a domain $D$, we define following notations.

$$I[u] := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + \lambda u^2 \, dx - \frac{1}{p+1} \int_{\mathbb{R}^N} u_+^{p+1} \, dx$$ for $u \in H_0^1(D) \subset H_0^1(\mathbb{R}^N)$,

$$M(D) := \{u \in H_0^1(D) \setminus \{0\}; \int_D |\nabla u|^2 + \lambda u^2 \, dx = \int_D u_+^{p+1}\},$$

$$\alpha(\Omega) := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I_D[\gamma(t)],$$

$$\Gamma := \{\gamma \in C([0,1]; H_0^1(D)); \gamma(0) = 0, I_D[\gamma(1)] \leq 0\}.$$ It is well-known that the mountain pass energy $\alpha(D)$ is well-defined and is equal to a least energy. i.e.
Lemma 2.1. Let $D$ be a domain. Suppose that $D$ satisfying Poincare's inequality or $\lambda > 0$. Then

$$\alpha(D) = \inf_{u \in M(D)} I_D[u]$$

and all nontrivial critical point $v$ of $I_D$ satisfies $I_D[v] \geq \alpha(D)$.

(cf. [9]).

Lemma 2.2. If $\Omega$ satisfies \((\Omega 1)\). Then Poincare's inequality holds. i.e. There exists a constant $C > 0$ such that

$$\int_{\Omega} u^2 dx \leq C \int_{\Omega} |\nabla u|^2 dx.$$ 

By Lemma 2.2, we can use the norm

$$||v||_{H_0^1(\Omega)}^2 = \int_{\Omega} |\nabla v|^2 dx.$$ 

Lemma 2.3. Let $K$ be a complete metric space, $K_0 \subset K$ be a closed set, $X$ be a Banach space and $\chi \in C(K_0, X)$. Define $\Gamma$ by

$$\Gamma := \{ \gamma \in C(K, X); \gamma(s) = \chi(s) \text{ if } s \in K_0 \}.$$ 

For $I \in C^1(X, \mathbb{R})$, put

$$c := \inf_{\gamma \in \Gamma} \max_{s \in K} I[\gamma(s)], \quad c := \max_{v \in K_0} I[\chi(v)].$$ 

If $c > c_1$ then for all $\epsilon > 0$ and $\gamma \in \Gamma$ with $\max_{s \in K} I[\gamma(s)] \leq c + \epsilon$, there exists $v \in X$ such that

$$c - \epsilon < I[v] < \max_{s \in K} I[\gamma(s)], \quad \dist(v, g(K)) \leq \epsilon^{\frac{1}{2}}, \quad |I'[v]| \leq \epsilon^{\frac{1}{2}}.$$ 

Especially, there is a Palais-Smale sequence.

For the proof of this Lemma, see [8, Theorem 4.3].

Proposition 2.4 (Concentration Compactness). Suppose \((\Omega 1)\). Let $\{u_n\}_{n=1}^{\infty}$ be nonnegative Palais-Smale $\beta$-sequence for $I_\Omega$ in $H_0^1(\Omega)$. i.e.

$$I_\Omega[u_n] = \beta + o(1), \quad I_\Omega'[u_n] = o(1) \quad \text{as } n \to \infty.$$
Then there exist a non-negative number \( l, k_1, \ldots, k_l \in \{1, \ldots, k\} \), \( \{z_n^i\}_{n=1}^{\infty} \subset \Lambda_k, (\{x = (x', x_N); x' = 0\}) \), \( u^0 \in H_0^1(\Omega) \) with \( u \geq 0 \), \( u^i \in H_0^1(\Lambda_{k_i}(S)) \) with \( u^i > 0 \) for \( 1 \leq i \leq l \) such that

\[
\begin{align*}
\lim_{n \to \infty} u_n(x) &= u^0(x) + u^1(x - z_n^1) + \cdots + u^i(x - z_n^i) + o(1) \\
as n \to \infty in H_0^1(\mathbb{R}^N),
\end{align*}
\]

\( I_{\Omega}[u_n] = I_{\Omega}[u^0] + I_{\mathbb{R}^N}[u^1] + \cdots + I_{\mathbb{R}^N}[u^i] + o(1) \) as \( n \to \infty \),

\[
\begin{cases}
-\Delta u^0 + \lambda u^0 = (u^0)^p & \text{in } \Omega, \\
-\Delta u^i + \lambda u^i = (u^i)^p & \text{in } \Lambda_{k_i}(S),
\end{cases}
\]

\( |z_n^i| \to \infty \) as \( n \to \infty \).

We can give the proof of Lemma 2.4 by using same argument as in [7]. For reader's convenience, we give the proof in Appendix. To prove theorem A, we use the following functional. Take \( \phi \in C(\mathbb{R}^N, [-1, 1]) \) satisfying

\[
\phi(x) = \begin{cases}
1 & x \in \Lambda_i(S_0), i: \text{odd}, \\
-1 & x \in \Lambda_i(S_0), i: \text{even}, \\
0 & \text{otherwise}.
\end{cases}
\]

Define the functional \( h : L^2(\mathbb{R}^N) \setminus \{0\} \to [-1, 1] \) by

\[
h[u] := \frac{1}{\|u\|_{L^2(\mathbb{R}^N)}^2} \int_{\mathbb{R}^N} \phi(x)|u(x)|^2 dx \quad \text{for } u \in L^2(\mathbb{R}^N) \setminus \{0\}.
\]

\( h \) is a continuous function in the following sense.

**Lemma 2.5.** There is a constant \( C > 0 \) such that

\[
|h[u + v] - h[u]| \leq C(\|u\|_{L^2(\mathbb{R}^N)} + \|v\|_{L^2(\mathbb{R}^N)}) \|v\|_{L^2(\mathbb{R}^N)}
\]

for all \( u, v \in L^2(\mathbb{R}^N) \) with \( u \neq 0 \) and \( u + v \neq 0 \). Especially, \( |h[u + v] - h[u]| \leq C\|v\|_{L^2(\mathbb{R}^N)}/\|u\|_{L^2(\mathbb{R}^N)} \) if \( \|v\|_{L^2(\mathbb{R}^N)} < \|u\|_{L^2(\mathbb{R}^N)} \).

We can show Lemma 2.5 by elementary calculus. We omit the proof of
3 Proof of Theorem A and Theorem B

To prove Theorem A, we consider the following mountain-pass value $\alpha_0(\Omega)$.
Put
\[
H = \{ u \in H_0^1(\Omega); h[u] = 0 \} \cup \{ 0 \},
\]
\[
\alpha_0(\Omega) := \inf_{\gamma \in \Gamma_0} \sup_{t \in [0,1]} I[\gamma(t)],
\]
\[
\Gamma_0 := \{ \gamma \in C([0,1], H); g(0) = 0, I[g(1)] \leq 0 \}.
\]
Here, it is easy to see that $H$ is a closed subspace of $H_0^1(\Omega)$. By the definition of $\alpha_0(\Omega)$, $\alpha(\Omega) \leq \alpha_0(\Omega)$ holds. It is well-known that $0 < \alpha(\Omega) \leq \alpha(S_d)$ if $\Omega$ satisfies $(\Omega 1)$ because of $\alpha(\hat{S}_d) = \alpha(S_d)$. So one of following cases holds.

(a) $\alpha(\Omega) < \alpha(S_d)$.

(b) $\alpha(\Omega) = \alpha(S_d)$ and $\alpha_0(\Omega) = \alpha(S_d)$.

(c) $\alpha(\Omega) = \alpha(S_d)$ and $\alpha_0(\Omega) > \alpha(S_d)$.

Proposition 3.1. Suppose that $(\Omega 1)$. If the case (a) or (b) holds then (1) has a positive solution.

Proposition 3.1 is proved by standard arguments by using concentration compactness principle. We omit the proof of it. By Proposition 3.1, it is enough to show that Theorem A in the case (c). Hereafter, we suppose (c) and $N = 2$. For the proof of Theorem A, the least energy solution on $S_d$ plays important role. Let $v \in H_0^1(S_d)$ be a least energy solution on $S_d$. i.e.

\[
\begin{cases}
-\Delta v + \lambda v = v_+^p & \text{in } S_d, \\
v > 0 & \text{on } \partial S_d, \\
I[v] = \alpha(S_d).
\end{cases}
\]

The existence of such solution is well-known. By the moving plain method, we can assume that

\[ v(x) = v(x_1, x_2) = v(|x_1|, |x_2|) \quad \text{for all } x \in S_d. \]

By the equation, we see

\[
\int_{S_d} |\nabla v|^2 + \lambda v^2 \, dx = \int_{S_d} v_+^{p+1} \, dx. \tag{3}
\]
Since $(Ω3)$, $Ψ := Φ^{-1}$ is well-defined. Define $v_t$, $u_t$ by

$$v_t(x) := v(Ψ_1(x), Ψ_2(x) - t), \quad u_t(x) = s(t)v_t(x),$$

where $s(t)$ is uniquely determined positive constant satisfying $u_t(x) ∈ M(Ω)$ for each $t$. (see Lemma 4.1.)

**Lemma 3.2.** If $(d||κ||_{L∞(R)})^2 < 1 - 2^{(1-p)/(1+p)}$ then there exist constants $t_0, s_0 > 0$ such that

$$I[u_{±t_0}] < \frac{1}{2}(α(S) + α_0(Ω)), \quad (4)$$

$$h[u_{t_0}] > \frac{1}{2}, \quad h[u_{-t_0}] < -\frac{1}{2}, \quad (5)$$

$$I[sv_t] ≤ 0 \quad \text{if } s ≥ s_0, \quad (6)$$

$$I[u_t] < 2α(S) \quad \text{for all } t ∈ R. \quad (7)$$

**Proof.** By elementally calculation for Φ,

$$I[sv_t] = \frac{s^2}{2} \int_{S_d} \frac{1}{1 - (y_1κ(y_2) + t)^2} v_{y_2}^2(y_1, y_2 - t) + (1 - y_1κ(y_2))v_{y_1}^2(y_1, y_2 - t) \quad (8)$$

$$+ λ(1 - y_1κ(y_2))v^2(y_1, y_2 - t) dy - \int_{S_d} (1 - y_1κ(y_2))F(sv(y_1, y_2 - t)) dy.$$

Since $v$ is even function with respect to $y_1$ and $1/(1+t)+1/(1-t) = 2/(1-t^2)$, we have

$$I[sv_t] = \frac{s^2}{2} \int_{S_d} \frac{1}{1 - (y_1κ(y_2 + t))^2} v_{y_2}^2 + v_{y_1}^2 + λv^2 dy - \frac{1}{p+1} \int_{S_d} (sv)_+^{p+1} dy.$$

Since $\frac{d}{ds}I[sv_t]|_{s=s(t)} = 0$, we obtain

$$\int_{S_d} \frac{1}{1 - (y_1κ(y_2 + t))^2} v_{y_2}^2 + v_{y_1}^2 + λv^2 dy = s(t)^{p-1} \int_{S_d} v_+^{p+1} dy \quad (8)$$

Here, the right hand side is increasing with respect to $s$ and

$$\int_{S_d} \frac{1}{1 - (y_1κ(y_2 + t))^2} v_{y_2}^2 + v_{y_1}^2 + λv^2 dy > \int_{S_d} v_{y_2}^2 + v_{y_1}^2 + λv^2 dy = \int_{S_d} v^{p+1} dy \quad (9)$$

by (3). So we have

$$s(t) ≥ 1. \quad (10)$$
By using Lesbergue's convergence theorem, the left hand side of (9) tends to 
\[ \int_{S_{d}} |\nabla v|^{2} + \lambda v^{2} \, dy \] as \( t \to \pm \infty \). It and (3) mean \( s(t) \to 1 \) as \( t \to \pm \infty \). It asserts \( I[u_{t}] \to \alpha(S) \) as \( t \to \pm \infty \). So (4) holds for sufficiently large \( t_{0} \).

(8) and (3) assert
\[ s(t)^{p-1} \leq \frac{1}{1 - (d||\kappa||_{L^{\infty}(\mathbb{R})})^{2}}. \tag{11} \]

By (8), (11) and the assumption of Theorem A, we can obtain
\[
I[u_{t}] = (\frac{1}{2} - \frac{1}{p+1})s(t)^{2} \int_{S_{d}} \frac{1}{1 - (y_{1}\kappa(y_{2} + t))^{2}} v_{y_{2}}^{2} + v_{y_{1}}^{2} + \lambda v^{2} \, dy
\]
\[ \leq s(t)^{2} \frac{1}{1 - (d||\kappa||_{L^{\infty}(\mathbb{R})})^{2}} \alpha(S_{d}) \]
\[ \leq \left( \frac{1}{1 - (d||\kappa||_{L^{\infty}(\mathbb{R})})^{2}} \right)^{p+1} \alpha(S_{d}) \]
\[ < 2\alpha(S_{d}). \]

It means (7) holds for any \( t \in \mathbb{R} \). It is easy to see that
\[
I[sv_{t}] \leq \frac{s^{2}}{2} \frac{1}{1 - (d||\kappa||_{L^{\infty}(\mathbb{R})})^{2}} \int_{S_{d}} |\nabla v|^{2} + \lambda v^{2} \, dy - \frac{s^{p+1}}{p+1} \int_{S_{d}} v^{p+1} \, dy
\]
The right hand side is independent of \( t \) and tends to \( -\infty \) as \( s \to \infty \). So we obtain (6) for sufficiently large \( s_{0} \).

By the assumption (Ω3) and the definition of \( u_{t} \), we have
\[
||\chi_{\Lambda_{1}(S_{d})} v_{t} - v_{t}||_{L^{2}(\mathbb{R}^{2})} \to 0 \text{ and } ||\chi_{\Lambda_{1}(S_{d})} v_{t}||_{L^{2}(\mathbb{R}^{2})} \to ||v||_{L^{2}(\mathbb{R}^{2})} \neq 0.
\]
Since \( h[\chi_{\Lambda_{1}(S_{d})} v_{t}] = -1 \) and Lemma 2.5, we obtain
\[
h[u_{t}] \to -1 \text{ as } t \to -\infty.
\]
Similarly,
\[
h[v_{t}] \to 1 \text{ as } t \to \infty
\]
holds. It completes the proof of Lemma 3.2. \( \square \)

Put \( K := [0, s_{0}] \times [-t_{0}, t_{0}] \) and define \( \beta \) by
\[
\beta := \inf_{\gamma \in \Gamma_{1}} \max_{(s, t) \in K} I[g(s, t)],
\]
\( \Gamma_{1} := \{ \gamma \in C(S, H_{0}^{1}(\Omega)) ; g(s, t) = sv_{t} \text{ if } (s, t) \in \partial K \} \).

Then the following Lemma 3.4 and Lemma 3.3 hold.
Lemma 3.3. Suppose same assumptions as in Lemma 3.2 then
\[ \alpha(S) < \beta < 2\alpha(S). \]

Lemma 3.4. Suppose same assumptions as in Lemma 3.2. Then there is a Palais-Smale \( \beta \)-sequence \( \{u_n\}_{n=1}^{\infty} \). i.e.
\[ I[u_n] = \beta + o(1), \quad ||I[u_n]|| = o(1) \quad as \ n \to \infty. \]

Proof of Lemma 3.3. Put \( \gamma_0(s, t) = sv_t \) for \( (s, t) \in K \) then \( \gamma_0 \in \Gamma_1 \). By the assumption of \( f \), we have \( I[sv_t] \leq I[u_t] \). Lemma 3.2 asserts that \( I[\gamma_0(s, t)] \leq 2\alpha(S) \) for all \( (s, t) \in K \). Hence \( \beta < 2\alpha(S) \).

Fix any \( \gamma \in \Gamma_1 \), Lemma 3.2 and similar argument as in [10] show that there is a curve \( \tau : [0, 1] \to K \) such that \( \gamma \circ \tau \in \Gamma_0 \). So we have
\[ \max_{(s, t) \in K} I[\gamma(s, t)] \geq \max_{t \in (0, 1)} I[\gamma \circ \tau(t)] \geq \alpha_0(\Omega). \]

It means \( \alpha(S) < \beta \) by the condition (c). \( \square \)

Proof of Lemma 3.4. Put \( \gamma_0(s, t) = sv_t \) for \( (s, t) \in K \) then \( \gamma_0 \in \Gamma_1 \), Lemma 3.2 asserts
\[ \max_{(s, t) \in \partial K} I[\gamma_0(s, t)] < \frac{1}{2}(\alpha_0(\Omega) + \alpha(S)) < \beta. \]

So we can apply Lemma 2.3 to obtain the existence of Palais-Smale \( \beta \)-sequence. \( \square \)

Now we can prove Theorem B in the following Proposition.

Proposition 3.5. Suppose that same assumption as in Theorem A. Then there is a positive solution.

Proof. Let \( \{u_n\}_{n=1}^{\infty} \) be a Palais-Smale \( \beta \) sequence in Lemma 3.4. By Proposition 2.4, by passing to a subsequence if necessary, there is a nonnegative number \( l \) such that
\[ u_n(x) = u^0(x) + u^1(x - x^1_n) + \cdots + u^l(x - x^l_n) + o(1) \quad as \ n \to \infty \quad in \ H^1_0(\mathbb{R}^2), \]
\[ I[u_n] = I[u^0] + I[u^1] + \cdots + I[u^l] + o(1) \quad as \ n \to \infty. \]

If \( u^0 \equiv 0 \) then \( u^0 \) is a positive solution. So it is enough to show that \( u^0 \not\equiv 0 \).

Suppose \( u \equiv 0 \) then \( l \geq 1 \) and
\[ I[u_n] = I[u^1] + \cdots + I[u^l] + o(1) \geq \lambda \alpha(S) + o(1) \quad as \ n \to \infty. \]
Since Lemma 3.4, we have $\beta < 2\alpha(S)$. So we can obtain $l = 1$. It mean that

$$I[u_n] = I[u^1] + o(1) \quad \text{as } n \to \infty.$$  

Hence $I[u_1] = \beta$. So, we see that $u_1(\Lambda_{k_1}(x))$ is a critical point of $I$ in $H^1_0(\Lambda_{k_1}(S_d))$ with $I[u_1(\Lambda_{k_1}(x))] = \beta$. It contradicts to the uniqueness of nontrivial solutions on $\Lambda_{k_1}(S_d)$. Consequently, there exists a positive solution $u^0$.

\[\square\]

4 Appendix

In this section, we note well-known facts and give the proof of Proposition 2.4. First, we note some properties for $f$.

Lemma 4.1. Suppose that $D$ is a domain in $\mathbb{R}^N$. Fix $v \in H^1_0(D)$ with $v_+ \neq 0$ in $H^1_0(D)$. Then there is an uniquely determined constant $s_0 > 0$ such that

$$\frac{d}{ds}I[sv]|_{s=s_0} = 0.$$  

Moreover,

$$\max_{s>0}I[sv] = I[s_0v].$$

Proof. We see

$$\frac{d}{ds}I_D[sv] = \int_D |\nabla v|^2 + \lambda v^2 \, dx - s^{p-1} \int_D v_+^{p+1} \, dy$$

if $s > 0$. Second term of the right hand side is strictly decreasing with respect to $s$ on $(0, \infty)$. Moreover, second term equals to 0 if $s = 0$ and tends to $-\infty$ as $s \to \infty$. Consequently, we obtain this Lemma.  

\[\square\]

Proof of Proposition 2.4. By the assumption of $u_n$, we have

$$<I'[u_n], u_n> = ||u_n||_{H^1_0(\Omega)}^2 + \lambda ||u_n||_{L^2(\Omega)}^2 - \int_{\Omega} (u_n)_+^{p+1} \, dx = o(1)||u_n||_{H^1_0(\Omega)}$$

as $n \to \infty$.  

(12) So we have

$$C \geq I[u_n] = \left( \frac{1}{2} - \frac{1}{p+1} \right)(||u_n||_{H^1_0(\Omega)}^2 + \lambda ||u_n||_{L^2(\Omega)}^2) + o(1)||u_n||_{H^1_0(\Omega)}$$

as $n \to \infty$.  

(13)
So we see that $u_n$ is bounded in $H^1_0(\Omega)$. By using weak compactness for Hilbert space and Rellich's compactness, there exists $u^0 \in H^1_0(\Omega)$ such that

\[
\begin{align*}
    u_n &\rightharpoonup u^0 \quad \text{weakly in } H^1_0(\Omega) \text{ as } n \to \infty, \\
    u_n &\to u^0 \quad \text{in } L^p_{\text{loc}}(\Omega) \text{ as } n \to \infty, \\
    u_n &\to u^0 \quad \text{a.e. in } \Omega \text{ as } n \to \infty,
\end{align*}
\]

by passing to a subsequence if necessary. So we obtain

\[
I'[u_n] \rightharpoonup I'[u^0] \quad \text{weakly in } H^{-1}(\Omega).
\]

It means $u^0$ is a critical point of $I$. Put $\phi_n^1 := u_n - u_0$ then

\[
\begin{align*}
    \phi_n^1 &\rightharpoonup 0 \quad \text{weakly in } H^1_0(\Omega) \text{ as } n \to \infty, \quad (14) \\
    \phi_n^1 &\to 0 \quad \text{in } L^p_{\text{loc}}(\Omega) \text{ as } n \to \infty. \quad (15)
\end{align*}
\]

Moreover, we have

\[
\|\phi_n^1\|_{H^1_0(\Omega)}^2 = \|u_n\|_{H^1_0(\Omega)}^2 - \|u_0\|_{H^1_0(\Omega)}^2 + o(1) \quad \text{as } n \to \infty.
\]

We can apply Brezis-Lieb's theorem to obtain

\[
\int_{\Omega}(\phi_n^1)^{p+1} \, dx = \int_{\Omega}(u_n)^{p+1} \, dx - \int_{\Omega}(u^0)^{p+1} \, dx.
\]

By using Vitali's Lemma, we have

\[
I'[\phi_n^1] = I'[u_n] - I'[u^0] + o(1) = o(1) \quad \text{in } H^{-1}(\Omega) \text{ as } n \to \infty. \quad (16)
\]

Suppose $\phi_n^1 \to 0$ in $H^1_0(\Omega)$ as $n \to \infty$, by passing to a subsequence if necessary. Then the proof is complete since $u_n \rightharpoonup u^0$ in $H^1_0(\Omega)$ as $n \to \infty$. So, hear-after, we can assume $\phi_n^1$ is not convergence to 0 in $H^1_0(\Omega)$ for any subsequence. Put

\[
\begin{align*}
    Q_0 &:= \Omega \setminus (\Lambda_1(\hat{S}_d) \cup \ldots \Lambda_k(\hat{S}_d)), \\
    Q_m &:= \{x = (x', x_N) \in S; m-1 < x_N \leq m\}, \\
    Q_m^j &:= \Lambda_j(Q_m)
\end{align*}
\]

for $m \geq 1$, $1 \leq j \leq k$. Define $d_n$ and $\hat{d}_n$ by

\[
d_n := \max_{m \in \mathbb{N}, 1 \leq j \leq k} \|\phi_n^1\|_{L^2(Q_m^j)}, \quad \hat{d}_n := \max\{d_n, \|\phi_n^1\|_{L^2(Q_0)}\}
\]

and show that

\[
\liminf_{n \to \infty} \hat{d}_n > 0.
\]
Since $Q_n^j$ is congruence we can apply Sobolev's inequality to obtain

$$\|\phi_n^1\|_{L^r(Q_m^n)} \leq C(r)\|\phi_n^1\|_{H_0^1(Q_m^n)}$$

for $q + 1 < r \leq 2^*$ where $C(q)$ is a positive constant independent of $n, j$. By using interpolation it holds that

$$\|\phi_n^1\|_{L^{q+1}(Q_m^n)} \leq C(r)\|\phi_n^1\|_{L^2(Q_m^n)}^{(1-\theta)(q+1)}\|\phi_n^1\|_{H_0^1(Q_m^n)}^{\theta(q+1)}$$

where $1/(q+1) = (1-\theta)/2 + \theta/r$. Since $\theta \to 1$ as $r \to q+1$, $\theta(q+1) - 2 > 0$ for $r$ near $q + 1$. Fix such $r$ then we have

$$\int_{Q_m^n} |\phi_n^1|^p \, dx \leq C\|\phi_n^1\|_{H_0^1(Q_m^n)}^{\theta(q+1) - 2} \int_{Q_m^n} |\nabla \phi_n^1|^2 \, dx.$$  

Similarly for $Q_0$, we have

$$\int_{Q_0} |\phi_n^1|^p \, dx \leq C\|\phi_n^1\|_{H_0^1(Q_m^n)}^{\theta(q+1) - 2} \int_{Q_0} |\nabla \phi_n^1|^2 \, dx.$$  

By taking sum, we obtain

$$\int_{\Omega} |\phi_n^1|^p \, dx \leq C\|\phi_n^1\|_{H_0^1(Q_m^n)}^{\theta(q+1) - 2} \int_{\Omega} |\nabla \phi_n^1|^2 \, dx.$$  

If $\hat{d}_n \to 0$ as $n \to \infty$ for some subsequence then $\|\phi_n^1\|_{L^p(\Omega)} \to 0$ as $n \to \infty$. On the other hand, by (16),

$$o(1) = I'[\phi]\phi = \|\phi_n^1\|_{H_0^1(\Omega)}^2 + \lambda\|\phi_n^1\|_{L^2(\Omega)}^2 - \int_{\Omega} (\phi_n^1)^{p+1} \, dx.$$  

By Sobolev's inequality,

$$\int_{\Omega} (\phi_n^1)^{p+1} \, dx \leq \epsilon C\|\phi_n^1\|_{H_0^1(\Omega)}^2 + C(\epsilon)\|\phi_n^1\|_{L^{p+1}(\Omega)}^{p+1}.$$  

So, for sufficiently small $\epsilon$, we have

$$\|\phi_n^1\|_{H_0^1(\Omega)}^2 \leq C\|\phi_n^1\|_{L^{p+1}(\Omega)}^{p+1} = o(1) \quad \text{as } n \to \infty.$$  

It is contradiction. So we obtain $\liminf_{n \to \infty} \hat{d}_n > 0$.

Here, by passing to a subsequent if necessary, there is $j(n) \in \{1, \ldots, k\}$ and $m(n) \in \mathbb{N} \cup \{0\}$ such that $\|\phi_n^1\|_{Q_m^n(n)}$, where $Q_m^{j(n)}(n) = Q_0$ if $m(n) = 0$. We can assume $j(n) \equiv j$ by passing to a subsequence if necessary. We show
that $m(n) \to \infty$ as $n \to \infty$. Suppose that there is a constant $m_0$ such that $m(n) \leq m_0$ for all $n$. Then
\[
\frac{d^2_n}{dt^2} \leq \sum_{0 \leq m \leq m_0} \|\phi_n^1\|^2_{L^2(Q_m^j)} = \|\phi_n^1\|^2_{L^2(Q)}
\]
where $Q = \bigcup_{0 \leq m \leq m_0} Q_m^j$. As $n \to \infty$, it contradicts to (15). We can assume that $m(n)$ is increasing without loss of generality.
Define the map $\Lambda$ by
\[
\Lambda(x) := \Lambda_j(x', x_n + m(n) - 1).
\]
Then $\Lambda(Q_1) = Q_{m(n)}^j$, $\Lambda(S_d) = \sum_{m \geq m(n)} Q_m^j$. Put $\hat{\phi}_n^1 := \phi_n^1 \circ \Lambda$ then we have
\[
\|\hat{\phi}_n^1\|_{H^1(\mathbb{R}^N)} < C, \quad \|\hat{\phi}_n^1\|_{L^2(Q_1)} \geq d_n.
\]
By the weak compactness of $H^1(\mathbb{R}^N)$, there exists $\hat{u}^1 \in H^1(\mathbb{R}^N)$ such that
\[
\hat{\phi}_n^1 \rightharpoonup \hat{u}^1 \quad \text{weakly in } H^1(\mathbb{R}^N)
\]
by passing to a subsequence if necessary. Here, we can assume parallel transformation to $\Lambda_j$ are $\Lambda_{j+1}, \ldots, \Lambda_{j+\hat{j}}$ for some $\hat{j} \in \mathbb{N} \cup \{0\}$. So there is a cone $V$ such that $V \cap \Omega \subset V \cap (\Lambda_j(S_d) \cup \Lambda_{j+1}(S_d))$. It means that for $n_0 \in \mathbb{N},$
\[
\phi_n^1 = 0 \text{ on } \Lambda_j^{-1}(V \setminus (\Lambda_j(S_d) \cup \cdots \cup \Lambda_{j+\hat{j}}(S_d))) - (0, m(n_0) - 1)
\]
\[
= (\Lambda_j^{-1}V - (0, m(n_0) - 1)) \setminus (S_d \cup \Lambda_j^{-1} \circ \Lambda_{j+1}(S_d) \cdots \cup \Lambda_j^{-1} \circ \Lambda_{j+\hat{j}}(S_d))
\]
if $n \geq n_0$.
As $n \to \infty$, we obtain
\[
\hat{u}^1 \neq 0 \text{ on } (\Lambda_j^{-1}V - (0, m(n_0) - 1)) \setminus (S_d \cup \Lambda_j^{-1} \circ \Lambda_{j+1}(S_d) \cdots \cup \Lambda_j^{-1} \circ \Lambda_{j+\hat{j}}(S_d))
\]
As $n_0 \to \infty$, we have
\[
\hat{u}^1 = 0 \text{ on } \mathbb{R}^N \setminus (S_d \cup \Lambda_j^{-1} \circ \Lambda_{j+1}(S_d) \cdots \cup \Lambda_j^{-1} \circ \Lambda_{j+\hat{j}}(S_d))
\]
It means that there is $\hat{u}^{1,0} \in H^1_0(S_d), \hat{u}^{1,1} \in H^1_0(\Lambda_j^{-1} \circ \Lambda_{j+1}(S_d)), \ldots, \hat{u}^{1,\hat{j}} \in H^1_0(\Lambda_{j+\hat{j}}(S_d))$ such that $\hat{u}^i \neq 0$.
Fix any $\psi \in C_0^\infty(\Omega)$ such that $\hat{u}^i \neq 0$. As $n \to \infty$, $\Lambda(\text{supp} \psi) \subset \Omega$ for large $n$. So we have
\[
\left| \int_{\mathbb{R}^N} \nabla \phi_n^1 \nabla \psi + \lambda \phi_n^1 \psi - (\phi_n^1)^p \psi \, dx \right|
\]
\[
= \left| \int_{\mathbb{R}^N} \nabla \phi_n^1 \nabla (\psi \circ \Lambda) + \lambda \phi_n^1 (\psi \circ \Lambda) - (\phi_n^1)^p \psi \circ \Lambda \, dx \right|
\]
\[
= \langle T(\phi_n^1), \psi \circ \Lambda \rangle \leq o(1) \|\psi \circ \Lambda\|_{H^1(\mathbb{R}^N)} = o(1) \|\psi\|_{H^1(\mathbb{R}^N)}.
\]
As $n \to \infty$, we obtain
\[
\int_{\mathbb{R}^N} \nabla \hat{u} \nabla \psi + \lambda \hat{u} \psi - (\hat{u})_+^p \psi \, dx = 0.
\]

It means
\[
I'[\hat{u}^1] = 0 \quad \text{in } H^{-1}(S_d \cup \Lambda_j^{-1} \circ \Lambda_j+1(S_d) \cdots \cup \Lambda_j^{-1} \circ \Lambda_{j+j}(S_d)).
\]

Hence $\hat{u}^{1,i}$ is a weak solution of
\[
\begin{cases}
-\Delta \hat{u}^{1,i} + \lambda \hat{u}^{1,i} = (\hat{u}^{1,i})_+^p & \text{in } \Lambda_j^{-1} \circ \Lambda_{j+i}(S_d), \\
\hat{u}^{1,i} \in H_0^1(\Lambda_j^{-1} \circ \Lambda_{j+i}(S_d))
\end{cases}
\]
for $0 \leq i \leq \hat{j}$. Put $u^{i+1}(x) := \hat{u}^{i,i} \circ \Lambda_j^{-1}$ and $z^{i+1}_n := \Lambda_j(x', m(n) - 1)$ with $\Lambda_j(x', 0) \in \Lambda_{i+1}(\{y' = 0\})$. for $0 \leq i \leq \hat{j}$. Then
\[
\begin{cases}
-\Delta u^{i+1} + \lambda u^{i+1} = (u^{i+1})_+^p, u^{i+1} > 0 & \text{in } \Lambda_{j+i}(S), \\
u^{i+1} = 0 & \text{on } \partial \Lambda_{j+i}(S), \\
\phi^1_n(x) \rightharpoonup u^1(x - z^1_n) + \cdots + u^{1+\hat{j}}(x - z^{1+\hat{j}}_n) & \text{weakly in } H^1(\mathbb{R}^N), \\
\phi^1_n(x) \rightharpoonup u^1(x - z^1_n) + \cdots + u^{1+\hat{j}}(x - z^{1+\hat{j}}_n) & \text{weakly in } L^p_{\text{loc}}(\mathbb{R}^N), \\
\phi^1_n(x) \rightharpoonup u^1(x - z^1_n) + \cdots + u^{1+\hat{j}}(x - z^{1+\hat{j}}_n) & \text{a.e. in } \mathbb{R}^N \text{ as } n \to \infty
\end{cases}
\]
for $0 \leq i \leq \hat{j}$. If $\phi^1_n \rightharpoonup u^1(x - z^1_n) + \cdots + u^{1+\hat{j}}(x - z^{1+\hat{j}}_n)$ strongly in $H_0^1(\mathbb{R}^N)$ for some subsequence then the proof is complete.

If not, by using the argument above, inductively, by passing to a subsequence if necessary, we have
\[
\phi^1_n(x) = u_n(x) - u^0(x) - u^1(x - z^1_n) - \cdots - u^l(x - z^l_n) + o(1) \text{ weakly in } H_0^1(\mathbb{R}^N),
\]
\[
\|\phi^1_n\|_{H_0^1(\mathbb{R}^N)} = \|u_n\|_{H_0^1(\mathbb{R}^N)} - \|u^0\|_{H_0^1(\mathbb{R}^N)} - \|u^1\|_{H_0^1(\mathbb{R}^N)} - \cdots - \|u^l\|_{H_0^1(\mathbb{R}^N)} \quad \text{as } n \to \infty.
\]
Since $\|u^l\|_{H_0^1(\mathbb{R}^N)}, \ldots, \|u^l\|_{H_0^1(\mathbb{R}^N)} \geq c_0(S)$ and $\|u_n\|_{H_0^1(\mathbb{R}^N)}$ is uniformly bounded, there is some $l \geq 1$ such that $u_n(x) = u^0(x) + u^1(x - z^1_n) + \cdots + u^l(x - z^l_n) + o(1)$ strongly in $H_0^1(\mathbb{R}^N)$. It completes the proof.

\[\square\]

References


