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Asymptotic Properties and Initial Values of Solutions to Periodic Linear Equations

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1 Introduction

Let $\mathbb{C}^m$ be a d-dimensional complex Euclidean space and $\mathbb{R}$ the real line. Consider the linear differential equation of the form

$$\frac{d}{dt}x(t) = Ax(t) + f(t), \tag{1}$$

where A is a complex $m \times m$ matrix and $f : \mathbb{R} \to \mathbb{C}^m$ a periodic continuous function with a period $\tau > 0$ (τ-periodic function, simply). Set $\omega = 2\pi/\tau$.

The purpose of the paper is to give a complete classification of the set of initial values according to asymptotic behaviors of solutions of Eq.(1).

In Section 2, we shall show that the variation-of-constant formula for Eq.(1) is reformed into a sum of τ-periodic functions and exponential like functions. Its proof is based on a new representation of solutions of the difference equation corresponding to Eq.(1).

In Section 3, as applications of this result, the asymptotic behaviors of all solutions as well as the boundedness of solutions are shown to be completely determined by the initial values. Refer to [2] for the complete proofs of results obtained in this paper.

2 Solutions of Equation (1)

Put $\sigma(A) = \{\lambda_1, \cdots, \lambda_\ell\}$, the set of spectrum of A, and let n_r be the index of $\lambda_r \in \sigma(A)$. Denote by $P_r : \mathbb{C}^m \to \mathbf{M}_r$ the projection corresponding to the direct sum decomposition $\mathbb{C}^m = \mathbf{M}_1 \oplus \cdots \oplus \mathbf{M}_\ell$, where $\mathbf{M}_r := \mathcal{N}((A - \lambda_r I)^{n_r})$ is the generalized eigenspace corresponding to λ_r. The following properties for the projections are well known :

$$P_r\mathbb{C}^m = \mathbf{M}_r, \quad AP_r = P_rA, \quad P_rP_s = 0 \ (r \neq s), \quad P_r^2 = P_r, \quad P_1 + P_2 + \cdots + P_\ell = I.$$

For a given $b \in \mathbb{C}^m$, we consider the difference equation of the form

$$x_{n+1} = e^{\tau A} x_n + b. \tag{2}$$

The solution x_n of this equation such that $x_0 = w$ is given by

$$x_n = W_n(w, b) := e^{n\tau A} w + S_n(e^{\tau A}) b, \quad S_n(e^{\tau A}) = \sum_{j=0}^{n-1} e^{j\tau A}.$$

Since

$$e^{tA} = \sum_{r=1}^{\ell} e^{\lambda_r t} Q_r(t) P_r, \quad Q_r(t) = \sum_{k=0}^{n_r - 1} \frac{t^k}{k!} (A - \lambda_r I)^k,$$

it follows that

$$P_r S_n(e^{\tau A}) b = \sum_{j=0}^{n-1} e^{j\tau\lambda_r} Q_r(j\tau) P_r b \;=\; \sum_{j=0}^{n-1} \sum_{k=0}^{n_r - 1} e^{j\tau\lambda_r} j^k A_{k,r} P_r b, \tag{3}$$

where $A_{k,r} = \tau^k (A - \lambda_r I)^k / k!$.

A function $g(z) := (e^z - 1)^{-1}$ and Bernoulii's number B_k are used in the development of $W_n(w, b)$. Bernoulli's polynomial $B_k(x), k \in \mathbb{N} \cup \{0\}$, in the relation

$$\frac{te^{xt}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}$$

is given by

$$B_k(x) = \sum_{i=0}^{k} \binom{k}{i} B_i x^{k-i}, \quad \binom{k}{i} = \frac{k!}{i!(k-i)!},$$

where $B_k = B_k(0)$ (Bernoulli's numbers, $B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, \cdots$.) is defined by

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

The following facts are well known, cf. [4].

Lemma 2.1 *$B_k(x)$ and B_k have the following properties :*

1) $B_k(x+1) - B_k(x) = kx^{k-1}$ *and* $B_k = B_k(1)$ *if* $k \geq 2$.
2) $B_{2k+1} = 0$ *if* $k \geq 1$.
3) $B_k = \sum_{i=0}^{k} \binom{k}{i} B_i$(or $\sum_{i=0}^{k-1} \binom{k}{i} B_i = 0$) *if* $k \geq 2$, *and* $B_1 = B_1(1) - B_0$.
4) $B_0(x) \equiv 1$ *and*

$$B_k(x) = x^k - \frac{1}{k+1} \sum_{i=0}^{k-1} \binom{k+1}{i} B_i(x), (k = 1, 2, \cdots).$$

For $\tau\lambda_r \notin 2\pi i\mathbb{Z}$, we set

$$a_r^{(k)} := g^{(k)}(\tau\lambda_r), \quad (k = 0, 1, \cdots, n_r - 1),$$

where $g^{(k)}$ stands for the k-th derivative of g. Put

$$Y_r(w, b) = P_r w + \sum_{k=0}^{n_r-1} a_r^{(k)} A_{k,r} P_r b \quad \text{if} \quad e^{\tau\lambda_r} \neq 1, \tag{4}$$

and

$$Z_r(w, b) = A_{1,r} P_r w + \sum_{k=0}^{n_r-1} B_k A_{k,r} P_r b \quad \text{if} \quad e^{\tau\lambda_r} = 1. \tag{5}$$

First, we develop the components $P_r W_n(w, b)$ of the solution $W_n(w, b)$ of the difference equation (2) in the manner that we can see the dependence on n explicitly. The following result is a new representation of solutions of the difference equation corresponding to Eq.(1), which plays an essential role in this paper. Its proof is omitted.

Theorem 1 *Let $Y_r(w, b), Z_r(w, b)$ be defined as in* (4),(5), *respectively.*
1) *If $\lambda_r \notin i\omega\mathbb{Z}$, then*

$$\begin{aligned} P_r W_n(w, b) &= e^{n\tau\lambda_r} \sum_{j=0}^{n_r-1} n^j A_{j,r} Y_r(w, b) - Y_r(w, b) + P_r w & (6) \\ &= e^{n\tau A} Y_r(w, b) - Y_r(w, b) + P_r w, \quad n \geq 0. & (7) \end{aligned}$$

2) *If $\lambda_r \in i\omega\mathbb{Z}$, then*

$$P_r W_n(w, b) = \sum_{j=0}^{n_r-1} \frac{1}{j+1} n^{j+1} A_{j,r} Z_r(w, b) + P_r w, \quad n \geq 0. \tag{8}$$

Next, we transfer Theorem 1 to the solution of Eq.(1). If $x(t)$ is a solution of Eq.(1), then $P_r x(t) \in \mathbf{M}_r$ for $t \in \mathbb{R}$ and satisfies the following equation :

$$\frac{d}{dt} y(t) = Ay(t) + P_r f(t).$$

In general, if an $\mathbf{M}_r$ valued function $y(t)$ satisfies this equation, we say that $y(t)$ is a solution of Eq.(1) in $\mathbf{M}_r$. The following theorem gives a characterization of solutions of Eq.(1). Set

$$b_f = \int_0^\tau e^{(\tau-s)A} f(s) ds.$$

Theorem 2 *Let $x(t)$ be a solution of Eq.(1) such that $x(0) = w$.*

1) *If $\lambda_r \notin i\omega\mathbb{Z}$,*

$$P_r x(t) = e^{tA} Y_r(w, b_f) + u_r(t, w, b_f), \tag{9}$$

where

$$u_r(t, w, b_f) = e^{tA}(-Y_r(w, b_f) + P_r w) + \int_0^t e^{(t-s)A} P_r f(s) ds \tag{10}$$

is a τ-periodic solution of Eq.(1) in $\mathbf{M}_r$.

2) *If $\lambda_r \in i\omega\mathbb{Z}$,*

$$P_r x(t) = \frac{e^{\lambda_r t}}{\tau} \sum_{i=0}^{n_r - 1} \frac{t^{i+1}}{(i+1)!}(A - \lambda_r I)^i Z_r(w, b_f) + v_r(t, w, b_f), \tag{11}$$

where

$$v_r(t, w, b_f) = e^{tA} P_r w - \frac{e^{\lambda_r t}}{\tau} \sum_{i=0}^{n_r - 1} \frac{t^{i+1}}{(i+1)!}(A - \lambda_r I)^i Z_r(w, b_f) + \int_0^t e^{(t-s)A} P_r f(s) ds \tag{12}$$

is a τ-periodic function, which is not necessarily a solution of Eq.(1) in $\mathbf{M}_r$.

Proof Let n be a nonnegative integer. Since $W_n(w, b_f) = x(n\tau)$, we have that, for any $t \in \mathbb{R}$,

$$P_r x(t) = e^{(t-n\tau)A} P_r W_n(w, b_f) + \int_{n\tau}^t e^{(t-s)A} P_r f(s) ds.$$

Since $f(s)$ is τ-periodic, the integral in the right hand side becomes

$$\int_{n\tau}^t e^{(t-s)A} P_r f(s) ds = \int_0^{t-n\tau} e^{(t-n\tau-s)A} P_r f(s) ds.$$

Suppose that $\lambda_r \notin i\omega\mathbb{Z}$. Since, by Theorem 1,

$$\begin{aligned} e^{(t-n\tau)A} P_r W_n(w, b_f) &= e^{(t-n\tau)A}(e^{n\tau A} Y_r(w, b_f) - Y_r(w, b_f) + P_r w) \\ &= e^{tA} Y_r(w, b_f) + e^{(t-n\tau)A}(-Y_r(w, b_f) + P_r w), \end{aligned}$$

it follows that

$$P_r x(t) = e^{tA} Y_r(w, b_f) + u_{n,r}(t, w, b_f), \tag{13}$$

where

$$u_{n,r}(t, w, b_f) = e^{(t-n\tau)A}(-Y_r(w, b_f) + P_r w) + \int_0^{t-n\tau} e^{(t-n\tau-s)A} P_r f(s) dr.$$

Since $u_{n,r}(t,w,b_f) = P_r x(t) - e^{tA}Y_r(w,b_f)$, $u_n := u_{n,r}(t,w,b_f)$ is independent of $n = 0,1,2,\cdots$; that is, $u_0 = u_1 = u_2 = \cdots$. Furthermore, it is easy to see that $u_n(t-\tau) = u_{n+1}(t)$. Since $u_{n+1} = u_n$, it follows that $u_n(t-\tau) = u_n(t)$; $u_n(t)$ is τ-periodic. Thus $u_0(t)$ is τ-periodic, and $P_r x(t) = e^{tA}Y_r(w,b_f) + u_{0,r}(t,w,b_f)$. Since $e^{tA}Y_r(w,b_f)$ is a solution of the homogeneous equation, $u_{0,r}(t,w,b_f)$ is a τ-periodic solution of Eq.(1). Therefore we obtain Formula (9) by taking $u_r = u_{0,r}$.

Suppose that $\lambda_r \in i\omega\mathbb{Z}$. Using Theorem 1, we have

$$
\begin{aligned}
& e^{(t-n\tau)A}(P_r W_n(w,b_f) - P_r w) \\
= \;& e^{(t-n\tau)\lambda_r} \sum_{k=0}^{n_r-1} \frac{(t-n\tau)^k}{k!}(A-\lambda_r I)^k \sum_{j=0}^{n_r-1} \frac{1}{j+1} n^{j+1} \frac{\tau^j}{j!}(A-\lambda_r I)^j Z_r(w,b_f) \\
= \;& e^{\lambda_r t} \sum_{k=0}^{n_r-1} \sum_{j=0}^{n_r-1} \frac{(t-n\tau)^k n^{j+1}\tau^j}{k!(j+1)!}(A-\lambda_r I)^{j+k} Z_r(w,b_f) \\
= \;& \frac{e^{\lambda_r t}}{\tau} \sum_{i=0}^{n_r-1} \sum_{k+j=i} \frac{(t-n\tau)^k (n\tau)^{j+1}}{k!(j+1)!}(A-\lambda_r I)^i Z_r(w,b_f) \\
= \;& \frac{e^{\lambda_r t}}{\tau} \sum_{i=0}^{n_r-1} \sum_{k=0}^{i} \frac{(t-n\tau)^k (n\tau)^{i-k+1}}{k!(i-k+1)!}(A-\lambda_r I)^i Z_r(w,b_f) \\
= \;& \frac{e^{\lambda_r t}}{\tau} \sum_{i=0}^{n_r-1} \sum_{k=0}^{i+1} \frac{(t-n\tau)^k (n\tau)^{i+1-k}}{k!(i+1-k)!}(A-\lambda_r I)^i Z_r(w,b_f) \\
& - \frac{e^{\lambda_r t}}{\tau} \sum_{i=0}^{n_r-1} \frac{(t-n\tau)^{i+1}}{(i+1)!}(A-\lambda_r I)^i Z_r(w,b_f) \\
= \;& \frac{e^{\lambda_r t}}{\tau} \sum_{i=0}^{n_r-1} \frac{t^{i+1}}{(i+1)!}(A-\lambda_r I)^i Z_r(w,b_f) \\
& - \frac{e^{\lambda_r t}}{\tau} \sum_{i=0}^{n_r-1} \frac{(t-n\tau)^{i+1}}{(i+1)!}(A-\lambda_r I)^i Z_r(w,b_f).
\end{aligned}
$$

Therefore, we have

$$
P_r x(t) = \frac{e^{\lambda_r t}}{\tau} \sum_{i=0}^{n_r-1} \frac{t^{i+1}}{(i+1)!}(A-\lambda_r I)^i Z_r(w,b_f) + v_{n,r}(t,w,b_f),
$$

where

$$
\begin{aligned}
v_{n,r}(t,w,b_f) = \;& e^{(t-n\tau)A}P_r w - \frac{e^{\lambda_r t}}{\tau} \sum_{i=0}^{n_r-1} \frac{(t-n\tau)^{i+1}}{(i+1)!}(A-\lambda_r I)^i Z_r(w,b_f) \\
& + \int_0^{t-n\tau} e^{(t-n\tau-s)A}P_r f(s)dr.
\end{aligned}
$$

As in the case of $u_{n,r}$, we see that $v_{n,r}$ are independent of $n = 0, 1, 2, \cdots$, and τ-periodic. Hence

$$v_{0,r}(t, w, b_f) = P_r x(t) - \frac{e^{\lambda_r t}}{\tau} \sum_{i=0}^{n_r - 1} \frac{t^{i+1}}{(i+1)!} (A - \lambda_r I)^i Z_r(w, b_f)$$

is a τ-periodic function. Therefore, we obtain Formula (11) by taking $v_r = v_{0,r}$. □

If $\Re\lambda_r \neq 0$, the representation (9) of $P_r x(t)$ in Theorem 2 is given as follows.

Proposition 2.2 (1) *If $\Re\lambda_r > 0$, then*

$$Y_r(w, b_f) = P_r w + \int_0^\infty e^{-sA} P_r f(s) ds, \tag{14}$$

and the representation (9) of $P_r x(t)$ becomes

$$P_r x(t) = e^{tA} Y_r(w, b_f) - \int_t^\infty e^{(t-s)A} P_r f(s) ds. \tag{15}$$

(2) *If $\Re\lambda_r < 0$, then*

$$Y_r(w, b_f) = P_r w - \int_{-\infty}^0 e^{-sA} P_r f(s) ds, \tag{16}$$

and the representation (9) of $P_r x(t)$ becomes

$$P_r x(t) = e^{tA} Y_r(w, b_f) + \int_{-\infty}^t e^{(t-s)A} P_r f(s) ds. \tag{17}$$

3 Behavior of solutions

First, we can describe the asymptotic behavior of the solution by *an index of growth order* $(m_1, m_2, \cdots, m_\ell)$ for the initial value w defined as follows. If $\lambda_r \notin i\omega\mathbb{Z}$, $m_r = 0$ in the case that $Y_r(w, b_f) = 0$; otherwise, m_r is a positive integer such that

$$(A - \lambda_r I)^{m_r - 1} Y_r(w, b_f) \neq 0, \ (A - \lambda_r I)^{m_r} Y_r(w, b_f) = 0.$$

If $\lambda_r \in i\omega\mathbb{Z}$, $m_r = 0$ in the case that $Z_r(w, b_f) = 0$; otherwise, m_r is a positive integer such that

$$(A - \lambda_r I)^{m_r - 1} Z_r(w, b_f) \neq 0, \ (A - \lambda_r I)^{m_r} Z_r(w, b_f) = 0.$$

The following result can be easily proved by using Theorem 2 and Proposition 2.2. Set $\mathbb{R}_+ = [0, \infty)$.

Theorem 3 *Let $P_r x(t)$ be the projection of a solution $x(t)$ with $x(0) = w$ to $\mathbf{M}_r$.*

1) *The case where $\lambda_r \notin i\omega\mathbb{Z}$.*

(1) *Let $\Re\lambda_r > 0$. If $m_r = 0$, $P_r x(t)$ is τ-periodic ; if $m_r \geq 1$, $P_r x(t)$ is unbounded on $\mathbb{R}_+$ such that*

$$P_r x(t) = -\int_t^\infty e^{(t-s)A} P_r f(s) ds + e^{t\lambda_r} \frac{t^{m_r - 1}}{(m_r - 1)!} (A - \lambda_r I)^{m_r - 1} Y_r(w, b_f) + o(e^{t\lambda_r} t^{m_r - 1}) \quad (t \to \infty). \tag{18}$$

(2) *Let $\Re\lambda_r < 0$. If $m_r = 0$, $P_r x(t)$ is τ-periodic ; if $m_r \geq 1$, $P_r x(t)$ is asymptotically τ-periodic such that*

$$P_r x(t) = \int_{-\infty}^t e^{(t-s)A} P_r f(s) ds + e^{t\lambda_r} \frac{t^{m_r - 1}}{(m_r - 1)!} (A - \lambda_r I)^{m_r - 1} Y_r(w, b_f) + o(e^{t\lambda_r} t^{m_r - 1}) \quad (t \to \infty). \tag{19}$$

(3) *Let $\Re\lambda_r = 0$. If $m_r = 0$, $P_r x(t)$ is τ-periodic ; if $m_r = 1$, $P_r x(t)$ is quasi periodic ; if $m_r \geq 2$, $P_r x(t)$ is unbounded on $[0, \infty)$ as well as on $(-\infty, 0]$ such that*

$$P_r x(t) = e^{t\lambda_r} \frac{t^{m_r - 1}}{(m_r - 1)!} (A - \lambda_r I)^{m_r - 1} Y_r(w, b_f) + o(|t|^{m_r - 1}) \quad (|t| \to \infty).$$

2) *The case where $\lambda_r \in i\omega\mathbb{Z}$.*

(1) *If $m_r = 0$, $P_r x(t)$ is τ-periodic.*

(2) *If $m_r \geq 1$, $P_r x(t)$ is unbounded on $[0, \infty)$ as well as on $(-\infty, 0]$ such that*

$$P_r x(t) = e^{t\lambda_r} \frac{t^{m_r}}{m_r!} \frac{1}{\tau} (A - \lambda_r I)^{m_r - 1} Z_r(w, b_r) + o(|t|^{m_r}) \quad (|t| \to \infty).$$

Next, using Theorem 3, we can describe the necessary and sufficient conditions for the solution of Eq.(1) to be bounded on $\mathbb{R}_+$ or τ-periodic.

Proposition 3.1 *The solution $x(t)$ of Eq.(1) with $x(0) = w$ is bounded on $\mathbb{R}_+$ if and only if $w \in \mathbb{C}^m$ satisfies the following conditions : for every $\lambda_r \in \sigma(A)$,*

1) *if $\Re\lambda_r > 0$, then $m_r = 0$, i.e. $Y_r(w, b_f) = 0$;*
2) *if $\Re\lambda_r = 0$, $\lambda_r \notin i\omega\mathbb{Z}$, then $m_r = 1$, i.e. $A_{1,r} Y_r(w, b_f) = 0$; and*
3) *if $\lambda_r \in i\omega\mathbb{Z}$, then $m_r = 0$, i.e. $Z_r(w, b_f) = 0$.*

Corollary 3.2 *Assume that there is an $M > 0$ such that $\|e^{tA}\| \leq M$ for all $t \geq 0$. All solutions of Eq.(1) are bounded on $\mathbb{R}_+$ if and only if $P_r b_f = 0$ for all r such that $\lambda_r \in i\omega\mathbb{Z}$.*

Corollary 3.3 *The solution $x(t)$ of Eq.(1)with $x(0) = 0$ or b_f is bounded on $\mathbb{R}_+$ if and only if $b_f \in C^m$ satisfies the following conditions : for every $\lambda_r \in \sigma(A)$,*

1) *if $\Re\lambda_r > 0$, then $P_r b_f = 0$;*
2) *if $\Re\lambda_r = 0$, $\lambda_r \notin i\omega\mathbb{Z}$, then $P_r b_f \in \mathcal{N}(A - \lambda_r I)$; and*
3) *if $\lambda_r \in i\omega\mathbb{Z}$, then $P_r b_f = 0$.*

The condition for τ-periodic solutions becomes as follows.

Proposition 3.4 *The following statements are equivalent :*
1) *The solution $x(t)$ of Eq.(1) with $x(0) = w$ is τ-periodic.*
2) *For every $r = 1, 2, \cdots, \ell$, $m_r = 0$, that is,*
 (1) *if $\lambda_r \notin i\omega\mathbb{Z}$, then $Y_r(w, b_f) = 0$; and*
 (2) *if $\lambda_r \in i\omega\mathbb{Z}$, then $Z_r(w, b_f) = 0$.*
3)

$$(I - e^{\tau A})w = b_f. \tag{20}$$

Finally, we derive necessary and sufficient conditions for f such that Eq.(1) has bounded solutions on $\mathbb{R}_+$.

Theorem 4 *The following statements are equivalent :*
1)*Eq.(1) has a solution which is bounded on $\mathbb{R}_+$.*
2)*$P_r b_f \in (A - \lambda_r I)\mathbf{M}_r$ for every r such that $\lambda_r \in i\omega\mathbb{Z}$.*
3) *$b_f \in \mathcal{R}(A - \lambda_r I)$ for every r such that $\lambda_r \in i\omega\mathbb{Z}$.*

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