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Kyoto University
ON A SLOW-FAST SYSTEM IN $R^6$ WITH DUCKS

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Abstract. The singular perturbation problem in $R^n (n > 3)$ includes a possibility having a constrained surface with a 3-dimensional differentiable manifold. We will take up the system in $R^6$ having such a constrained surface. Although it is difficult to analyze these systems in general, we will show some sufficient conditions to make it possible. Furthermore, we will reduce the system to the problem in $R^3$ and show the existence of the duck solutions using Benoit’s criterion.

1. Introduction

S.A. Campbell, one of authors of [3], investigated first the coupled FitzHugh-Nagumo equations as a bifurcation problem. In the system, we have already proved the existence of the winding duck solutions in $R^4$ ([4]). As the associated slow-fast system (or singular perturbation problem) has a 2-dimensional slow manifold (constrained surface), it is able to reduce it to the slow-fast one in $R^3$. In this paper, we take up the system in $R^6$ with a 3-dimensional slow manifold. A typical example of this system is a 3-paralleled FitzHugh-Nagumo equations. There exists a 2-dimensional undefined region in the corresponding time scaled reduced system. This region makes it turn to have a projected slow-fast subsystem in $R^3$.

2. Preliminaries

Let consider a constrained system (2.1):

\[
\begin{align*}
\frac{dx}{dt} &= f(x, y, z, u), \\
\frac{dy}{dt} &= g(x, y, z, u), \\
h(x, y, z, u) &= 0,
\end{align*}
\]

where $u$ is a parameter (any fixed) and $f, g, h$ are defined in $R^3 \times R^1$. Furthermore, let consider the singular perturbation problem of the system (2.1):

\[
\begin{align*}
\frac{dx}{dt} &= f(x, y, z, u), \\
\frac{dy}{dt} &= g(x, y, z, u), \\
\frac{dz}{dt} &= h(x, y, z, u),
\end{align*}
\]
where $\epsilon$ is infinitesimally small.

We assume that the system (2.1) satisfies the following conditions $(A1) - (A5)$:

(A1) $f$ and $g$ are of class $C^1$ and $h$ is of class $C^2$.

(A2) The set $S = \{(x, y, z) \in \mathbb{R}^3|h(x, y, z, u) = 0\}$ is a 2-dimensional differentiable manifold and the set $S$ intersects the set $T = \{(x, y, z) \in \mathbb{R}^3|\partial h(x, y, z, u)/\partial z = 0\}$ transversely so that the pli set $PL = \{(x, y, z) \in S \cap T\}$ is a 1-dimensional differentiable manifold.

(A3) Either the value of $f$ or that of $g$ is nonzero at any point $p \in PL$.

Let $(x(t, u), y(t, u), z(t, u))$ be a solution of (2.1). By differentiating $h(x, y, z, u)$ with respect to the time $t$, the following equation holds:

\[
(2.3) \quad h_x(x, y, z, u)f(x, y, z, u) + h_y(x, y, z, u)g(x, y, z, u) + h_z(x, y, z, u)\frac{dz}{dt} = 0,
\]

where $h_i(x, y, z, u) = \partial h(x, y, z, u)/\partial i$, $i = x, y, z$. The above system (2.1) becomes the following system:

\[
(2.4) \quad \begin{align*}
\frac{dx}{dt} &= f(x, y, z, u), \\
\frac{dy}{dt} &= g(x, y, z, u), \\
\frac{dz}{dt} &= -\{h_x(x, y, z, u)f(x, y, z, u) + h_y(x, y, z, u)g(x, y, z, u)\}/h_z(x, y, z, u),
\end{align*}
\]

where $(x, y, z) \in S \setminus PL$. The system (2.1) coincides with the system (2.4) at any point $p \in S \setminus PL$. In order to study the system (2.4), let consider the following system:

\[
(2.5) \quad \begin{align*}
\frac{dx}{dt} &= -h_x(x, y, z, u)f(x, y, z, u), \\
\frac{dy}{dt} &= -h_x(x, y, z, u)g(x, y, z, u), \\
\frac{dz}{dt} &= h_x(x, y, z, u)f(x, y, z, u) + h_y(x, y, z, u)g(x, y, z, u).
\end{align*}
\]

As the system (2.5) is well defined at any point of $\mathbb{R}^3$, it is well defined indeed at any point of $PL$. The solutions of (2.4) coincide with those of (2.1) on $S \setminus PL$ except the velocity when they start from the same initial points.

(A4) For any $(x, y, z) \in S$, either of the following holds:

\[
(2.6) \quad h_y(x, y, z, u) \neq 0, h_x(x, y, z, u) \neq 0,
\]

that is, the surface $S$ can be expressed as $y = \varphi(x, z, u)$ or $x = \psi(y, z, u)$ in the neighborhood of $PL$. Let $y = \varphi(x, z, u)$ exist, then the projected system, which restricts the system (2.5) is obtained:

\[
(2.7) \quad \begin{align*}
\frac{dx}{dt} &= -h_x(x, \varphi(x, z, u), z, u)f(x, \varphi(x, z, u), z, u), \\
\frac{dz}{dt} &= h_x(x, \varphi(x, z, u), z, u)f(x, \varphi(x, z, u), z, u) + h_y(x, \varphi(x, z, u), z, u)g(x, \varphi(x, z, u), z, u).
\end{align*}
\]

(A5) All the singular points of (2.7) are nondegenerate, that is, the matrix induced from the linearized system of (2.7) at a singular point has two nonzero eigenvalues. Note that all the points contained in $PS = \{(x, y, z) \in PL|\frac{dz}{dt} = 0\}$, which is called pseudo singular points are the singular points of (2.5).
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3. SLOW-FAST SYSTEM IN $\mathbb{R}^6$

Let us consider the following slow-fast system:

$$
\begin{align*}
\epsilon \frac{dx_1}{dt} &= h_1(x, y, u), \\
\epsilon \frac{dx_2}{dt} &= h_2(x, y, u), \\
\epsilon \frac{dx_3}{dt} &= h_3(x, y, u), \\
dy_1 / dt &= f_1(x, y, u), \\
dy_2 / dt &= f_2(x, y, u), \\
dy_3 / dt &= f_3(x, y, u),
\end{align*}
$$

(3.1)

where $x^t = (x_1, x_2, x_3)$, $y^t = (y_1, y_2, y_3)$, are variables, $u \in \mathbb{R}$ is a parameter and $\epsilon$ is infinitesimal in the sense of non-standard analysis of Nelson. We put $h^t = (h_1, h_2, h_3)$, $f^t = (f_1, f_2, f_3)$, then assume that rank $[Jh] = 3$ with respect to $y$, that is, there exists $h_y^{-1}$. Then, $y$ is uniquely described like as $y = \psi(x, u)$. On the constrained surface, when $\epsilon$ tends to zero,

$$
\begin{align*}
\frac{dy}{dt} &= 0,
\end{align*}
$$

(3.2)

differentiating it by $t$,

$$
\begin{align*}
h_x \frac{dx}{dt} + h_y \frac{dy}{dt} &= 0.
\end{align*}
$$

(3.3)

By the above assumption, we can obtain

$$
\begin{align*}
\frac{dy}{dt} &= -h_y^{-1}h_x \frac{dx}{dt} \\
&= f(x, y, u).
\end{align*}
$$

(3.4)

In the system (3.4), we can reduce it to the time scaled reduced system:

$$
\begin{align*}
\frac{dx}{dt} &= -\det(h_y^{-1}h_x)(h_y^{-1}h_x)^{-1}f(x, \psi(x, u), u).
\end{align*}
$$

(3.5)

The singular point of the system (3.5) is called a generalized pseudo-singular point (GPS). Here, we assume that the generalized pli set GPL:

$$
\begin{align*}
\text{GPL} &= \{(x, u) | \det(h_y^{-1}h_x) = 0\}
\end{align*}
$$

(3.6)

gives a 2-dimensional differential manifold. Note that this is well defined in the original system (3.1).

Let us consider the following slow-fast system induced from the system (3.5):

$$
\begin{align*}
x_1' &= -\left[\det(h_y^{-1}h_x)(h_y^{-1}h_x)^{-1}f(x, \psi(x, u), u)\right]_1 = k_1(x, u), \\
x_2' &= -\left[\det(h_y^{-1}h_x)(h_y^{-1}h_x)^{-1}f(x, \psi(x, u), u)\right]_2 = k_2(x, u), \\
\epsilon x_3' &= \det(h_y^{-1}h_x) = k_0(x, u),
\end{align*}
$$

(3.7)

where $dx_i / dt = \dot{x}_i$ and $[\ast]^i_i = k_i(x, u)$ denotes the $i$th component of the vector $[\ast]$. Here, we assume that $|x_1 - \dot{x}_2|$, $|\dot{x}_2 - \dot{x}_3|$, and $|\dot{x}_3 - x_1|$ are limited.

The above system is a projected subsystem into $\mathbb{R}^3$ as an approximation of the original system (3.1). We suppose further that this system satisfies the assumptions $(A1)-(A5)$ in the section 2 and the intersection of the set $PS_3$ of the pseudo-singular points in the system (3.7) and the set GPS is not empty. Especially, the assumption $(A2)$ makes an important role in this framework, that is, $S = \{(x, u) | k_0(x, u) = 0\}$ intersects $T_3 = \{(x, u) \partial k_0(x, u) / \partial x_3 = 0\}$ transversely.
Let $\lambda_1, \lambda_2$ be the eigenvalues of the linearized system of the time scaled reduced system. If the PS has a saddle ($\lambda_1 > 0, \lambda_2 < 0$ or $\lambda_1 < 0, \lambda_2 > 0$) or a node point ($\lambda_1 > 0, \lambda_2 > 0$ or $\lambda_1 < 0, \lambda_2 < 0$ and $\lambda_1/\lambda_2 \notin Q$), there exist duck solutions in the slow-fast system.

Let us consider the projected subsystems:

$$\frac{dx_i}{dt} = -[\det(h_y^{-1}h_x)](h_y^{-1}h_x)^{-1}f(x, \psi(x, u), u)]_i = k_i(x, u), (i \neq j),$$
$$\epsilon \frac{dx_j}{dt} = k_0(x, u).$$

Here, in the above systems, we assume that there exists the number $j$ such that the set $S$ intersects the set $T_j$ transversely and the intersection of the set $GPS$ and the set $PS_j$ is not empty.

**Theorem 2.** In the $j$ th system with $(A1)-(A5)$, if $PS_j$ has a saddle or a node point satifying the above, there exist duck solutions in the original system (3.1).

### 4. A 3-PARALLELED FITZHUGH-NAGUMO EQUATIONS

Let $h, f$ in the section 3 be

$$(h_1) = \begin{pmatrix} y_1 - x_1^3/3 + x_2 - x_3 \\ y_2 - x_2^3/3 + x_1 \\ y_3 - x_3^3/3 - x_2 + 2x_3 \end{pmatrix},$$

$$(f_1) = \begin{pmatrix} -(x_1 + by_1 - a_1) \\ -(x_2 + by_2 - a_2) \\ -(x_3 + by_3 - a_3) \end{pmatrix},$$

then we can obtain the following system like as the system (3.4)

$$(x_1^2 - 1 1 - 1 \begin{pmatrix} x_2^2 \end{pmatrix}) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

and then the set $GPL$:

$$GPL = \{(x, u)|\det(h_y^{-1}h_x) = (x_3^2 - 2)(x_1^2x_2^2 - 1) - 1 = 0\}$$

has 2-dimensional differentiable manifold satisfying the assumption $(A4)$. When putting the matrix in the equation (4.3) into $A$ simply, the time scaled reduced system:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (A_{ji}) \begin{pmatrix} bx_1^3/3 + x_1 - bx_2 + bx_3 - a_1 \\ -bx_1 + bx_2^3/3 + x_2 - a_2 \\ bx_2 + bx_3^3/3 + (1 - 2b)x_3 - a_3 \end{pmatrix},$$

where $(A_{ij})$ is the cofactor matrix of $A$, gives a pseudo singular point $(0, 0, a_1)$ ($3 < a_1 < 4, a_2 = 0, a_3 = 1$) of $GPS$ on $GPL$, when $b = -1$. 
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On the constrained surface: \( \det A = 0 \), which has a 2-dimensional differentiable manifold, let us consider a slow-fast subsystem projected in \( R^3 \): putting \( b = -1 \),

\[
\begin{align*}
\dot{x}_1 &= x_2^2(x_3^2 - 2)(-x_1^3/3 + x_1 + x_2 - x_3 - a_1) \\
&\quad + (x_3^2 - 1)(x_1 - x_2^3/3 + x_2) + x_2^2(x_2 + x_3^3/3 - 3x_3 + 1), \\
\epsilon \dot{x}_2 &= (x_3^2 - 2)(x_1^2x_2^2 - 1) - 1, \\
\dot{x}_3 &= x_1^3/3 - x_1 - x_2 + x_3 + a_1 - x_1^2(x_1 - x_2^3/3 + x_2) \\
&\quad + (x_1^2x_2^2 - 1)(-x_2 - x_3^3/3 + 3x_3 - 1).
\end{align*}
\]

Note that we use the variable \( x_2 \) instead of the variable \( x_3 \) in the system (3.7), that is, \( j = 2 \). The set \( T_2 \) is denoted as

\[
T_2 = \{(x, -1) | \partial k_0 / \partial x_2 = 2(x_3^2 - 2)x_2^2 = 0 \}.
\]

On the set PL, the sign of \( \partial k_0 / \partial x_2 \) changes when the variable \( x_2 \) increases from \(-\) to \(+\). Therefore, the set \( S \) intersects the set \( T_2 \) transversely. In case we choose the set \( T_1 \), it is possible to get the same result as the case choosing the set \( T_2 \). However, in case we do the set \( T_3 \), it is impossible to get the projected subsystem, because the set \( T_3 \):

\[
T_3 = \{(x, -1) | \partial k_0 / \partial x_3 = 2(x_1^2x_2^2 - 1)x_3 = 0 \},
\]

does not intersect the set \( S \) transversely. In fact, as the third component of the pseudo-singular point is positive (\( 3 < a_1 < 4 \)), the sign of \( \partial k_0 / \partial x_3 \) does not change on the set \( PL \). The time scaled reduced system of the equation (3.5) is

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{pmatrix} = \begin{pmatrix}
F(x, u) \\
G(x, u) \\
H(x, u)
\end{pmatrix},
\]

where

\[
\begin{align*}
F(x, u) &= x_2^2(x_3^2 - 2)(-x_1^3/3 + x_1 + x_2 - x_3 - a_1) \\
&\quad + (x_3^2 - 1)(x_1 - x_2^3/3 + x_2) + x_2^2(x_2 + x_3^3/3 - 3x_3 + 1), \\
G(x, u) &= (x_3^2 - 2)(-x_1^3/3 + x_1 + x_2 - x_3 - a_1) \\
&\quad + x_2^2(x_3^2 - 2)(x_1 - x_2^3/3 + x_2) + x_2 + x_3^3/3 - 3x_3 + 1, \\
H(x, u) &= x_1^3/3 - x_1 - x_2 + x_3 + a_1 - x_1^2(x_1 - x_2^3/3 + x_2) \\
&\quad + (x_1^2x_2^2 - 1)(-x_2 - x_3^3/3 + 3x_3 - 1).
\end{align*}
\]
Then first derivatives of the equations (4.10) are

\[
\frac{\partial F}{\partial x_1} = x_2^2(x_3^2 - 2)(-x_1^2 + 1) + x_3^2 - 1,
\]
\[
\frac{\partial F}{\partial x_2} = (x_3^2 - 2)(-2x_3x_2/3 + 2x_1x_2 + 3x_2^2 - 2x_1x_2 - 2a_1x_2) + (x_3^2 - 1)(-x_2^2 + 1) + 3x_2^2 + 2x_3^2x_2 - 6x_3x_2 + 2x_2,
\]
\[
\frac{\partial F}{\partial x_3} = 2x_2x_3(-x_1^3/3 + x_1 + x_2 - x_3 - a_1) - x_2^2(x_3^2 - 2) + 2x_3(x_1 - x_2^3/3 + x_2) + x_2^2(x_3^2 - 3),
\]
\[
\frac{\partial F}{\partial x_1} = (x_3^2 - 2)(-x_1^2 + 1) + (x_3^2 - 2)(3x_1^2 - 2x_2^3x_1/3 + 2x_2x_1) + (x_3^2 - 2) - x_1^2(x_3^2 - 2)(x_2^2 - 1) + 1,
\]
\[
\frac{\partial F}{\partial x_3} = 2x_3(-x_1^3/3 + x_1 + x_2 - x_3 - a_1) - (x_3^2 - 2) + 2x_2^2x_3(x_1 - x_2^3/3 + x_2) + x_2^2 - 3,
\]
\[
\frac{\partial G}{\partial x_1} = x_3^2 - 1 - 2x_1(x_1 - x_2^3/3 + x_2) - x_1^2 + 2x_1x_2^2(-x_2 - x_3^3/3 + 3x_3 - 1) - (x_1^2x_2^2 - 1),
\]
\[
\frac{\partial H}{\partial x_1} = -1 - x_1^2(-x_1^2 + 1) + 2x_2x_1^2(-x_2 - x_3^3/3 + 3x_3 - 1) + (x_1^2x_2^2 - 1),
\]
\[
\frac{\partial H}{\partial x_3} = 1 + (x_1^2x_2^2 - 1)(-x_3^2 + x_3).
\]

The corresponding Jacobian matrix $J_p$ at the pseudo singular point $p = (0, 0, a_1) \in PS \cap GPS$ is

\[
(4.12)
J_p = \begin{pmatrix}
a_1^2 - 1 & a_1^2 - 1 & 0 \\
a_1^2 - 2 & a_1^2 - 1 & -1 \\
-1 & 0 & a_1^2 - 2
\end{pmatrix}.
\]

For some constant $a_1(3 < a_1 < 4)$, the above eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are positive and $\lambda_1/\lambda_2 \not\in Q, \lambda_2/\lambda_3 \not\in Q$. Even if the parameter $b$ changes, it is established almost everywhere. Therefore, in the time scaled reduced system projected into $R^2$, the eigenvalues ensures that the pseudo singular points are node at around $b = -1$.

**Theorem3.** In the system (3.1) with the equations (4.1), (4.2), putting the parameter $b = -1$, it has duck solutions for some constant $3 < a_1 < 4$, $a_2 = 0$ and $a_3 = 1$.

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REFERENCES


1-28-1 Tamazutsumi Setagaya-Ku, Tokyo, 158-0087, Japan