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Kyoto University
Kneser's property for a semilinear parabolic partial differential equation with Dirichlet boundary condition

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1. Introduction. We consider an initial and boundary value problem

\[ \begin{align*}
\frac{\partial u}{\partial t} &= \Delta u + F(t, x, u) \quad \text{for} \ 0 < t \leq T, \ x \in D, \ u \in \mathbb{R} \\
\quad u(0, x) &= u_0(x) \quad \text{for} \ x \in \overline{D}, \\
\quad u(t, x) &= 0 \quad \text{for} \ 0 < t \leq T, \ x \in \partial D,
\end{align*} \]

(E_n)

where \( T > 0 \) is a given constant, \( D = (0,1)^n \subset \mathbb{R}^n \), \( F : [0, T] \times \overline{D} \times \mathbb{R} \to \mathbb{R} \) is continuous and \( u_0 \in C(\overline{D}, \mathbb{R}) \) satisfies \( u_0(x) = 0 \) on \( \partial D \). A continuous function \( u(t, x) \) defined on \([0, \tau] \times \overline{D}\) will be called a (mild) solution of (E_n) when \( u \) is expressed by

\[
\begin{align*}
u(t, x) &= \int_D G(t, x, y)u_0(y) \, dy + \int_0^t ds \int_D G(t-s, x, y)F(s, y, u(s, y)) \, dy,
\end{align*}
\]

where \( G \) is the fundamental solution of \( \partial u/\partial t = \Delta u \) with \( u = 0 \) on \( \partial D \).

We shall discuss the Kneser's property for solutions of (E_n). In [2] and [3], we proved that solutions of (E_n) have Kneser's property, where the boundary condition is replaced with Neumann boundary condition and \( D \) is assumed to be a bounded domain with smooth boundary.

In this article, we always assume the following assumption (A) to the function \( F \).

(A) \( F(t, x, y) \) is expressed by

\[
F(t, x, u) = f(t, x, u) + g(t, x, u),
\]

where \( f \) and \( g \) are continuous functions on \([0, T] \times \overline{D} \times \mathbb{R}\) and satisfy

\[
\begin{align*}
f(t, x, u) &= 0 \quad \text{for} \ 0 \leq t \leq T, \ x \in \partial D, \ u \in \mathbb{R}, \\
g(t, x, -u) &= -g(t, x, u) \quad \text{for} \ 0 \leq t \leq T, \ x \in \overline{D}, \ u \in \mathbb{R}.
\end{align*}
\]
Only for simplicity of notations, we shall state our results in the case where $n = 1$, and hence, (E$_n$) will be reduced to the problem
\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + F(t, x, u) \quad \text{for} \quad 0 < t \leq T, \quad x \in \overline{D} = [0, 1], \quad u \in \mathbb{R}, \\
u(0, x) &= u_0(x) \quad \text{for} \quad x \in \overline{D} = [0, 1], \\
u(t, 0) &= u(t, 1) = 0 \quad \text{for} \quad 0 < t \leq T,
\end{aligned}
\]
where $u_0$ is a continuous function satisfying $u_0(0) = u_0(1) = 0$. The following example shows that solutions of (E$_1$) are not always unique.

\textbf{Example.} Consider the following problem for $t > 0, x \in [0, 1]$ and $u \in \mathbb{R}$.
\[
\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \sqrt{\frac{x^4 - 2x^3 + x}{12}} \sqrt{|u|} + \frac{12u}{1 + x - x^2}, \\
u(0, x) &= 0, \\
u(t, 0) &= u(t, 1) = 0.
\end{aligned}
\]
It is clear that (E) admits the zero solution $u(t, x) \equiv 0$. Furthermore, it is not difficult to see that
\[
u(t, x) = \frac{t^2(x^2 - x)(x^2 - 1)}{48} = \frac{t^2}{4} \cdot \frac{x^4 - 2x^3 + x}{12}
\]
is also a solution of (E).

\textbf{Remark.} The function $F$ in (E) satisfies assumption (A).

2. \textbf{Compactness of solutions.} It is well known (e.g. [1]) that the fundamental solution $G$ for $\partial u/\partial t = \partial^2 u/\partial x^2$ with $u(t, 0) = u(t, 1) = 0$ is expressed by
\[
G(t, x, y) = \sum_{k=-\infty}^{k=\infty} \{E(t, x - y + 2k) - E(t, x + y + 2k)\},
\]
where $E(t, \xi) = (4\pi t)^{-1/2} \exp(-\xi^2/4t)$ for $t > 0, \xi \in \mathbb{R}$.

Let $X$ be any metric space. We shall denote by $BC(X, \mathbb{R})$ the Banach space of all bounded and continuous functions on $X$ with the norm $\| \cdot \|$ defined by
\[
\|v\| = \sup\{|v(x)|; x \in X\}
\]
for $v \in BC(X, \mathbb{R})$. Similarly, for any compact metric space $X$, we shall denote by
$C(X, \mathbb{R})$ the Banach space of all continuous functions on $X$ with the norm $\| \cdot \|$ given by (3).

By assumption (A), the functions $f$ and $g$ admit a continuous and nondecreasing function $\varphi : [0, \infty) \to (0, \infty)$ with the property that

$$|f(t, x, u)| \leq \varphi(|u|), \quad |g(t, x, u)| \leq \varphi(|u|)$$

for $(t, x, u) \in [0, T] \times [0, 1] \times \mathbb{R}$.

Now we shall define several extensions of the functions $u_0(x), u(t, x), f(t, x, u)$ and $g(t, x, u)$ in the following way. For a function $u_0 \in C([0, 1], \mathbb{R})$ with $u_0(0) = u_0(1) = 0$, we can easily construct a continuous extension $\hat{u}_0 : \mathbb{R} \to \mathbb{R}$ of $u$ which satisfies that $\hat{u}_0(x)$ is an odd mapping and is 2-periodic. Similarly, for $\tau \in (0, T]$ and for a function $u = u(t, x) \in C([0, \tau] \times [0, 1], \mathbb{R})$ satisfying $u(t, 0) = u(t, 1) = 0$ on $[0, \tau]$, let $\tilde{u} = \tilde{u}(t, x) \in C([0, \tau] \times \mathbb{R}, \mathbb{R})$ be a continuous extension of $u$ which is an odd mapping and 2-periodic in $x$ for each $t \in [0, \tau]$, while let $\tilde{u} = \tilde{u}(t, x) \in C([0, \tau] \times \mathbb{R}, \mathbb{R})$ be a continuous extension of $u$ which is an even mapping and 2-periodic in $x$ for each $t \in [0, \tau]$. Finally, for the functions $f$ and $g$ satisfying (1), let $\hat{f} = \hat{f}(t, x, u) \in C([0, T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ be an extension of $f$ which is an odd mapping and 2-periodic in $x$ for each $(t, u) \in [0, T] \times \mathbb{R}$, while $\tilde{g} = \tilde{g}(t, x, u) \in C([0, T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ be an extension of $g$ which is an even mapping and 2-periodic in $x$ for each $(t, u) \in [0, T] \times \mathbb{R}$. Here, notice that $\tilde{g}(t, x, u)$ is an odd mapping in $u$ because of (1).

**Lemma 1.** For a function $u_0 \in C([0, 1], \mathbb{R})$ with $u_0(0) = u_0(1) = 0$, we have

$$\int_D G(t, x, y)u_0(y) \, dy = \int_{\mathbb{R}} E(t, x - y)\hat{u}_0(y) \, dy.$$

**Proof.** It follows from (2) that

$$\int_D G(t, x, y)u_0(y) \, dy$$

$$= \sum_{k = -\infty}^{k = \infty} \left\{ \int_0^1 E(t, x - y + 2k)u_0(y) \, dy - \int_0^1 E(t, x + y + 2k)u_0(y) \, dy \right\} \, dy$$

$$= \sum_{k = -\infty}^{k = \infty} \left\{ \int_{-2k}^{1-2k} E(t, x - z)u_0(z + 2k) \, dz + \int_{-2k}^{-1-2k} E(t, x - z)u_0(-z - 2k) \, dz \right\}$$

$$= \sum_{k = -\infty}^{k = \infty} \left\{ \int_{-2k}^{1-2k} E(t, x - z)\hat{u}_0(z) \, dz + \int_{-1-2k}^{-2k} E(t, x - z)\hat{u}_0(z) \, dz \right\} \, dz$$
Lemma 2. Suppose that (A) holds and that \( \tau \in (0, T] \). Then for a function \( u \in C([0, \tau] \times [0, 1], \mathbb{R}) \) satisfying \( u(t, 0) = u(t, 1) = 0 \) for \( t \in [0, \tau] \), it follows, for \( 0 \leq s \leq t \leq \tau \), that

\[
\int_D G(t - s, x, y)f(s, y, u(s, y))\,dy = \int_{\mathbb{R}} E(t - s, x - y)\hat{f}(s, y, \tilde{u}(s, y))\,dy
\]

and

\[
\int_D G(t - s, x, y)g(s, y, u(s, y))\,dy = \int_{\mathbb{R}} E(t - s, x - y)\tilde{g}(s, y, \hat{u}(s, y))\,dy.
\]

Proof. It is easy to observe that the following equalities hold for each \((s, y) \in [0, \tau] \times \mathbb{R}\).

\[
\hat{f}(s, -y, \hat{u}(s, -y)) = -\hat{f}(s, y, \hat{u}(s, y)), \quad \hat{f}(s, y + 2, \hat{u}(s, y + 2)) = \hat{f}(s, y, \hat{u}(s, y)),
\]

\[
\tilde{g}(s, -y, \hat{u}(s, -y)) = -\tilde{g}(s, y, \hat{u}(s, y)), \quad \tilde{g}(s, y + 2, \hat{u}(s, y + 2)) = \tilde{g}(s, y, \hat{u}(s, y)).
\]

By using the similar arguments as in the proof of Lemma 1, we can easily prove the assertion of the lemma.

Let \( h : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) be a continuous function satisfying

\[
|h(t, x, u)| \leq \varphi(|u|) \quad \text{for} \quad (t, x, u) \in [0, T] \times \mathbb{R} \times \mathbb{R},
\]

where \( \varphi : [0, \infty) \rightarrow (0, \infty) \) is a continuous and nondecreasing function introduced in the above. For this function \( h, \tau \in (0, T] \) and for \( u \in BC([0, \tau] \times \mathbb{R}, \mathbb{R}) \), define a function \( H(h, u, \tau) \) on \([0, \tau] \times \mathbb{R}\) by

\[
[H(h, u, \tau)](t, x) = \int_0^t ds \int_{\mathbb{R}} E(t - s, x - y)h(s, y, u(s, y))\,dy.
\]

By using similar arguments as in the proof of Lemma 1.5 in [2], we can prove the following lemma.

Lemma 3. For any \( \tau \in (0, T] \), \( u \in BC([0, \tau] \times \mathbb{R}, \mathbb{R}) \) and for any function \( h \) satisfying (5), we have

\[
|[H(h, u, \tau)](t, x) - [H(h, u, \tau)](t', x')|
\leq 8M\sqrt{t}\sqrt{t' - t} + M(t' - t) + 2\sqrt{2}M\sqrt{t}|x - x'|.
\]
for any $0 \leq t < t' \leq \tau$ and $x, x' \in \mathbb{R}$, where $M = \sup \{|h(t, u(t, x))|; t \in [0, \tau], x \in \mathbb{R}\} \leq \varphi(||u||) < \infty$.

**Theorem 1 (Existence).** Suppose that (A) holds. Then for any function $u_0 \in C([0, 1], \mathbb{R})$ with $u_0(0) = u_0(1) = 0$, there exists at least one solution $u(t, x)$ of ($E_1$) on $[0, \tau] \times [0, 1]$ for some $\tau > 0$.

**Proof.** Put $||u_0|| = M_0$ and take a number $L$ satisfying $L > M_0$. Then we can choose a number $\tau > 0$ so that an inequality

$$M_0 + 2\varphi(L)\tau \leq L$$

holds. We denote by $V$ the set of all functions $u \in C([0, \tau] \times [0, 1], \mathbb{R})$ which satisfy that $||u|| \leq L$, $u(t, 0) = u(t, 1) = 0$ and that $u(0, x) = u_0(x)$ for $x \in [0, 1]$. Then $V$ is a closed and convex subset of $C([0, \tau] \times [0, 1], \mathbb{R})$. For every $v \in V$, we define a mapping $\Psi v : [0, \tau] \times [0, 1] \rightarrow \mathbb{R}$ by $[\Psi v](0, x) = u_0(x)$ for $x \in [0, 1]$ and

$$[\Psi v](t, x) = \int_D G(t, x, y)u_0(y)dy + \int_0^t ds \int_D G(t-s, x, y)F(s, y, v(s, y))dy$$

for $0 < t \leq \tau$, $x \in [0, 1]$. Then $\Psi v$ belongs to $C([0, \tau] \times [0, 1], \mathbb{R})$ and $[\Psi v](t, 0) = [\Psi v](t, 1) = 0$ for $t \in (0, \tau]$. It follows from Lemmas 1 and 2 that

$$[\Psi v](t, x) = \int_\mathbb{R} E(t, x-y)\hat{u}_0(y)dy$$

$$+ \int_0^t ds \int_\mathbb{R} E(t-s, x-y)\hat{f}(s, y, \tilde{v}(s, y))dy$$

$$+ \int_0^t ds \int_\mathbb{R} E(t-s, x-y)\tilde{g}(s, y, \hat{v}(s, y))dy,$$

thus we have

$$||[\Psi v](t, x)|| \leq M_0 + \int_0^t ds \int_\mathbb{R} E(t-s, x-y)\varphi(||\hat{v}||)dy$$

$$+ \int_0^t ds \int_\mathbb{R} E(t-s, x-y)\varphi(||\tilde{v}||)dy$$

$$\leq M_0 + 2\varphi(L)\tau \leq L$$

because $\int_\mathbb{R} E(t, x-y)dy = 1$. Therefore, we obtain that $\Psi(V) \subset V$. It follows from (6) and Lemma 3 that $\Psi(V)$ is relatively compact, and hence, we can find a fixed point $u$ in $V$ by Shauder's fixed point theorem. Clearly, $u$ is a solution of ($E_1$), which completes the proof. $\square$
Lemma 4. Suppose that (A) holds. Then there exist two numbers $\tau > 0$ and $M > 0$ such that every solution $u$ of (E) exists and satisfies $|u(t, x)| \leq M$ on $[0, \tau] \times [0, 1]$.

Proof. Put $||u_0|| = M_0$. Then any solution $u$ of (E) satisfies

$$|u(t, x)| \leq M_0 + 2 \int_0^t \int_{\mathbb{R}} E(t-s, x-y) \varphi(||u(s)||) dy ds$$

for $t > 0$ and $x \in [0, 1]$ as long as $u$ exists, where $||u(s)|| = \sup\{|u(s, y)|; y \in [0, 1]\}$. Therefore, it follows that

$$||u(t)|| \leq M_0 + 2 \int_0^t \varphi(||u(s)||) ds.$$

If we put $v(t) := ||u(t)||$ and $w(t) := M_0 + 2 \int_0^t \varphi(v(s)) ds$ for $t > 0$, then we have $v(t) \leq w(t)$ and $w'(t) = 2\varphi(v(t)) \leq 2\varphi(w(t))$. By the comparison theorem in the theory of ordinary differential equations, the maximal solution $p(t)$ of $p' = 2\varphi(p)$ with $p(0) = M_0$ exists on $[0, \tau]$ for some $\tau > 0$ and an inequality $p(\tau) \geq p(t) \geq w(t)$ holds on $[0, \tau]$. By putting $M = p(\tau)$, we have the assertion. \qed

3. Kneser's property. For the functions $f$ and $g$ satisfying (1) and for $m \in \mathbb{N}$, we put

$$f_m(t, x, u) = \frac{m}{2} \int_{u-\frac{1}{m}}^{u+\frac{1}{m}} f(t, x, v) dv, \quad g_m(t, x, u) = \frac{m}{2} \int_{u-\frac{1}{m}}^{u+\frac{1}{m}} g(t, x, v) dv.$$ 

Then $f_m(t, x, u) = 0$ for $x = 0, 1$, while $g_m(t, x, -u) = -g_m(t, x, u)$ by virtue of (1). It is easy to see that $\{f_m\}$ and $\{g_m\}$ converge, respectively, to $f$ and $g$ uniformly on every compact set in $[0, T] \times [0, 1] \times \mathbb{R}$. Clearly, $f_m$ and $g_m$ are locally Lipschitz continuous in $u$. Moreover, by the mean value theorem in integration, we have

$$|f_m(t, x, u)| \leq \frac{m}{2} \int_{u-\frac{1}{m}}^{u+\frac{1}{m}} |f(t, x, v)| dv \leq \frac{m}{2} \int_{u-\frac{1}{m}}^{u+\frac{1}{m}} \varphi(|v|) dv = \varphi(|u + \theta/m|) \leq \varphi(|u| + 1),$$

where $\theta$ is a suitable number satisfying $-1 < \theta < 1$. By replacing $\varphi(s + 1)$ by $\varphi(s)$, we may assume that $|f_m(t, x, u)| \leq \varphi(|u|)$. Similarly, we may also assume that $|g_m(t, x, u)| \leq \varphi(|u|)$. 


Theorem 2. Suppose that (A) holds and that \( u_0 \in C([0, 1], \mathbb{R}) \) is an arbitrary function satisfying \( u_0(0) = u_0(1) = 0 \). Then a family

\[
\mathcal{F} = \{ u \in C([0, \tau] \times [0, 1], \mathbb{R}); u \text{ is a solution of } (E_1) \}
\]

is compact and connected in \( C([0, \tau] \times [0, 1], \mathbb{R}) \) when \( \tau > 0 \) is sufficiently small.

Proof. By Lemma 4, there exist \( \tau > 0 \) and \( M > 0 \) such that every solution \( u \) of \( (E_1) \) exists and satisfies \( |u(t, x)| \leq M \) on \( [0, \tau] \times [0, 1] \). For this \( \tau > 0 \), we shall prove the assertion of the theorem.

It suffices to show that \( \mathcal{F} \) is connected because the compactness of \( \mathcal{F} \) is obvious by Lemma 3. Suppose that \( \mathcal{F} \) is not connected. Then there exist an open set \( \mathcal{O} \) and two nonempty compact sets \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) in \( C([0, \tau] \times [0, 1], \mathbb{R}) \) such that

\[
\mathcal{F}_1 \cup \mathcal{F}_2 = \mathcal{F}, \quad \mathcal{F}_1 \subset \mathcal{O}, \quad \mathcal{F}_2 \cap \overline{\mathcal{O}} = \emptyset.
\]

Let \( u_1 \) and \( u_2 \) be any elements in \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \), respectively. Then, for each \( m \in \mathbb{N} \), \( u_i \) is a solution of

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + H_i(t, x, u), \quad (i = 1, 2),
\]

where

\[
H_i(t, x, u) = F(t, x, u_i(t, x)) - F_m(t, x, u_i(t, x)) + F_m(t, x, u)
\]

and

\[
F_m(t, x, u) = f_m(t, x, u) + g_m(t, x, u).
\]

Let \( m \) be fixed. For any \( \theta \in [0, 1] \), define \( \Phi_\theta(t, x, u) \) by

\[
\Phi_\theta(t, x, u) = (1 - \theta)H_1(t, x, u) + \theta H_2(t, x, u).
\]

Then \( \Phi_\theta(t, x, u) \) is expressed by

\[
\Phi_\theta(t, x, u) = G_m(t, x) + f_m(t, x, u) + g_m(t, x, u),
\]

where

\[
G_m(t, x) = (1 - \theta)\{F(t, x, u_1(t, x)) - F_m(t, x, u_1(t, x))\}
\]

\[
+ \theta\{F(t, x, u_2(t, x)) - F_m(t, x, u_2(t, x))\}.
\]

Here, we notice that \( G_m(t, 0) = G_m(t, 1) = 0 \). Since \( \{G_m(t, x)\} \) converges to 0 uniformly on \( [0, \tau] \times [0, 1] \) as \( m \to \infty \), we may assume that \( |G_m(t, x)| \leq 1 \) for \( m \in \mathbb{N} \).
by taking a subsequence if necessary. Therefore, we may also assume that

$$|G_m(t, x) + f_m(t, x, u)| \leq \varphi(|u|)$$

by replacing $1 + \varphi(s)$ by $\varphi(s)$.

For any fixed $m \in \mathbb{N}$, a problem

$$\begin{aligned}
& \left\{ \begin{array}{lcl}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \Phi_\theta(t, x, u) & \text{for } 0 < t \leq \tau, x \in [0, 1], u \in \mathbb{R}, \\
 u(0, x) = u_0(x) & \text{for } x \in [0, 1], \\
 u(t, 0) = u(t, 1) = 0 & \text{for } 0 < t \leq \tau
\end{array} \right.
\end{aligned}$$

(E_\theta)

has a unique solution $v_\theta(t, x)$ because $\Phi_\theta(t, x, u)$ is locally Lipschitz continuous in $u$. Evidently, $v_0 = u_1$ and $v_1 = u_2$. Moreover, it is not difficult to verify that a mapping $\theta \mapsto v_\theta$ is continuous from $[0, 1]$ into $C([0, \tau] \times [0, 1], \mathbb{R})$, and hence, there exists a $\theta \in [0, 1]$ such that $v_\theta \in \partial O$. We denote these $\theta$ and $v_\theta$ by $\theta_m$ and $u_m$, respectively. Then $u_m$ is a solution of (E_{\theta_m}) and a relation $u_m \in \partial O$ holds. It follows from Lemma 3 that $\{u_m\}$ is equicontinuous on $[0, \tau] \times [0, 1]$, and hence, we may assume that $\{u_m\}$ converges uniformly to some $u \in C([0, \tau] \times [0, 1], \mathbb{R})$ by taking a subsequence if necessary. Since $\{\Phi_{\theta_m}\}$ converges to $f + g$ uniformly on every compact set in $[0, \tau] \times [0, 1] \times \mathbb{R}$, $u$ is a solution of (E_1), which implies that $u \in \partial O$ and $u \in \mathcal{F}$. This is a contradiction. \hfill \Box

The following corollary is a direct consequence of Theorem 2.

Corollary. Under the same assumptions as in Theorem 2, a set

$$\mathcal{F} = \{u(\tau) \in C([0, 1], \mathbb{R}); u \text{ is a solution of (E_1)}\}$$

is compact and connected in $C([0, 1], \mathbb{R})$ when $\tau > 0$ is sufficiently small.

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