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<td>Sagaki, Daisuke; Naito, Satoshi</td>
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Kyoto University
Path Model for a Level-Zero Extremal Weight Module over a Quantum Affine Algebra

佐垣 大輔 (Daisuke SAGAKI)  内藤 聡 (Satoshi NAITO)
筑波大学 数学系
Institute of Mathematics,
University of Tsukuba
sagaki@math.tsukuba.ac.jp  naito@math.tsukuba.ac.jp

0 Introduction.

Let \( \mathfrak{g} \) be a symmetrizable \( K \)-\( \mathcal{X} \)-\( \mathbf{M} \) algebra over \( \mathbb{Q} \) with the Cartan subalgebra \( \mathfrak{h} \) and the Weyl group \( W \). We fix an integral weight lattice \( P \subset \mathfrak{h}^* := \text{Hom}_\mathbb{Q}(\mathfrak{h}, \mathbb{Q}) \) that contains all simple roots of \( \mathfrak{g} \). Let \( \lambda \in P \) be an integral weight. In [L1] and [L2], Littelmann introduced the notion of Lakshmibai–Seshadri paths of shape \( \lambda \), which are piecewise linear, continuous maps \( \pi : [0, 1] \to P \) parametrized by pairs of a sequence of elements of \( W \lambda \) and a sequence of rational numbers satisfying a certain condition, called the chain condition. Denote by \( \mathcal{B}(\lambda) \) the set of Lakshmibai–Seshadri paths of shape \( \lambda \). Littelmann proved that \( \mathcal{B}(\lambda) \) has a normal crystal structure in the sense of [Kas3], and that if \( \lambda \) is a dominant integral weight, then the formal sum \( \sum_{\pi \in \mathcal{B}(\lambda)} e(\pi(1)) \) is equal to the character \( \text{ch} L(\lambda) \) of the integrable highest weight \( \mathfrak{g} \)-module \( L(\lambda) \) of highest weight \( \lambda \). Then he conjectured that \( \mathcal{B}(\lambda) \) for dominant \( \lambda \in P \) would be isomorphic to the crystal base of the integrable highest weight module of highest weight \( \lambda \) as crystals. This conjecture was affirmatively proved independently by Kashiwara [Kas4] and Joseph [J].

In [Kas2] and [Kas5], Kashiwara introduced an extremal weight module \( V(\lambda) \) of extremal weight \( \lambda \in P \) over the quantized universal enveloping algebra \( U_q(\mathfrak{g}) \) over \( \mathbb{Q}(q) \), and showed that it has a crystal base \( \mathcal{B}(\lambda) \). The extremal weight module is a natural generalization of an integrable highest (lowest) weight module. In fact, we know from [Kas2, §8] that if \( \lambda \in P \) is dominant (resp. anti-dominant), then the extremal weight module \( V(\lambda) \) is isomorphic to the integrable highest (resp. lowest) weight module of highest (resp. lowest) weight \( \lambda \), and the crystal base \( \mathcal{B}(\lambda) \) of \( V(\lambda) \) is isomorphic to the crystal base of the integrable highest (resp. lowest) weight module as a crystal.
Now, we assume that $g$ is of affine type. Let $I$ be the index set of the simple roots of $g$, and fix a special vertex $0 \in I$ as in [Kas5, §5.2]. In this paper, as an extension of the isomorphism theorem due to Kashiwara and Joseph, we prove that if $\lambda$ is a level-zero fundamental weight $\varpi_i \in P$ for $i \in I_0 := I \setminus \{0\}$ (see [Kas5, §5.2]; note that $\varpi_i$ is not dominant), then the connected component $B_0(\varpi_i)$ of $B(\varpi_i)$ containing $\pi_{\varpi_i}(t) := t \varpi_i$ is isomorphic to the crystal base $B(\varpi_i)$ of the extremal weight module $V(\varpi_i)$ as crystals. Namely, we prove the following:

**Theorem 1.** Assume that $g$ is of affine type. There exists a unique isomorphism $\Phi_{\varpi_i} : B(\varpi_i) \cong B_0(\varpi_i)$ of crystals such that $\Phi_{\varpi_i}(\varpi_i) = \pi_{\varpi_i}$, where $\varpi_i \in B(\varpi_i)$ is the unique extremal weight element of weight $\varpi_i$.

Let $g_s$ be the Levi subalgebra corresponding to a proper subset $S$ of the index set $I$, and let $U_q(g_s) \subset U_q(g)$ be the quantized universal enveloping algebra of $g_s$. By restriction, we can regard the crystals $B(\varpi_i)$ and $B_0(\varpi_i)$ for $U_q(g)$ as crystals for $U_q(g_s)$. We show the following branching rule for $B(\varpi_i)$ and $B_0(\varpi_i)$ as crystals for $U_q(g_s)$:

**Theorem 2.** As crystals for $U_q(g_s)$, $B(\varpi_i)$ and $B_0(\varpi_i)$ decompose as follows:

$$B(\varpi_i) \cong \bigsqcup_{\pi \in B(\varpi_i) \text{ $g_s$-dominant}} B_S(\pi(1)), \quad B_0(\varpi_i) \cong \bigsqcup_{\pi \in B_0(\varpi_i) \text{ $g_s$-dominant}} B_S(\pi(1)).$$

where $B_S(\lambda)$ is the set of Lakshmibai–Seshadri paths of shape $\lambda$ for $U_q(g_s)$, and $\pi \in B(\varpi_i)$ is said to be $g_s$-dominant if $(\pi(t))(\alpha_i^\vee) \geq 0$ for all $t \in [0, 1]$ and $i \in S$.

We also show that the extremal weight module $V(\varpi_i)$ of extremal weight $\varpi_i$ is completely reducible as a $U_q(g_s)$-module. Then, as an application of Theorems 1 and 2 above, we obtain the following branching rule for $V(\varpi_i)$:

**Theorem 3.** The extremal weight module $V(\varpi_i)$ of extremal weight $\varpi_i$ is completely reducible as a $U_q(g_s)$-module, and the decomposition of $V(\varpi_i)$ as a $U_q(g_s)$-module is given by:

$$V(\varpi_i) \cong \bigoplus_{\pi \in B_0(\varpi_i) \text{ $g_s$-dominant}} V_S(\pi(1)),$$

where $V_S(\lambda)$ is the integrable highest weight $U_q(g_s)$-module of highest weight $\lambda$.

Assume that $\varpi_i$ is minuscule, i.e., $\varpi_i(\alpha_i^\vee) \in \{\pm 1, 0\}$ for every dual real root $\alpha_i^\vee$ of $g$. Then we can check that $B(\varpi_i)$ is connected, and hence $B(\varpi_i) = B_0(\varpi_i)$.
In this case, we get the following decomposition rule of Littelmann type for the concatenation $\mathcal{B}(\lambda) \ast \mathcal{B}(\varpi_i)$. Here we note that unlike Theorems 2 and 3, this theorem does not necessarily imply the decomposition rule for tensor products of corresponding $U_q(\mathfrak{g})$-modules.

**Theorem 4.** Let $\lambda$ be a dominant integral weight which is not a multiple of the null root $\delta$ of $\mathfrak{g}$, and assume that $\varpi_i$ is minuscule. Then, the concatenation $\mathcal{B}(\lambda) \ast \mathcal{B}(\varpi_i)$ decomposes as follows:

$$\mathcal{B}(\lambda) \ast \mathcal{B}(\varpi_i) \cong \bigoplus_{\pi \in \mathcal{B}(\varpi_i)} \mathcal{B}(\lambda + \pi(1)),$$

where $\pi \in \mathcal{B}(\varpi_i)$ is said to be $\lambda$-dominant if $(\lambda + \pi(t))(\alpha_i^\vee) \geq 0$ for all $t \in [0,1]$ and $i \in I$.

**Remark.** The reader should compare Theorems 1 and 4 with the corresponding results [G, Theorems 1.5 and 1.6] of Greenstein for bounded modules.

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1 Preliminaries and Notation.

1.1 Quantized universal enveloping algebras. Let $A = (a_{ij})_{i,j \in I}$ be a symmetrical Cartan matrix, and $\mathfrak{g} := \mathfrak{g}(A)$ the Kac–Moody algebra over $\mathbb{Q}$ associated to the generalized Cartan matrix $A$. Denote by $\mathfrak{h}$ the Cartan subalgebra, by $\Pi := \{\alpha_i\}_{i \in I} \subset \mathfrak{h}^*$ and $\Pi^\vee := \{\alpha_i^\vee\}_{i \in I} \subset \mathfrak{h}$ the set of simple roots and simple coroots, and by $W = \langle r_i \mid i \in I \rangle$ the Weyl group. We take (and fix) an integral weight lattice $P \subset \mathfrak{h}^*$ such that $\alpha_i \in P$ for all $i \in I$.

Denote by $U_q(\mathfrak{g})$ the quantized universal enveloping algebra of $\mathfrak{g}$ over the field $\mathbb{Q}(q)$ of rational functions in $q$, and by $U_q^{-}(\mathfrak{g})$ (resp. $U_q^{+}(\mathfrak{g})$) the negative (resp. positive) part of $U_q(\mathfrak{g})$. We denote by $\widetilde{U}_q(\mathfrak{g}) = \bigoplus_{\lambda \in P} U_q(\mathfrak{g}) a_{\lambda}$ the modified quantized universal enveloping algebra of $\mathfrak{g}$, where $a_{\lambda}$ is a formal element of weight $\lambda$ (cf. [Kas2, §1.2]).

1.2 Affine Lie algebras. Assume that $\mathfrak{g}$ is of affine type. Let

$$\delta = \sum_{i \in I} a_i \alpha_i \in \mathfrak{h}^* \quad \text{and} \quad c = \sum_{i \in I} a_i^\vee \alpha_i^\vee \in \mathfrak{h}$$

(1.2.1)
be the null root and the canonical central element of $\mathfrak{g}$. We denote by $(\cdot, \cdot)$ the bilinear form on $\mathfrak{h}^*$, which is normalized by: $a_i^\vee = \frac{\langle \alpha_i, \alpha_i \rangle}{2} a_i$ for all $i \in I$. Set $\mathfrak{h}_0^* := \bigoplus_{i \in I} \mathbb{Q} \alpha_i \subset \mathfrak{h}^*$, and let $\text{cl} : \mathfrak{h}_0^* \rightarrow \mathfrak{h}_0^*/\mathbb{Q} \delta$ the canonical map from $\mathfrak{h}_0^*$ onto the quotient space $\mathfrak{h}_0^*/\mathbb{Q} \delta$. We have a bilinear form (also denoted by $(\cdot, \cdot)$) on $\mathfrak{h}_0^*/\mathbb{Q} \delta$ induced from the bilinear form $(\cdot, \cdot)$, which is positive-definite.

We take (and fix) a special vertex $0 \in I$ as in [Kas5, §5.2], and set $I_0 := I \setminus \{0\}$. For $i \in I_0$, let $\varpi_i$ be a unique element in $\bigoplus_{i \in I_0} \mathbb{Q} \alpha_i$ such that $\varpi_i(\alpha_j^\vee) = \delta_{i,j}$ for all $j \in I_0$. Notice that $\Lambda_i := \varpi_i + a_i^\vee \Lambda_0$ is an $i$-th fundamental weight for $\mathfrak{g}$, where $\Lambda_0$ is a 0-th fundamental weight for $\mathfrak{g}$. So, we may assume that all the $\varpi_i$'s are contained in the integral weight lattice $P$.

1.3 Crystal bases. Let $B(\infty)$ be the crystal base of the negative part $U_q^{-}(\mathfrak{g})$ with $u_\infty$ the highest weight element. Denote by $e_i$ and $f_i$ the raising and lowering Kashiwara operator on $B(\infty)$, respectively, and define $\varepsilon_i : B(\infty) \rightarrow \mathbb{Z}$ and $\varphi_i : B(\infty) \rightarrow \mathbb{Z}$ by

$$
\varepsilon_i(b) := \max\{n \geq 0 \mid e_i^n b \neq 0\}, \quad \varphi_i(b) := \varepsilon_i(b) + (\text{wt}(b))(\alpha_i^\vee).
$$

Denote by $\ast : B(\infty) \rightarrow B(\infty)$ the $\ast$-operation on $B(\infty)$ (cf. [Kas1, Theorem 2.1.1] and [Kas3, §8.3]). We put $e_i^* := \ast \circ e_i \circ \ast$ and $f_i^* := \ast \circ f_i \circ \ast$ for each $i \in I$.

**Theorem 1.3.1** (cf. [Kas1, Theorem 2.2.1]). For each $i \in I$, there exists an embedding $\Psi_i^{-} : B(\infty) \hookrightarrow B(\infty) \otimes B_i$ of crystals that maps $u_\infty$ to $u_\infty \otimes b_i(0)$, where $B_i := \{b_i(n) \mid n \in \mathbb{Z}\}$ is a crystal in [Kas1, Example 1.2.6]. In addition, if $b = (f_i^*)^k b_0$ for some $k \in \mathbb{Z}_{\geq 0}$ and $b_0 \in B(\infty)$ such that $e_i^* b_0 = 0$, then $\Psi_i^{-}(b) = b_0 \otimes b_i(-k)$.

We denote by $B(-\infty)$ the crystal base of the positive part $U_q^{+}(\mathfrak{g})$ with $u_{-\infty}$ the lowest weight vector, and by $e_i$ and $f_i$ the raising and lowering Kashiwara operator on $B(-\infty)$, respectively. We set

$$
\varepsilon_i(b) := \varphi_i(b) - (\text{wt}(b))(\alpha_i^\vee), \quad \varphi_i(b) := \max\{n \geq 0 \mid f_i^n b \neq 0\}.
$$

We also have the $\ast$-operation $\ast : B(-\infty) \rightarrow B(-\infty)$ on $B(-\infty)$. We can easily show that there exists an embedding $\Psi_i^{+} : B(-\infty) \hookrightarrow B_i \otimes B(-\infty)$ of crystals with properties similar to $\Psi_i^{-}$ in Theorem 1.3.1.

Let $B(\tilde{U}_q(\mathfrak{g})) = \bigsqcup_{\lambda \in \mathcal{P}} B(U_q(\mathfrak{g})a_\lambda)$ be the crystal base of the modified quantized universal enveloping algebra $\tilde{U}_q(\mathfrak{g})$ with $u_\lambda$ the element of $B(U_q(\mathfrak{g})a_\lambda)$ corresponding to $a_\lambda \in U_q(\mathfrak{g})a_\lambda$ (cf. [Kas2, Theorem 2.1.2]). We denote by $e_i$ and $f_i$ the raising
and lowering Kashiwara operator on \( B(\tilde{U}_q(g)) \), and define \( \epsilon_i : B(\tilde{U}_q(g)) \to \mathbb{Z} \) and \( \varphi_i : B(\tilde{U}_q(g)) \to \mathbb{Z} \) by

\[
\epsilon_i(b) := \max\{n \geq 0 \mid e_i^n b \neq 0\}, \quad \varphi_i(b) := \max\{n \geq 0 \mid f_i^n b \neq 0\}. \tag{1.3.3}
\]

We know the following theorem from [Kas2, Theorem 3.1.1].

**Theorem 1.3.2.** There exists an isomorphism \( \Xi_\lambda : B(U_q(g)a_\lambda) \cong B(\infty) \otimes T_\lambda \otimes B(-\infty) \) of crystals such that \( \Xi_\lambda(u_\lambda) = u_\infty \otimes t_\lambda \otimes u_{-\infty} \), where \( T_\lambda := \{ t_\lambda \} \) is a crystal consisting of a single element \( t_\lambda \) of weight \( \lambda \) (cf. [Kas3, Example 7.3]).

We also denote by \( * : B(\tilde{U}_q(g)) \to B(\tilde{U}_q(g)) \) the *-operation on \( B(\tilde{U}_q(g)) \) (cf. [Kas2, Theorem 4.3.2]). We know the following theorem from [Kas2, Corollary 4.3.3].

**Theorem 1.3.3.** Let \( b \in B(U_q(g)a_\lambda) \), and assume that \( \Xi_\lambda(b) = b_1 \otimes t_\lambda \otimes b_2 \) with \( b_1 \in B(\infty) \) and \( b_2 \in B(-\infty) \). Then, \( b^* \) is contained in \( B(U_q(g)a_{\lambda'}) \), where \( \lambda' := -\lambda - \text{wt}(b_1) - \text{wt}(b_2) \), and \( \Xi_{\lambda'}(b^*) = b_1^* \otimes t_{\lambda'} \otimes b_2^* \).

### 1.4 The crystal base of an extremal weight module.

Since \( B(\tilde{U}_q(g)) \) is a normal crystal, we can define an action of the Weyl group \( W \) on \( B(\tilde{U}_q(g)) \) (see [Kas2, §7.1]); for \( i \in I \), we define an action of the simple reflection \( r_i \) by

\[
r_i b := \begin{cases} f_i^n b & \text{if } n := (\text{wt}(b))(\alpha_i^\vee) \geq 0 \\ e_i^{-n} b & \text{if } n := (\text{wt}(b))(\alpha_i^\vee) \leq 0 \end{cases} \quad \text{for } b \in B(\tilde{U}_q(g)). \tag{1.4.1}
\]

An element \( b \in B(\tilde{U}_q(g)) \) is said to be extremal if the elements \( \{ wb \}_{w \in W} \subset B(\tilde{U}_q(g)) \) satisfy the following condition for all \( i \in I \):

\[
\begin{align*}
&\text{if } (\text{wt}(wb))(\alpha_i^\vee) \geq 0, \text{ then } e_i(wb) = 0, \\
&\text{and if } (\text{wt}(wb))(\alpha_i^\vee) \leq 0, \text{ then } f_i(wb) = 0. \tag{1.4.2}
\end{align*}
\]

For \( \lambda \in P \), we define a subcrystal \( B(\lambda) \) of \( B(U_q(g)a_\lambda) \) by

\[
B(\lambda) := \{ b \in B(U_q(g)a_\lambda) \mid b^* \text{ is extremal} \}. \tag{1.4.3}
\]

Remark that \( u_\lambda \in B(U_q(g)a_\lambda) \) is contained in \( B(\lambda) \). We know from [Kas2, Proposition 8.2.2] and [Kas5, §3.1] that \( B(\lambda) \) is the crystal base of the extremal weight module \( V(\lambda) \) of extremal weight \( \lambda \) over \( U_q(g) \).
2 Some Tools for Crystal Bases.

2.1 Multiple maps. We know the following theorem.

Theorem 2.1.1 ([Kas4, Theorem 3.2]). Let $m \in \mathbb{Z}_{>0}$. There exists a unique injective map $S_{m,\infty} : B(\infty) \hookrightarrow B(\infty)$ such that for each $b \in B(\infty)$ and $i \in I$, we have

\[
\mathrm{wt}(S_{m,\infty}(b)) = m \mathrm{wt}(b), \quad \epsilon_i(S_{m,\infty}(b)) = m \epsilon_i(b), \quad \varphi_i(S_{m,\infty}(b)) = m \varphi_i(b), 
\]

(2.1.1)

\[
S_{m,\infty}(u_\infty) = u_\infty, \quad S_{m,\infty}(e_i b) = e_i^m S_{m,\infty}(b), \quad S_{m,\infty}(f_i b) = f_i^m S_{m,\infty}(b). 
\]

(2.1.2)

Proposition 2.1.2. We set $S_{m,\infty}^* := \ast \circ S_{1n,\infty} \circ \ast$. Then we have $S_{m,\infty}^* = S_{m,\infty}$ on $B(\infty)$. Namely, the $\ast$-operation commutes with the map $S_{m,\infty} : B(\infty) \hookrightarrow B(\infty)$.

The proposition above can be shown in a way similar to [NS2, Theorem 2.3.1]. Before giving a proof of the proposition, we show the following lemma.

Lemma 2.1.3. The following diagram is commutative:
\[
\begin{array}{ccc}
B(\infty) & \xrightarrow{\Psi_j^{-}} & B(\infty) \otimes B_j \\
S_{m,\infty}^* & \downarrow & S_{m,\infty}^* \otimes S_{m,j} \\
B(\infty) & \xrightarrow{\Psi_j} & B(\infty) \otimes B_j.
\end{array}
\]

(2.1.3)

Here $S_{m,j} : B_j \to B_j$ is a map defined by $S_{m,j}(b_j(n)) := b_j(mn)$.

Proof. For $b \in B(\infty)$, there exists $b_0 \in B(\infty)$ such that $b = (f_j^*)^k b_0$ for some $k \in \mathbb{Z}_{\geq 0}$ and $e_j^* b_0 = 0$. Then, by Theorem 1.3.1, we have $\Psi_j^{-}(b) = b_0 \otimes b_j(-k)$, and hence

\[
(S_{\infty}^* \otimes S_{m,j})(\Psi_j^{-}(b)) = S_{\infty}^*(b_0) \otimes b_j(-mk).
\]

On the other hand, we see that $S_{m,\infty}^*(b) = (f_j^*)^k S_{m,\infty}^*(b_0)$. If $e_j^* S_{m,\infty}^*(b_0) \neq 0$, then we have $\epsilon_j(S_{m,\infty}(b_0)) \geq 1$. Since $\epsilon_j(S_{m,\infty}(b)) = m \epsilon_j(b) \in m \mathbb{Z}$ for all $b \in B(\infty)$, we deduce that $\epsilon_j(S_{m,\infty}(b_0)) \geq m$, and hence $(e_j^*)^m S_{m,\infty}^*(b_0) \neq 0$. However, since $e_j^* b_0 = 0$, we get $(e_j^*)^m S_{m,\infty}^*(b_0) = S_{m,\infty}^*(e_j^* b_0) = 0$, which is a contradiction. Therefore, we conclude that $e_j^* S_{m,\infty}^*(b_0) = 0$. It follows from Theorem 1.3.1 that

\[
\Psi_j^{-}(S_{m,\infty}^*(b)) = \Psi_j^{-}((f_j^*)^k S_{m,\infty}^*(b_0)) = S_{m,\infty}^*(b_0) \otimes b_j(-mk).
\]

Hence we have $(S_{m,\infty}^* \otimes S_{m,j})(\Psi_j^{-}(b)) = \Psi_j^{-}(S_{m,\infty}^*(b))$. This completes the proof of the lemma. \qed
Proof of Proposition 2.1.2. We will prove that $S_{n,\infty}^{*}(b) = S_{m,\infty}(b)$ for $b \in \mathcal{B}(\infty)$ by induction on the height $\text{ht}(\xi)$ of $\xi$ (note that $-\text{wt}(b) \in \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_{i}$ for all $b \in \mathcal{B}(\infty)$). If $\text{ht}(\xi) = 0$, then $b$ is the highest weight element $u_{\infty} \in \mathcal{B}(\infty)$, and hence the assertion is obvious.

Assume that $\text{ht}(\xi) \geq 1$. Then, there exists some $i \in I$ such that $b_{1} := e_{i}b \neq 0$. If $e_{j}^{*}b_{1} = 0$ for all $j \in I$, then $b_{1} = u_{\infty}$, and hence $b = f_{i}u_{\infty}$. Because $f_{i}^{k}u_{\infty}$ is a unique element of weight $-k\alpha_{i}$ for each $k \in \mathbb{Z}_{\geq 0}$, and $\text{wt}(b^{*}) = \text{wt}(b)$ for all $b \in \mathcal{B}(\infty)$, we deduce that $b^{*} = b$, and hence that

$$S_{m,\infty}^{*}(b) = (S_{m,\infty}(b^{*}))^{*} = (S_{m,\infty}(b))^{*} = f_{i}^{m}u_{\infty} = S_{m,\infty}(b).$$

So, we may assume that there exists $j \in I$ such that $e_{j}^{*}b_{1} \neq 0$. Let $b_{2} \in \mathcal{B}(\infty)$ be such that $e_{j}^{*}b_{2} = 0$ and $b_{1} = (f_{j}^{*})^{k}b_{2}$ for some $k \in \mathbb{Z}_{\geq 1}$. Namely, $b = f_{i}(f_{j}^{*})^{k}b_{2}$ for some $k \geq 1$ and $b_{2} \in \mathcal{B}(\infty)$ such that $e_{j}^{*}b_{2} = 0$.

**Case 1: $i \neq j$.** We show that $\Psi_{j}^{-}(S_{m,\infty}^{*}(b)) = \Psi_{j}^{-}(S_{m,\infty}(b))$ (recall that $\Psi_{j}^{-} : \mathcal{B}(\infty) \hookrightarrow \mathcal{B}(\infty) \otimes \mathcal{B}_{j}$ is an embedding of crystals). We have

$$\Psi_{j}^{-}(b) = \Psi_{j}^{-}(f_{i}(f_{j}^{*})^{k}b_{2}) = f_{i}\Psi_{j}^{-}((f_{j}^{*})^{k}b_{2}) = f_{i}(b_{2} \otimes b_{j}(-k))$$

$$= f_{i}b_{2} \otimes b_{j}(-k).$$

Here the last equality immediately follows from the definition of the tensor product of crystals (see, for example, [Kas3, §7.3]) and the condition that $i \neq j$. Therefore, we obtain

$$\Psi_{j}^{-}(S_{m,\infty}^{*}(b)) = (S_{m,\infty} \otimes S_{m,j})(\Psi_{j}^{-}(b)) \quad \text{by Lemma 2.1.3}$$

$$= S_{m,\infty}(f_{i}b_{2}) \otimes b_{j}(-mk)$$

$$= S_{m,\infty}(f_{i}b_{2}) \otimes b_{j}(-mk) \quad \text{by the inductive assumption}$$

$$= f_{i}^{m}S_{m,\infty}(b_{2}) \otimes b_{j}(-mk).$$

On the other hand,

$$S_{m,\infty}(b) = S_{m,\infty}(f_{i}(f_{j}^{*})^{k}b_{2}) = f_{i}^{m}S_{m,\infty}((f_{j}^{*})^{k}b_{2})$$

$$= f_{i}^{m}(f_{j}^{*})^{mk}S_{m,\infty}(b_{2}) \quad \text{by the inductive assumption}.$$ 

As in the proof of Lemma 2.1.3, we deduce that $e_{j}^{*}S_{m,\infty}(b_{2}) = 0$, and hence $e_{j}^{*}S_{m,\infty}(b_{2}) = e_{j}^{*}S_{m,\infty}(b_{2}) = 0$ by the inductive assumption. Therefore,

$$\Psi_{j}^{-}(S_{m,\infty}(b)) = \Psi_{j}^{-}(f_{i}^{m}(f_{j}^{*})^{mk}S_{m,\infty}(b_{2})) = f_{i}^{m}\Psi_{j}^{-}((f_{j}^{*})^{mk}S_{m,\infty}(b_{2}))$$

$$= f_{i}^{m}(S_{m,\infty}(b_{2}) \otimes b_{j}(-mk)) = (f_{i}^{m}S_{m,\infty}(b_{2})) \otimes b_{j}(-mk).$$
Here the last equality immediately follows again from the definition of the tensor product of crystals and the condition that \( i \neq j \). Thus, we get that \( \Psi_j^{-}(S_{m,\infty}(b)) = \Psi_j^{-}(S_{m,\infty}(b)) \), and hence \( S_{m,\infty}^{*}(b) = S_{m,\infty}(b) \).

**Case 2 :** \( i = j \). As in Case 1, we have \( \Psi_j^{-}(b) = f_i(b_2 \otimes b_i(-k)) \). We deduce from the definition of the tensor product of crystals that

\[
\Psi_i^{-}(b) = f_i(b_2 \otimes b_i(-k)) = \begin{cases} 
  f_i b_2 \otimes b_i(-k) & \text{if } \phi_i(b_2) > k, \\
  b_2 \otimes b_i(-k - 1) & \text{if } \phi_i(b_2) \leq k.
\end{cases}
\]

Hence, as in Case 1, we get

\[
\Psi_i^{-}(S_{m,\infty}^{*}(b)) = \begin{cases} 
  f_i^m S_{m,\infty}(b_2) \otimes b_i(-mk) & \text{if } \phi_i(b_2) > k, \\
  S_{m,\infty}(b_2) \otimes b_i(-mk - m) & \text{if } \phi_i(b_2) \leq k.
\end{cases}
\]

On the other hand, in exactly the same way as in Case 1, we can show that \( \Psi_i^{-}(S_{m,\infty}(b)) = f_i^m(S_{m,\infty}(b_2) \otimes b_i(-mk)) \). Because \( \phi_i(S_{m,\infty}(b_2)) = m\phi_i(b_2) \) by (2.1.1), we deduce from the definition of the tensor product of crystals that

\[
f_i^m(S_{m,\infty}(b_2) \otimes b_i(-mk)) = \begin{cases} 
  f_i^m S_{m,\infty}(b_2) \otimes b_i(-mk) & \text{if } \phi_i(b_2) > k, \\
  S_{m,\infty}(b_2) \otimes b_i(-mk - m) & \text{if } \phi_i(b_2) \leq k.
\end{cases}
\]

Therefore, we obtain that \( \Psi_i^{-}(S_{m,\infty}^{*}(b)) = \Psi_i^{-}(S_{m,\infty}(b)) \), and hence \( S_{m,\infty}^{*}(b) = S_{m,\infty}(b) \). Thus, we have proved the proposition. \( \square \)

**Remark 2.1.4.** A similar result holds for the crystal base \( B(-\infty) \). Namely, for each \( m \in \mathbb{Z}_{>0} \), there exists a unique injective map \( S_{m,-\infty} : B(-\infty) \hookrightarrow B(-\infty) \) with properties similar to \( S_{m,\infty} \) in Theorem 2.1.1, and it commutes with the *-operation on \( B(-\infty) \).

For \( m \in \mathbb{Z}_{>0} \), we define an injective map \( \tilde{S}_{m,\lambda} : B(U_q(\mathfrak{g})a_{\lambda}) \hookrightarrow B(U_q(\mathfrak{g})a_{m\lambda}) \) as in the following commutative diagram (cf. Theorem 1.3.2):

\[
\begin{array}{ccc}
B(U_q(\mathfrak{g})a_{\lambda}) & \xrightarrow{\Xi_{\lambda}} & B(\infty) \otimes T_\lambda \otimes B(-\infty) \\
\tilde{S}_{m,\lambda} & \downarrow \sim & S_{m,\infty} \otimes \tau_{m,\lambda} \otimes S_{m,-\infty} \\
B(U_q(\mathfrak{g})a_{m\lambda}) & \xleftarrow{\Xi_{m\lambda}^{-1}} & B(\infty) \otimes T_{m\lambda} \otimes B(-\infty),
\end{array}
\]

(2.1.4)

where \( \tau_{m,\lambda} : T_\lambda \to T_{m\lambda} \) is defined by \( \tau_{m,\lambda}(t_\lambda) := t_{m\lambda} \). We define \( \tilde{S}_m : \tilde{U}_q(\mathfrak{g}) \hookrightarrow \tilde{U}_q(\mathfrak{g}) \) as the direct sum of all the \( \tilde{S}_{m,\lambda} \)'s.
Proposition 2.1.5. The maps $\tilde{S}_{m,\lambda}: B(U_q(\mathfrak{g})a_\lambda) \hookrightarrow B(U_q(\mathfrak{g})a_{m\lambda})$ and $\tilde{S}_m : B(\tilde{U}_q(\mathfrak{g})) \hookrightarrow B(\tilde{U}_q(\mathfrak{g}))$ have properties similar to $S_{m,\infty}$ in Theorem 2.1.1. In addition, the map $\tilde{S}_m$ commutes with the $*$-operation on $B(\tilde{U}_q(\mathfrak{g}))$.

Proof. The first assertion immediately follows from Theorem 2.1.1, Remark 2.1.4, and the definition of the tensor product of crystals (see also [Kas5, Appendix B]). Let us prove the second assertion. We set $\tilde{S}_m^* := \ast \circ \tilde{S}_m \circ \ast$. It suffices to show the following:

Claim. Let $\lambda \in P$, and $b \in B(U_q(\mathfrak{g})a_\lambda)$. Then, we have that $\tilde{S}_m^*(b) \in B(U_q(\mathfrak{g})a_{m\lambda})$, and that $\Xi_{m\lambda}(\tilde{S}_m^*(b)) = \Xi_{m\lambda}(\tilde{S}_m(b))$.

Assume that $\Xi_{\lambda}(b) = b_1 \otimes t_\lambda \otimes b_2$ with $b_1 \in B(\infty)$ and $b_2 \in B(-\infty)$. Then we see by the definition of $\tilde{S}_m$ that

$$
\Xi_{m\lambda}(\tilde{S}_m(b)) = (S_{m,\infty} \otimes \tau_{m,\lambda} \otimes S_{m,-\infty})(\Xi_{\lambda}(b)) = S_{m,\infty}(b_1) \otimes t_{m\lambda} \otimes S_{m,-\infty}(b_2).
$$

On the other hand, we know from Theorem 1.3.3 that $b^* \in B(U_q(\mathfrak{g})a_{\lambda'})$ and $\Xi_{\lambda'}(b^*) = b_1^* \otimes t_{\lambda'} \otimes b_2^*$, where $\lambda' := -\lambda - \mathrm{wt}(b_1) - \mathrm{wt}(b_2)$. Hence we have

$$
\Xi_{m\lambda}(\tilde{S}_m^*(b^*)) = (S_{m,\infty} \otimes \tau_{m,\lambda} \otimes S_{m,-\infty})(\Xi_{\lambda'}(b^*)) = S_{m,\infty}(b_1^*) \otimes t_{m\lambda} \otimes S_{m,-\infty}(b_2^*).
$$

We deduce again from Theorem 1.3.3 that $\tilde{S}_m^*(b) = (\tilde{S}_m(b^*))^* \in B(U_q(\mathfrak{g})a_{m\lambda})$, and that

$$
\Xi_{m\lambda}(\tilde{S}_m^*(b)) = S_{m,\infty}^*(b_1) \otimes t_{m\lambda} \otimes S_{m,-\infty}^*(b_2)
$$

by Proposition 2.1.2 and Remark 2.1.4.

Thus, we obtain $\Xi_{m\lambda}(\tilde{S}_m^*(b)) = \Xi_{m\lambda}(\tilde{S}_m(b))$, as desired. \hfill \Box

Theorem 2.1.6. Let $m \in \mathbb{Z}_{>0}$. There exists an injective map $S_{m,\lambda} : B(\lambda) \hookrightarrow B(m\lambda)$ such that $S_{m,\lambda}(u_\lambda) = u_{m\lambda}$ and such that for each $b \in B(\infty)$ and $i \in I$, we have

$$
\mathrm{wt}(S_{m,\lambda}(b)) = m \mathrm{wt}(b), \quad \epsilon_i(S_{m,\lambda}(b)) = m \epsilon_i(b), \quad \varphi_i(S_{m,\lambda}(b)) = m \varphi_i(b), \quad (2.1.5)
$$

$$
S_{m,\lambda}(e_i b) = e_i^m S_{m,\lambda}(b), \quad S_{m,\lambda}(f_i b) = f_i^m S_{m,\lambda}(b). \quad (2.1.6)
$$

Proof. Set $S_{m,\lambda} := \tilde{S}_m|_{B(\lambda)}$. Then, it is obvious from Proposition 2.1.5 that $S_{m,\lambda}(B(\lambda)) \subset B(U_q(\mathfrak{g})a_{m\lambda})$. Hence we need only show that $(S_{m,\lambda}(b))^*$ is extremal for every $b \in B(\lambda)$. We can easily check that the action of the Weyl group $W$ commutes with $S_{m,\lambda}$. So, it follows from Proposition 2.1.5 that

$$
w((S_{m,\lambda}(b))^*) = w S_{m,\lambda}(b^*) = S_{m,\lambda}(wb^*) \quad \text{for all} \quad b \in B(\lambda) \text{ and } w \in W.
$$
Assume that $\text{wt}(b^*) = \mu$. Then we see that $\text{wt}((S_{m,\lambda}(b))^*) = m\mu$. Suppose that $(w(m\mu))(\alpha_i^\vee) \geq 0$ and $e_i(w((S_{m,\lambda}(b))^*)) \neq 0$. As in the proof of Lemma 2.1.3, we deduce that $e_i^m(w(S_{m,\lambda}(b))) \neq 0$. Hence we have

$$S_{m,\lambda}(e_i(wb^*)) = e_i^mS_{m,\lambda}(wb^*) = e_i^m(wS_{m,\lambda}(b^*)) = e_i^m(w((S_{m,\lambda}(b))^*)) \neq 0.$$ 

However, since $(w(\mu))(\alpha_i^\vee) \geq 0$ and $b^*$ is extremal, we have $e_i(wb^*) = 0$, and hence $S_{m,\lambda}(e_i(wb^*)) = 0$, which is a contradiction. Therefore, we obtain that $e_i(w(S_{m,\lambda}(b))) = 0$. Similarly, we can prove that if $(w(m\mu))(\alpha_i^\vee) \leq 0$, then $f_i(w((S_{m,\lambda}(b))^*)) = 0$. This completes the proof of the theorem.

\section{2.2 Embedding into tensor products.}

In this subsection, we assume that $g$ is an affine Lie algebra (for the notation, see §1.2). We know the following theorem from [B, §2], [N, §3] in the symmetric case, and from [BN, §4] in the nonsymmetric case.

\begin{theorem}
We have an embedding $G_{m,\omega_i} : B_0(m\omega_i) \hookrightarrow B(\omega_i)^{\otimes m}$ of crystals that maps $u_{m\omega_i}$ to $u_{\omega_i}^{\otimes m}$.
\end{theorem}

\begin{remark}
In [BN, they take a vertex $0 \in I$ such that $a_0 = 1$ (see [BN, §2.1]). So, in the case of $A_2^{(2)}$, the choice of the vertex 0 is different from that in [Kas5, §5.2], and hence from ours. However, this does not cause a serious problem. For details, see the comment after [BN, Theorem 2.15].
\end{remark}

Since $B(\omega_i)$ is connected (see [Kas5, Theorem 5.5]), we see that $S_{m,\omega_i}(B(\omega_i)) \subset B_0(m\omega_i)$. Hence we can define $\sigma_{m,\omega_i} : B(\omega_i) \hookrightarrow B(\omega_i)^{\otimes m}$ by $\sigma_{m,\omega_i} := G_{m,\omega_i} \circ S_{m,\omega_i}$ for each $m \in \mathbb{Z}_{>0}$. Remark that $\sigma_{m,\omega_i}$ has the following properties:

$$\text{wt}(\sigma_{m,\omega_i}(b)) = m \text{wt}(b), \quad \epsilon_j(\sigma_{m,\omega_i}(b)) = m\epsilon_j(b), \quad \varphi_j(\sigma_{m,\omega_i}(b)) = m\varphi_j(b), \quad (2.2.1)$$

$$\sigma_{m,\omega_i}(u_{\omega_i}) = u_{\omega_i}^{\otimes m}, \quad \sigma_{m,\omega_i}(e_jb) = e_j^m\sigma_{m,\omega_i}(b), \quad \sigma_{m,\omega_i}(f_jb) = f_j^m\sigma_{m,\omega_i}(b). \quad (2.2.2)$$

\begin{lemma}
Let $m, n \in \mathbb{Z}_{>0}$. Then we have $\sigma_{m,\omega_i} = \sigma_{n,\omega_i}^{\otimes m} \circ \sigma_{m,\omega_i}$.
\end{lemma}

\begin{proof}
Since $B(\omega_i)$ is connected, every $b \in B(\omega_i)$ is of the form

$$b = x_{j_1}x_{j_2}\cdots x_{j_k}u_{\omega_i},$$

for some $j_1, j_2, \ldots, j_k \in I$, where $x_j$ is either $e_j$ or $f_j$. We will show by induction on $k$ that $\sigma_{m,\omega_i}(b) = \sigma_{n,\omega_i}^{\otimes m} \circ \sigma_{m,\omega_i}(b)$ for all $b \in B(\omega_i)$. If $k = 0$, then the assertion is obvious, since $b = u_{\omega_i}$. Assume that $k \geq 1$. We set $b' := x_{j_2}\cdots x_{j_k}u_{\omega_i}$, and $\sigma_{m,\omega_i}(b') = u_1 \otimes u_2 \otimes \cdots \otimes u_m \in B(\omega_i)^{\otimes m}$. Assume that

$$\sigma_{m,\omega_i}(b) = x_{j_1}^m\sigma_{m,\omega_i}(b') = x_{j_1}^k u_1 \otimes x_{j_1}^k u_2 \otimes \cdots \otimes x_{j_1}^k u_m$$


for some \( k_1, k_2, \ldots, k_m \in \mathbb{Z}_{\geq 0} \). Then we have

\[
\sigma_{m,\varpi}^\otimes b = x_{j_1}^n \sigma_{m,\varpi}(u_1) \otimes x_{j_2}^m \sigma_{m,\varpi}(u_2) \otimes \cdots \otimes x_{j_{k+1}}^n \sigma_{m,\varpi}(u_m). 
\]

Here we remark (cf. [Kasl, Lemma 1.3.6]) that for all \( u_1 \otimes u_2 \otimes \cdots \otimes u_m \in B(\varpi_i)^{\otimes m} \),

\[
x_j(u_1 \otimes u_2 \otimes \cdots \otimes u_m) = u_1 \otimes u_2 \otimes \cdots \otimes x_j u_1 \otimes \cdots \otimes u_m
\]

if and only if

\[
x_j^n(\sigma_{m,\varpi}(u_1) \otimes \cdots \otimes \sigma_{m,\varpi}(u_m)) = \sigma_{m,\varpi}(u_1) \otimes \cdots \otimes x_j^n \sigma_{m,\varpi}(u_m).
\]

So we obtain

\[
\sigma_{m,\varpi}^\otimes \sigma_{m,\varpi}(b) = x_{j_1}^m (\sigma_{m,\varpi}(u_1) \otimes \cdots \otimes \sigma_{m,\varpi}(u_m))
\]

\[
= x_{j_1}^m (\sigma_{m,\varpi}^\otimes \sigma_{m,\varpi}(b')).
\]

We see that \( \sigma_{m,\varpi}^\otimes \sigma_{m,\varpi}(b') = \sigma_{m,\varpi}(b') \) by the inductive assumption, and that \( \sigma_{m,\varpi}(b') = x_{j_1}^m \sigma_{m,\varpi}(b') \). Therefore, we obtain \( \sigma_{m,\varpi}^\otimes \sigma_{m,\varpi}(b) = \sigma_{m,\varpi}(b) \).

For each \( w \in W \), we set \( u_{w\varpi} := u_{w1\varpi} \in B(\varpi_i) \). By [Kas5, Proposition 5.8], we see that \( u_{w1\lambda} \) is well-defined. We can easily show the following lemma.

**Lemma 2.2.4.** For each \( m \in \mathbb{Z}_{>0} \) and \( w \in W \), we have \( \sigma_{m,\varpi}(u_{w\varpi}) = (u_{w\varpi})^\otimes m \).

**Proposition 2.2.5.** Let \( b \in B(\varpi_i) \). Assume that \( b = x_{j_1} x_{j_2} \cdots x_{j_k} u_{\varpi}, \) where \( x_j \) is either \( e_j \) or \( f_j \), and set \( b_l := x_{j_1} x_{j_2} \cdots x_{j_l} u_{\varpi} \) for \( l = 1, 2, \ldots, k+1 \). Then there exists sufficiently large \( m \in \mathbb{Z} \) such that for every \( l = 1, 2, \ldots, k+1 \),

\[
\sigma_{m,\varpi}(b_l) = u_{w_{l,1}\varpi} \otimes u_{w_{l,2}\varpi} \otimes \cdots \otimes u_{w_{l,m}\varpi} 
\]

for some \( w_{l,1}, w_{l,2}, \ldots, w_{l,m} \in W \).

**Proof.** We show the assertion by induction on \( k \). If \( k = 0 \), then the assertion is obvious. Assume that \( k \geq 1 \). By the inductive assumption, there exists \( m \in \mathbb{Z}_{>0} \) such that \( \sigma_{m,\varpi}(b_l) \) is of the desired form for every \( l = 2, \ldots, k+1 \). Assume that

\[
\sigma_{m,\varpi}(b_1) = \sigma_{m,\varpi}(x_{j_1} b_2) = x_{j_1}^m \sigma_{m,\varpi}(b_2)
\]

\[
= x_{j_1}^{c_1} u_{w_{2,1}\varpi} \otimes x_{j_1}^{c_2} u_{w_{2,2}\varpi} \otimes \cdots \otimes x_{j_1}^{c_m} u_{w_{2,m}\varpi}
\]
for some $c_1, c_2, \ldots, c_m \in \mathbb{Z}_{\geq 0}$. We can easily check by Lemma 2.2.4 and [Kas1, Lemma 1.3.6] that if $n_p \in \mathbb{Z}_{>0}$ satisfies the condition that $(w_{2,p}w_i)(\alpha_{j_i}^*) | n_pc_p$, then 
$$
\sigma_{n_p,w_i}(\alpha_{j_i}^*)u_{w_{2,p}w_i} = u_{w_1w_i} \otimes u_{w_{2}w_i} \otimes \cdots \otimes u_{w_{n}w_i}
$$
for some $w_1, w_2, \ldots, w_n \in W$. Therefore, by Lemma 2.2.4, we see that there exists $N \gg 0$ (for example, put $N = \prod_{p=1}^{m} n_p$) such that

$$(\sigma_{N,w_i})^\otimes m \circ \sigma_{m,w_i}(b_1) = u_{w_{1.1}w_i} \otimes u_{w_{1.2}w_i} \otimes \cdots \otimes u_{w_{1.Nm}w_i}$$

for some $w_{1.1}, w_{1.2}, \ldots, w_{1.Nm} \in W$. Furthermore, we deduce from Lemma 2.2.4 that $(\sigma_{N,w_i})^\otimes m \circ \sigma_{m,w_i}(b_1)$ is of the desired form for every $l = 2, \ldots, k + 1$. It follows from Lemma 2.2.3 that $(\sigma_{N,w_i})^\otimes m \circ \sigma_{m,w_i} = \sigma_{N,m,w_i}$. Thus we have proved the proposition.

\section{Preliminary Results.}

\subsection{Some tools for path models.}

A path is, by definition, a piecewise linear, continuous map $\pi : [0, 1] \to \mathbb{Q} \otimes \mathbb{Z} P$ such that $\pi(0) = 0$. We regard two paths $\pi$ and $\pi'$ as equivalent if there exist piecewise linear, nondecreasing, surjective, continuous maps $\psi, \psi' : [0, 1] \to [0, 1]$ (reparametrization) such that $\pi \circ \psi = \pi' \circ \psi$. We denote by $P$ the set of paths (modulo reparametrization) such that $\pi(1) \in P$, and by $e_i$ and $f_i$ the raising and lowering root operator (see [L2, §1]). By using root operators, we can endow $P$ with a normal crystal structure (see [L2, §1 and §2]); we set $\text{wt}(\pi) := \pi(1)$, and define $\epsilon_i : P \to \mathbb{Z}$ and $\varphi_i : P \to \mathbb{Z}$ by

$$
\epsilon_i(\pi) := \max\{n \geq 0 \mid e_i^n\pi \neq 0\}, \quad \varphi_i(\pi) := \max\{n \geq 0 \mid f_i^n\pi \neq 0\}. \quad (3.1.1)
$$

Let $\lambda \in P$ be an (arbitrary) integral weight. We denote by $B(\lambda) \subset P$ the set of Lakshmibai–Seshadri paths of shape $\lambda$ (see [L2, §4]), and set $\pi_{\lambda}(t) := t\lambda \in B(\lambda)$. Denote by $B_0(\lambda)$ the connected component of $B(\lambda)$ containing $\pi_{\lambda}$. We obtain the following lemma by [L2, Lemma 2.4].

\begin{lemma}
For $\pi \in P$, we define $S_m : P \hookrightarrow P$ by $S_m(\pi) := m\pi$, where $(m\pi)(t) := m\pi(t)$ for $t \in [0, 1]$. Then we have $S_m(B_0(\lambda)) = B_0(m\lambda)$. In addition, the map $S_m$ has properties similar to $S_{m,\infty}$ in Theorem 2.1.1.
\end{lemma}

For paths $\pi_1, \pi_2 \in P$, we define a concatenation $\pi_1 \ast \pi_2 \in P$ as in [L2, §1]. Because $\pi_{\lambda} \ast \pi_{\lambda} \ast \cdots \ast \pi_{\lambda}$ ($m$-times) is just $\pi_{m\lambda}$ modulo reparametrization, we obtain the following lemma.
Lemma 3.1.2. We have a canonical embedding $G_{m,\lambda}: B_0(m\lambda) \hookrightarrow B(\lambda)^{sm}$ of crystals that maps $\pi_{m,\lambda}$ to $\pi_m^{\lambda}$, where $B(\lambda)^{sm} := \{\pi_1 \ast \pi_2 \ast \cdots \ast \pi_m | \pi_i \in B(\lambda)\}$, and $\pi_m^{\lambda} := \pi_\lambda \ast \pi_\lambda \ast \cdots \ast \pi_\lambda \in B(\lambda)^{sm}$.

By combining Lemmas 3.1.1 and 3.1.2, we get an embedding $\sigma_{m,\lambda}: B_0(\lambda) \hookrightarrow B(\lambda)^{sm}$ defined by $\sigma_{m,\lambda} := G_{m,\lambda} \circ S_m$. It can easily be seen that this map has properties similar to (2.2.1) and (2.2.2).

Since $B(\lambda)$ is a normal crystal, we can define an action of the Weyl group $W$ on $B(\lambda)$ (cf. (1.4.1); see also [L2, Theorem 8.1]). We set $\pi_{w\lambda} := w\pi_\lambda$ for $w \in W$. Note that $(w\pi_\lambda)(t) = t(w\lambda)$ for each $w \in W$. Using [L2, Lemma 2.7], we can prove the following proposition in a way similar to Proposition 2.2.5.

Proposition 3.1.3. Let $\pi \in B_0(\lambda)$. Assume that $\pi = x_{j_1}x_{j_2}\ldots x_{j_k}\pi_\lambda$, where $x_j$ is either $e_j$ or $f_j$, and set $\pi_l := x_{j_l}x_{j_l+1}\ldots x_{j_k}\pi_\lambda$ for $l = 1, 2, \ldots, k + 1$ (here $\pi_{k+1} := \pi_\lambda$). Then, there exists sufficiently large $m \in \mathbb{Z}$ such that for every $l = 1, 2, \ldots, k + 1$,

$$\sigma_{m,\lambda}(\pi_l) = \pi_{w_{l,1}\lambda} \ast \pi_{w_{l,2}\lambda} \ast \cdots \ast \pi_{w_{l,m}\lambda}$$

for some $w_{l,1}, w_{l,2}, \ldots, w_{l,m} \in W$.

3.2 Preliminary lemmas. In this subsection, $g$ is assumed to be of affine type (for the notation, see §1.2). By using [L2, Lemma 2.1 c)], we can easily show the following lemma.

Lemma 3.2.1. Let $i \in I_0$. For each $w \in W$ and $j \in I$, we have $\text{wt}(\pi_{w\varpi_0}) = \text{wt}(u_{w\varpi_0})$, $\epsilon_j(\pi_{w\varpi_0}) = \epsilon_j(u_{w\varpi_0})$, and $\varphi_j(\pi_{w\varpi_0}) = \varphi_j(u_{w\varpi_0})$.

It follows from [Kas1, Lemma 1.3.6], [L2, Lemma 2.7], and Lemma 3.2.1 that

$$x_j^k(u_{w_1\varpi_1} \otimes u_{w_2\varpi_2} \otimes \cdots \otimes u_{w_m\varpi_m}) = x_j^{k_1}u_{w_1\varpi_1} \otimes x_j^{k_2}u_{w_2\varpi_2} \otimes \cdots \otimes x_j^{k_m}u_{w_m\varpi_m}$$

for some $k_1, k_2, \ldots, k_m \in \mathbb{Z}_{\geq 0}$ if and only if

$$x_j^k(\pi_{w_1\varpi_1} \ast \pi_{w_2\varpi_2} \ast \cdots \ast \pi_{w_m\varpi_m}) = x_j^{k_1}\pi_{w_1\varpi_1} \ast x_j^{k_2}\pi_{w_2\varpi_2} \ast \cdots \ast x_j^{k_m}\pi_{w_m\varpi_m}$$

for every $k \in \mathbb{Z}_{\geq 0}$, $m \in \mathbb{Z}_{>0}$ and $w_1, w_2, \ldots, w_m \in W$. So, we obtain the following lemma.

Lemma 3.2.2. (1) Let $b = x_{j_1}x_{j_2}\ldots x_{j_k}u_{\varpi_1} \in B(\varpi_1)$. Take $m \in \mathbb{Z}_{>0}$ such that the assertion of Proposition 2.2.5 holds, and assume that $\sigma_{m,\varpi_1}(b) = u_{w_1\varpi_1} \otimes$
Then we have $\pi := x_{j_1}x_{j_2} \cdots x_{j_k}\pi_{\omega_i} \neq 0$, and $\sigma_{m,\omega_i}(\pi) = \pi_{w_1}\omega_i * \pi_{w_2}\omega_i * \cdots * \pi_{w_m}\omega_i$.

(2) The converse of (1) holds. Namely, let $\pi = x_{j_1}x_{j_2} \cdots x_{j_k}\pi_{\omega_i} \in \mathcal{B}(\omega_i)$. Take $m \in \mathbb{Z}_{>0}$ such that the assertion of Proposition 3.1.3 holds, and assume that $\sigma_{m,\omega_i}(\pi) = \pi_{w_1}\omega_i * \pi_{w_2}\omega_i * \cdots * \pi_{w_m}\omega_i$. Then we have $b := x_{j_1}x_{j_2} \cdots x_{j_k}\omega_i \neq 0$, and $\sigma_{m,\omega_i}(b) = u_{w_1}\omega_i \otimes u_{w_2}\omega_i \otimes \cdots \otimes u_{w_m}\omega_i$.

4 Main Results.

4.1 Isomorphism theorem. From now on, we assume that $g$ is an affine Lie algebra. We can carry out the proof of our isomorphism theorem, following the general line of that for [Kas5, Theorem 4.1].

Theorem 4.1.1. There exists a unique isomorphism $\Phi_{\omega_i} : \mathcal{B}(\omega_i) \to \mathcal{B}_0(\omega_i)$ of crystals such that $\Phi_{\omega_i}(u_{\omega_i}) = \pi_{\omega_i}$.

Proof. It suffices to prove that for $j_1, j_2, \ldots, j_p \in I$ and $k_1, k_2, \ldots, k_q \in I$,

(1) $x_{j_1}x_{j_2} \cdots x_{j_p}\omega_i = x_{k_1}x_{k_2} \cdots x_{k_q}\omega_i \Leftrightarrow x_{j_1}x_{j_2} \cdots x_{j_p}\pi_{\omega_i} = x_{k_1}x_{k_2} \cdots x_{k_q}\pi_{\omega_i}$,

(2) $x_{j_1}x_{j_2} \cdots x_{j_p}\omega_i = 0 \Leftrightarrow x_{j_1}x_{j_2} \cdots x_{j_p}\pi_{\omega_i} = 0$.

Part (2) has already been proved in Lemma 3.2.2. Let us show the direction $(\Rightarrow)$ of part (1). Take $m \in \mathbb{Z}_{>0}$ such that the assertion of Proposition 2.2.5 holds for both $b_1 := x_{j_1}x_{j_2} \cdots x_{j_p}\omega_i$ and $b_2 := x_{k_1}x_{k_2} \cdots x_{k_q}\omega_i$:

$\sigma_{m,\omega_i}(b_1) = u_{w_1}\omega_i \otimes u_{w_2}\omega_i \otimes \cdots \otimes u_{w_m}\omega_i$,

$\sigma_{m,\omega_i}(b_2) = u_{w'_1}\omega_i \otimes u_{w'_2}\omega_i \otimes \cdots \otimes u_{w'_m}\omega_i$.

Since $b_1 = b_2$, we get $u_{w_1}\omega_i = u_{w'_1}\omega_i$, and hence $w_l\omega_i = w'_l\omega_i$ for all $l = 1, 2, \ldots, m$.

By Lemma 3.2.2 (1), we see that

$\sigma_{m,\omega_i}(\pi_1) = \pi_{w_1}\omega_i * \pi_{w_2}\omega_i * \cdots * \pi_{w_m}\omega_i$,

$\sigma_{m,\omega_i}(\pi_2) = \pi_{w'_1}\omega_i * \pi_{w'_2}\omega_i * \cdots * \pi_{w'_m}\omega_i$,

where $\pi_1 := x_{j_1}x_{j_2} \cdots x_{j_p}\pi_{\omega_i}$ and $\pi_2 := x_{k_1}x_{k_2} \cdots x_{k_q}\pi_{\omega_i}$. Since $w_l\omega_i = w'_l\omega_i$ and $\pi_{w_l\omega_i}(t) = t(w\omega_i)$ for all $w \in W$, we get $\sigma_{m,\omega_i}(\pi_1) = \sigma_{m,\omega_i}(\pi_2)$. Since $\sigma_{m,\omega_i}$ is injective, we conclude that $\pi_1 = \pi_2$.

We show the reverse direction $(\Leftarrow)$ of part (1). Take $m \in \mathbb{Z}_{>0}$ such that the assertion of Proposition 3.1.3 holds for both $\pi_1 := x_{j_1}x_{j_2} \cdots x_{j_p}\pi_{\omega_i}$ and $\pi_2 :=
\[ x_{k_{1}}x_{k_{2}} \cdots x_{k_{q}} \pi_{\omega_{i}}. \]

\[ \sigma_{m,\omega_{i}}(\pi_{1}) = \pi_{w_{1}\omega_{i}} * \pi_{w_{2}\omega_{i}} * \cdots * \pi_{w_{m}\omega_{i}}, \]
\[ \sigma_{m,\omega_{i}}(\pi_{2}) = \pi_{w'_{1}\omega_{i}} * \pi_{w'_{2}\omega_{i}} * \cdots * \pi_{w'_{m}\omega_{i}}. \]

Since \( \pi_{1} = \pi_{2} \), and hence \( \sigma_{m,\omega_{i}}(\pi_{1}) = \sigma_{m,\omega_{i}}(\pi_{2}) \) in \( \mathbb{P} \), the two paths \( \pi_{w_{1}\omega_{i}} * \pi_{w_{2}\omega_{i}} * \cdots * \pi_{w_{m}\omega_{i}} \) and \( \pi_{w'_{1}\omega_{i}} * \pi_{w'_{2}\omega_{i}} * \cdots * \pi_{w'_{m}\omega_{i}} \) are identical modulo reparametrization. 

Hence we can deduce that \( w_{l}\omega_{i} = w'_{l}\omega_{i} \) for all \( l = 1, 2, \ldots, m \) from the fact that if \( a\omega_{j} \in W\omega_{i} \) for some \( a \in \mathbb{Q}_{\geq 0} \) and \( i, j \in I_{0} \), then \( i = j \) and \( a = 1 \). By Lemma 3.2.2 (2), we have
\[
\sigma_{m,\omega_{i}}(b_{1}) = u_{w_{1}\omega_{i}} \otimes u_{w_{2}\omega_{i}} \otimes \cdots \otimes u_{w_{m}\omega_{i}},
\]
\[
\sigma_{m,\omega_{i}}(b_{2}) = u_{w'_{1}\omega_{i}} \otimes u_{w'_{2}\omega_{i}} \otimes \cdots \otimes u_{w'_{m}\omega_{i}}. \]

Since \( w_{l}\omega_{i} = w'_{l}\omega_{i} \) for all \( l = 1, 2, \ldots, m \), it follows from [Kas5, Proposition 5.8 (i)] that \( u_{w_{l}\omega_{i}} = u_{w'_{l}\omega_{i}} \) for all \( l = 1, 2, \ldots, m \). Therefore we have \( \sigma_{m,\omega_{i}}(b_{1}) = \sigma_{m,\omega_{i}}(b_{2}) \). Since \( \sigma_{m,\omega_{i}} \) is injective, we conclude that \( b_{1} = b_{2} \). \( \square \)

**Remark 4.1.2.** In general, an isomorphism of crystals between \( B(\lambda) \) and \( \mathbb{B}_{0}(\lambda) \) does not exist, even if \( B(\lambda) \) is connected. For example, let \( \mathfrak{g} \) be of type \( A_{2}^{(1)} \), and \( \lambda = \varpi_{1} + \varpi_{2} \) (we know from [Kas5, Proposition 5.4] that \( B(\lambda) \) is connected). If \( B(\lambda) \cong \mathbb{B}_{0}(\lambda) \) as crystals, then we would have \( w\lambda = w'\lambda \) in \( B(\lambda) \) for every \( w, w' \in W \) with \( w\lambda = w'\lambda \), but we have an example of \( w, w' \in W \) such that \( w\lambda \neq w'\lambda \) in \( B(\lambda) \) and \( w\lambda = w'\lambda \) (see [Kas5, Remark 5.10]).

**Remark 4.1.3.** In [G], Greenstein proved that if \( \mathfrak{g} \) is of type \( A_{2}^{(1)} \), then the connected component \( \mathbb{B}_{0}(m\varpi_{i} + n\delta) \) is a path model for a certain bounded module \( L(\ell, m, n) \). He also showed a decomposition rule for tensor products, which seems to be closely related to Theorem 4.3.3 below.

### 4.2 Branching rule for \( V(\varpi_{i}) \)

**Lemma 4.2.1.** For every \( \pi \in B(\varpi_{i}) \), we have \( (\pi(1), \pi(1)) \leq (\varpi_{i}, \varpi_{i}) \).

**Proof.** Let \( \pi = (\nu_{1}, \nu_{2}, \ldots, \nu_{s} ; a_{0}, a_{1}, \ldots, a_{s}) \) with \( \nu_{j} \in W\varpi_{i} \) and \( a_{j} \in [0, 1] \) be a Lakshmibai-Seshadri path of shape \( \varpi_{i} \) (cf. [L2, §4]). By the definition of a Lakshmibai-Seshadri path, we see that \( \pi(1) = \sum_{j=1}^{s}(a_{j} - a_{j-1})\nu_{j} \). Hence we have
\[
(\pi(1), \pi(1)) = \sum_{j=1}^{s}(a_{j} - a_{j-1})^{2}(\nu_{j}, \nu_{j}) + 2 \sum_{1 \leq k < l \leq s}(a_{k} - a_{k-1})(a_{l} - a_{l-1})(\nu_{k}, \nu_{l})
\]
\[
= \sum_{j=1}^{s}(a_{j} - a_{j-1})^{2}(\varpi_{i}, \varpi_{i}) + 2 \sum_{1 \leq k < l \leq s}(a_{k} - a_{k-1})(a_{l} - a_{l-1})(\varpi_{i}, w_{kl}\varpi_{i})
\]
for some $w_{kl} \in W$. By [Kac, Proposition 6.3], we deduce that $w_{kl} \varpi_{i} = \varpi_{i} - \beta_{kl} + n_{kl} \delta$ for some $\beta_{kl} \in \sum_{i \in I_{0}} \mathbb{Z}_{\geq 0} \alpha_{i}$ and $n_{kl} \in \mathbb{Z}$. Therefore, we have (note that $\varpi_{i}$ is of level 0)

$$(\pi(1), \pi(1)) = \sum_{j=1}^{s} (a_{j} - a_{j-1})^{2} (\varpi_{i}, \varpi_{i}) + 2 \sum_{1 \leq k < l \leq s} (a_{k} - a_{k-1})(a_{l} - a_{l-1})(\varpi_{i}, \beta_{kl})$$

$= \sum_{j=1}^{s} (a_{j} - a_{j-1})^{2} (\varpi_{i}, \varpi_{i}) - 2 \sum_{1 \leq k < l \leq s} (a_{k} - a_{k-1})(a_{l} - a_{l-1})(\varpi_{i}, \beta_{kl})$

Since $(\varpi_{i}, \beta_{kl}) \geq 0$ for all $1 \leq k < l \leq s$, we deduce that $(\pi(1), \pi(1)) \leq (\varpi_{i}, \varpi_{i})$, as desired. \hfill \Box

Let $S$ be a proper subset of $I$, i.e., $S \subset I$. Let $\mathfrak{g}_{S}$ be the Levi subalgebra of $\mathfrak{g}$ corresponding to $S$, and $U_{q}(\mathfrak{g}_{S}) \subset U_{q}(\mathfrak{g})$ the quantized universal enveloping algebra of $\mathfrak{g}_{S}$. Note that a crystal for $U_{q}(\mathfrak{g})$ can be regarded as a crystal for $U_{q}(\mathfrak{g}_{S})$ by restriction.

**Theorem 4.2.2.** As crystals for $\mathfrak{g}_{S}$, $\mathcal{B}(\varpi_{i})$ and $\mathcal{B}_{0}(\varpi_{i})$ decompose as follows:

$$\mathcal{B}(\varpi_{i}) \cong \bigsqcup_{\pi \in \mathcal{B}(\varpi_{i}), \pi: \mathfrak{g}_{S}-\text{dominant}} \mathcal{B}_{S}(\pi(1)),$$

$$\mathcal{B}_{0}(\varpi_{i}) \cong \bigsqcup_{\pi \in \mathcal{B}_{0}(\varpi_{i}), \pi: \mathfrak{g}_{S}-\text{dominant}} \mathcal{B}_{S}(\pi(1)), \quad (4.2.1)$$

where $\mathcal{B}_{S}(\lambda)$ is the set of Lakshmibai-Seshadri paths of shape $\lambda$ for $U_{q}(\mathfrak{g}_{S})$, and a path $\pi$ is said to be $\mathfrak{g}_{S}$-dominant if $(\pi(t))(\alpha_{i}^{\vee}) \geq 0$ for all $t \in [0, 1]$ and $i \in S$.

**Proof.** We will show only the first equality in (4.2.1), since the second one can be shown in the same way. As in [Kas1, §9.3], we deduce, using Lemma 4.2.1, that each connected component of $\mathcal{B}(\varpi_{i})$ (as a crystal for $U_{q}(\mathfrak{g}_{S})$) contains an extremal weight element $\pi'$ with respect to $W_{S} := \langle r_{j} \mid j \in S \rangle$. Because $\mathfrak{g}_{S}$ is a finite-dimensional reductive Lie algebra, there exists $w \in W_{S}$ such that $((w\pi')(1))(\alpha_{i}^{\vee}) \geq 0$ for all $j \in S$. Put $\pi := w\pi'$ for this $w \in W_{S}$. Since $\pi$ is also extremal, we have that $e_{j}\pi = 0$ for all $j \in S$. Because $\pi$ is a Lakshmibai-Seshadri path of shape
\( \varpi_i \), we deduce from [L2, Lemmas 2.2 b) and 4.5 d)] that \((\pi(t))(\alpha_j^\vee) \geq 0 \) for all \( t \in [0, 1] \) and \( j \in S \), i.e., \( \pi \) is \( g_S \)-dominant. We see from [L2, Theorem 7.1] that the connected component containing \( \pi \) as a crystal for \( U_q(g_S) \) is isomorphic to \( B_S(\pi(1)) \), thereby completing the proof of the theorem.

\[ \square \]

**Theorem 4.2.3.** (1) The extremal weight module \( V(\varpi_i) \) of extremal weight \( \varpi_i \) is completely reducible as a \( U_q(g_S) \)-module.

(2) The decomposition of \( V(\varpi_i) \) as a \( U_q(g_S) \)-module is given by:

\[
V(\varpi_i) \cong \bigoplus_{\pi \in B_0(\varpi_i)} V_S(\pi(1)), \tag{4.2.2}
\]

where \( V_S(\lambda) \) is the integrable highest weight \( U_q(g_S) \)-module of highest weight \( \lambda \).

**Proof.** (1) First we prove that \( U := U_q(g_S)u \) is finite-dimensional for each weight vector \( u \in V(\varpi_i) \). To prove this, it suffices to show that the weight system \( \text{Wt}(U) \) of \( U \) is a finite set, since each weight space of \( V(\varpi_i) \) is finite-dimensional (see [Kas5, Proposition 5.16 (iii)]). Remark that if \( \mu, \nu \in P \) are weights of \( U \), then \( \mu, \nu \in h_0^* \), and \( \mu - \nu \in Q_S := \sum_{i \in S} \mathbb{Z} \alpha_i \). Hence the canonical map \( \text{cl} : h_0^* \to h_0^*/Q\delta \) is injective on \( \text{Wt}(U) \), since \( k\delta \notin Q_S \) for any \( k \in \mathbb{Z} \setminus \{0\} \). Since \( \text{Wt}(U) \) is contained in the weight system \( \text{Wt}(V(\varpi_i)) \) of \( V(\varpi_i) \), it follows from Theorem 4.1.1 and Lemma 4.2.1 that

\[
\text{cl}(\text{Wt}(U)) \subset \text{cl}(\text{Wt}(V(\varpi_i))) = \text{cl}(\{\pi(1) | \pi \in B_0(\varpi_i)\}) \quad \text{by Theorem 4.1.1}
\]

\[
\subset \{\mu' \in h_0^*/Q\delta | (\mu', \mu') \leq (\text{cl}(\varpi_i), \text{cl}(\varpi_i)) \} \quad \text{by Lemma 4.2.1}.
\]

Because the bilinear form \((\cdot, \cdot)\) on \( h_0^*/Q\delta \) is positive-definite, the set \( \text{cl}(\text{Wt}(U)) \) is discrete and contained in a compact set with respect to the usual metric topology on \( \mathbb{R} \otimes_{\mathbb{Q}} (h_0^*/Q\delta) \) defined by \((\cdot, \cdot)\). Therefore, we see that \( \text{cl}(\text{Wt}(U)) \) is a finite set, and hence so is \( \text{Wt}(U) \). Thus, we conclude that \( U = U_q(g_S)u \) is finite-dimensional.

Since \( q \) is assumed to generic, the finite-dimensional \( U_q(g_S) \)-module \( U_q(g_S)u \) is completely reducible for each weight vector \( u \in V(\varpi_i) \). Because \( V(\varpi_i) \) is a sum of all such modules \( U_q(g_S)u \), we deduce that \( V(\varpi_i) \) is also completely reducible.

(2) Because each weight space of \( V(\varpi_i) \) is finite-dimensional, we can define the formal character \( \text{ch} V(\varpi_i) \) of \( V(\varpi_i) \). By Theorem 4.2.2, we have

\[
\text{ch} V(\varpi_i) = \sum_{\pi \in B_0(\varpi_i)} \text{ch} V_S(\pi(1)).
\]
Therefore, in order to prove part (2), we need only show that this is the unique way of writing $\text{ch} V(\varpi_i)$ as a sum of the characters of integrable highest weight $U_q(\mathfrak{g}_S)$-modules. Assume that

$$\text{ch} V(\varpi_i) = \sum_{\lambda \in P} c_{\lambda} \text{ch} V_S(\lambda) \quad \text{and} \quad \text{ch} V(\varpi_i) = \sum_{\lambda \in P} c'_{\lambda} \text{ch} V_S(\lambda)$$

with $c_{\lambda}, c'_{\lambda} \in \mathbb{Z}$ for $\lambda \in P$. Then we have $\sum_{\lambda \in P} (c_{\lambda} - c'_{\lambda}) \text{ch} V_S(\lambda) = 0$. Suppose that there exists $\lambda \in P$ such that $c_{\lambda} - c'_{\lambda} \neq 0$, and set $X := \{ \lambda \in P \mid c_{\lambda} - c'_{\lambda} \neq 0 \}(\neq \emptyset)$. Note that $X$ is contained in the weight system $\text{Wt}(V(\varpi_i))$ of $V(\varpi_i)$. As in the proof of part (1), we deduce that

$$\text{cl}(\text{Wt}(V(\varpi_i))) \subset \{ \mu' \in \mathfrak{h}_0^* / \mathbb{Q} \delta \mid (\mu', \mu') \leq (\text{cl}(\varpi_i), \text{cl}(\varpi_i)) \},$$

and hence $\text{Wt}(V(\varpi_i))$ modulo $\mathbb{Z} \delta$ is a finite set.

Now, we define a partial order $\geq_S$ on $P$ as follows:

$$\mu \geq_S \nu \quad \text{for} \quad \mu, \nu \in P \quad \iff \quad \mu - \nu \in (Q_S)_+ := \sum_{i \in S} \mathbb{Z}_{\geq 0} \alpha_i.$$

Let us show that the set $X$ has a maximal element with respect to this order $\geq_S$. Let $\mu \in X$. Then $\text{Wt}(V(\varpi_i)) \cap (\mu + Q_S)$ is a finite set. Indeed, if this is not a finite set, then there exist elements $\nu, \nu'$ of it such that $\nu - \nu' = k \delta$ with $k \in \mathbb{Z} \setminus \{0\}$, since $\text{Wt}(V(\varpi_i))$ modulo $\mathbb{Z} \delta$ is a finite set. However, since $\nu - \nu' \in Q_S$ and $k \delta \not\in Q_S$ for any $k \in \mathbb{Z} \setminus \{0\}$, this is a contradiction. Therefore, we see that $X \cap (\mu + (Q_S)_+)$ is also a finite set, and hence that $X$ has a maximal element of the form $\mu + \beta$ for some $\beta \in (Q_S)_+$.

Let $\nu \in X$ be a maximal element with respect to this order $\geq_S$. We can easily see that the coefficient of $e(\nu)$ in $\sum_{\lambda \in P} (c_{\lambda} - c'_{\lambda}) \text{ch} V_S(\lambda)$ is equal to $c_{\nu} - c'_{\nu}$. Since $\nu \in X$, we have $c_{\nu} - c'_{\nu} \neq 0$, which contradicts $\sum_{\lambda} (c_{\lambda} - c'_{\lambda}) \text{ch} V_S(\lambda) = 0$. This completes the proof of the theorem.

\[ \square \]

### 4.3 Decomposition rule for tensor products

In this subsection, we assume that $\varpi_i$ is minuscule, i.e., $\varpi_i(\alpha^v) \in \{ \pm 1, 0 \}$ for every dual real root $\alpha^v$ of $\mathfrak{g}$.

**Remark 4.3.1.** The following is the list of minuscule weights (cf. [H, p. 174]). We use the numbering of vertices of the Dynkin diagrams in [Kac, Ch. 4]:
\begin{array}{c|c}
A^{(1)}_{\ell} (\ell \geq 1) & A^{(2)}_{2\ell-1} (\ell \geq 3) \\
B^{(1)}_{\ell} (\ell \geq 3) & A^{(2)}_{2\ell} (\ell \geq 2) \\
C^{(1)}_{\ell} (\ell \geq 2) & D^{(2)}_{\ell+1} (\ell \geq 2) \\
D^{(1)}_{\ell} (\ell \geq 4) & \end{array}

Remark 4.3.2. If \( \varpi_i \) is minuscule, then, for any \( \mu, \nu \in W \varpi_i \) and rational number \( 0 < a < 1 \), there does not exist an \( a \)-chain for \( (\mu, \nu) \). Hence it follows from the definition of Lakshmibai–Seshadri paths that \( \mathcal{B}(\varpi_i) = \{ \pi_{w\varpi_i} \mid w \in W \} \). Since \( w\varpi_i = \pi_{w\varpi_i} \), we see that \( \mathcal{B}(\varpi_i) \) is connected, and hence \( \mathcal{B}(\varpi_i) = \mathcal{B}_0(\varpi_i) \).

Theorem 4.3.3. Let \( \lambda \) be a dominant integral weight which is not a multiple of the null root \( \delta \) of \( \mathfrak{g} \). Then, the concatenation \( \mathcal{B}(\lambda) \ast \mathcal{B}(\varpi_i) \) decomposes as follows:

\[
\mathcal{B}(\lambda) \ast \mathcal{B}(\varpi_i) \cong \bigcap_{\pi \in \mathcal{B}(\varpi_i)} \mathcal{B}(\lambda + \pi(1)),
\]

where \( \pi \in \mathcal{B}(\varpi_i) \) is said to be \( \lambda \)-dominant if \( (\lambda + \pi(t))(\alpha_i^\vee) \geq 0 \) for all \( t \in [0,1] \) and \( i \in I \).

Proof. We will prove that each connected component contains a (unique) path of the form \( \pi_{\lambda} \ast \pi \) for a \( \lambda \)-dominant path \( \pi \in \mathcal{B}(\varpi_i) \). Then the assertion of the theorem follows from [L2, Theorem 7.1].

Let \( \pi_1 \ast \pi_2 \in \mathcal{B}(\lambda) \ast \mathcal{B}(\varpi_i) \). It can easily be seen that \( e_{i_1}e_{i_2} \cdots e_{i_k}(\pi_1 \ast \pi_2) = \pi_\lambda \ast \pi_2' \) for some \( i_1, i_2, \ldots, i_k \in I \), where \( \pi_2' \in \mathcal{B}(\varpi_i) \) (cf. [G, §5.6]). Set \( S := \{ i \in I \mid \lambda(\alpha_i^\vee) = 0 \} \) (note that \( S \subsetneq I \), since \( \lambda \) is not a multiple of \( \delta \)), and let \( \mathcal{B} \) be the set of paths of the form \( e_{j_1}e_{j_2} \cdots e_{j_l}(\pi_\lambda \ast \pi_2') \) for \( j_1, j_2, \ldots, j_l \in S \). Remark that if \( e_{j_1}e_{j_2} \cdots e_{j_l}(\pi_\lambda \ast \pi_2') \neq 0 \), then \( e_{j_1}e_{j_2} \cdots e_{j_l}(\pi_\lambda \ast \pi_2') = \pi_\lambda \ast (e_{j_1}e_{j_2} \cdots e_{j_l}\pi_2') \). As in the proof of part (2) of Theorem 4.2.3, we deduce that

\[
\{ \pi(1) \mid \pi \in \mathcal{B}(\varpi_i) \} \cap (\pi_2'(1) + (Q_S)_+) = \text{Wt}(V(\varpi_i)) \cap (\pi_2'(1) + (Q_S)_+)
\]

is a finite set. Hence we have \( \pi_\lambda \ast \pi_2'' \in \mathcal{B} \) for some \( \pi_2'' \in \mathcal{B}(\varpi_i) \) such that \( e_j(\pi_\lambda \ast \pi_2'') = 0 \) for all \( j \in S \). Because \( \varpi_i \) is minuscule and \( \pi_2'' = \pi_{w\varpi_i} \) for some \( w \in W \) (cf. Remark 4.3.2), we see that \( e_j(\pi_\lambda \ast \pi_2'') = 0 \) for all \( j \in I \setminus S \). Therefore, we conclude that \( \pi_2'' \in \mathcal{B}(\varpi_i) \) is \( \lambda \)-dominant. Thus, we have completed the proof of the theorem. \( \square \)
Remark 4.3.4. Unlike Theorems 4.2.2 and 4.2.3, this theorem does not necessarily imply the decomposition rule for tensor products of corresponding $U_q(\mathfrak{g})$-modules.

References


[G] J. Greenstein, Littelmann’s path crystal and combinatorics of certain integrable $\overline{\mathfrak{g}}_{\ell+1}$ modules of level zero, math.QA/0206263.


