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On Centralizers of Parabolic Subgroups in Coxeter Groups

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Abstract

We describe the structure of the centralizer of an arbitrary parabolic subgroup in any finitely generated Coxeter group.

1 Introduction

Let $(W, S)$ be a Coxeter system such that $S$ is a finite set. A parabolic subgroup $W_I$ of $W$ is the subgroup generated by a subset $I$ of $S$. In this paper, we determine the structure of the centralizer $C_W(W_I)$ of $W_I$ in $W$ for arbitrary finite $S$ and $I$. In particular, we do not assume that $W$ is finite.

The structure of $C_W(W_I)$ has been known in certain cases, such as $I = S$ (in this case, $C_W(W_I)$ is the center of $W$) and $|I| = 1$ (this case is examined by Brink [1]). However, no corresponding general result for an arbitrary $I$ has been known. Our result generalizes these results to the general case.

$W$ has a well known faithful reflection representation in a real vector space with a symmetric bilinear form (which may not be positive definite in general), and the notion of root system in this vector space (see Section 2.1). Using this terminology, we decompose $C_W(W_I)$ (in Section 3) as $W_{iso} \times (W_I^+ \rtimes G_I)$, where $W_{iso}$ is the parabolic subgroup generated by the elements of $I$ which are isolated in the Coxeter graph of $I$, $W_I^+$ is the subgroup of $W$ generated by reflections fixing all simple roots in $W_I$, and $G_I$ is a certain subgroup defined in that section. We have $W_{iso} \simeq (\mathbb{Z}/2\mathbb{Z})^{|I_{iso}|}$, and by a result by Deodhar [4] or by Dyer [5], $W_I^+$ is a Coxeter group, whose Coxeter generators and Coxeter relations can be determined if the root system of $W$ is well understood. Our main objective in this paper is to determine the structure of $G_I$.

To describe $G_I$, we define a groupoid (see Section 2.2 for the notion of groupoids) $H$ on the set $S^{(N)}$, $N = |I|$, where $S^{(N)} = \{x = (x_1, x_2, \ldots, x_N) \in S^N | x_i \neq x_j$ for all $i \neq j\}$,
such that its vertex group $H_{x,x}$ is a normal subgroup of $G_I$ whenever $I = \{x_1, x_2, \ldots, x_N\}$, and introduce a graph $\mathcal{G}$, which we call the transition diagram, based on information on the subsets of $S$ of finite type. Note that $W_{I^{\text{iso}}}, W_{I}^\perp, G_I$ are denoted by $W_{[a]^{\text{iso}}}, W_{x}^\perp, G_{x,x}$ respectively in the text, by taking such $x \in S^{(N)}$. Then we present $H$ as a quotient groupoid of the fundamental groupoid of $\mathcal{G}$ (which is a free groupoid), and give a method to specify the generators of the kernel of the quotient map in terms of the directed paths of $\mathcal{G}$. As a consequence, a presentation of $H_{x,x}$ is also obtained. (These are done in Section 4.) Moreover, in Section 5, coset representatives of $G_{x,x}/H_{x,x}$ and their products in $G_{x,x}$ are described using the graph $\mathcal{G}$.

Section 6 deals with certain examples; we compute $C_W(W_I)$ for an affine Coxeter group according to the result of previous sections. Further, in that section, we also consider the case of maximal parabolic subgroups; that is, $I$ is a maximal proper subset of $S$.

Finally, note that all the proofs of the results are omitted in this paper, by reason of space. The detailed proofs will be given in [8].

2 Background material

2.1 Coxeter groups

In this paper, the basic notations and well-known facts about Coxeter groups are based on Humphreys [7]. Here we do not yet assume that $S$ is a finite set.

A pair $(W, S)$ is called a Coxeter system (or simply, $W$ is called a Coxeter group) if $W$ is a group presented as

$$W = \langle S \mid s^2 = 1 \text{ for all } s \in S, (ss')^{m_{s,s'}} = 1 \text{ for some } s \neq s' \rangle,$$

where $m_{s,s'}$ denotes the (possibly infinite) order of $ss'$ in $W$. (In this paper $(W, S)$ always denotes a Coxeter system.) Then the structure of $W$ is described also by its Coxeter graph $\Gamma$; this is a simple, undirected graph on $S$ which has an edge between $s$ and $s'$ labeled $m_{s,s'}$ if and only if $m_{s,s'} \geq 3$. Note that labels $m_{s,s'}$ are usually omitted if $m_{s,s'} = 3$.

$(W, S)$ has a well-known geometric representation (over $\mathbb{R}$) as follows. Let $V$ be a real vector space with basis $\Pi = \{\alpha_s\}_{s \in S}$ having a symmetric bilinear form $\langle \, , \rangle$ determined by $\langle \alpha_s, \alpha_s' \rangle = -\cos(\pi/m_{s,s'})$, where $\pi/\infty$ is interpreted as 0. Then $W$ acts on $V$ by $s \cdot v = v - 2 \langle v, \alpha_s \rangle \alpha_s$ for all $s \in S$, and this action is faithful and preserves the bilinear form. The orbit $\Phi = W \cdot \Pi$ is called the root system of $(W, S)$, and its elements are called roots of $(W, S)$. Obviously, each element of $\Pi$ is a root; this is called a simple root. Further, a root $\gamma$ is called positive, negative, denoted by $\gamma > 0, \gamma < 0$, if $\gamma$ is a linear
combination of simple roots with all coefficients nonnegative, nonpositive respectively. For \( \Psi \subset \Phi \), let \( \Psi^+, \Psi^- \) denote the set of all positive, negative roots of \( \Psi \) respectively. Then it is well known (see [7]) that \( \Phi^- = -\Phi^+ \cup \Phi^- \) (disjoint union).

Let \( \gamma = w \cdot \alpha_s \) be a root with \( s \in S \), \( w \in W \). Then the reflection \( s_\gamma = ws w^{-1} \) about \( \gamma \) is determined independently on the choice of \( w, s \), and acts on \( V \) by \( s_\gamma \cdot v = v - 2(v, \gamma) \gamma \).

For \( w \in W \), the length \( \ell(w) \) of \( w \) is the minimal number \( k \) such that \( w = s_1 s_2 \cdots s_k \) for some \( s_i \in S \). Then it is also well known that \( \ell(w) = \#\Phi_w^+ \), where \( \Phi_w^+ \) denotes the set of all positive roots \( \gamma \in \Phi \) such that \( w \cdot \gamma < 0 \).

For \( I \subset S \), let \( W_I = \langle I \rangle \) denote the parabolic subgroup of \( W \) generated by \( I \) and let \( \Pi_I = \{ \alpha_s \}_{s \in I}, V_I = \text{span}_\mathbb{R} \Pi_I \) and \( \Phi_I = W_I \cdot \Pi_I \). Then \( (W_I, I) \) is also a Coxeter system with geometric representation \( V_I \), root system \( \Phi_I \) and simple roots \( \Pi_I \). Let \( \Gamma_I \) denote the Coxeter graph of \( (W_I, I) \). We often say that \( I \) is connected instead of that \( \Gamma_I \) is connected. For example, \( (W, S, D) \) is called irreducible if \( S \) is connected.

For \( I \subset S \), we say that \( I \) is of finite type if \( W_I \) is a finite group. Then \( W_I \) has the (unique) element \( w_0(I) \) with maximal length, called the longest element of \( W_I \), if and only if \( I \) is of finite type. Further, the following theorem holds:

**Theorem 2.1** (see [9]). If \( I \subset S \) is of finite type, then \( w_0(I) \cdot \Pi_I = -\Pi_I \). \( \square \)

Note that \( w_0(I) \) is involutive. Owing to Theorem 2.1, we define a permutation \( \sigma_I \) on \( I \) by \( w_0(I) \cdot \alpha_s = -\alpha_{\sigma_I(s)} \) for each \( s \in I \). Then \( \sigma_I(s) = w_0(I)sw_0(I) \) and so \( \sigma_I \) is an involutive automorphism of \( \Gamma_I \). Figure 1 shows the table of all finite irreducible Coxeter systems (see, for example, [7] for the classification of finite irreducible Coxeter systems) and the action of \( \sigma \) for each Coxeter system.

### 2.2 Groupoids

In this paper, the basic notations and well-known facts about groupoids are based on Higgins [6] or Brown [2].

A family \( X = \{ X_{i,j} \}_{i,j \in V(X)} \) of sets \( X_{i,j} \) is called a graph on vertex set \( V(X) \). We write \( x \in X \) when \( x \in X_{i,j} \) for some \( i, j \). Now a groupoid, certain generalization of a group, \( G \) is a graph (in the above sense) satisfying the following axioms:

\((\mathbf{G}1)\) For \( x \in G_{i,j} \) and \( y \in G_{j,k} \), the composition (or multiplication) \( xy \in G_{i,k} \) is defined.  
\((\mathbf{G}2)\) \( (xy)z = x(yz) \) holds for all \( x \in G_{i,j}, y \in G_{j,k}, z \in G_{k,l} \).  
\((\mathbf{G}3)\) For \( i \in V(G) \), there exists a unit \( 1_i \in G_{i,i} \) such that \( 1_i x = x \) for all \( x \in G_{i,j}, j \in V(G) \) and \( y 1_i = y \) for all \( y \in G_{k,i}, k \in V(G) \).  
\((\mathbf{G}4)\) For \( x \in G_{i,j} \), there exists an inverse \( x^{-1} \in G_{j,i} \) of \( x \) such that \( xx^{-1} = 1_i \) and
Figure 1: Finite irreducible Coxeter systems

Each $s_i$ is the numbering on $S$, and $\overline{s}_i = \sigma_S(s_i)$.

$A_n (n \geq 1)$

$B_n (n \geq 2)$

$D_n (n \geq 4, n \text{ is even})$

$D_n (n \geq 4, n \text{ is odd})$

$E_6$

$E_7$

$E_8$

$F_4$

$H_3$

$H_4$

$I_2(m)$ (m $\geq 5$, $m$ odd)
The unit is unique for each $i \in V(G)$ and $x^{-1}$ is unique for each $x$, similarly to the case of groups. Note that each $G_{i,j}$, $i \in V(G)$ is a group, called a vertex group of $G$. On the other hand, for a generalization of semigroups, any graph satisfying all axioms above except (G4) is called a category. (In the context of the usual category theory, a ‘category’ defined above is indeed a small category, with objects $V(G)$ and morphisms $G_{i,j}$. Further, a groupoid is now a small category such that all morphisms are invertible.)

**Example 2.2.** Now we define the fundamental groupoid $\overline{P} = \overline{P}(G)$ of any undirected graph $G$, which is one of the important examples of groupoids.

Let $P_{i,j} = P_{i,j}(G)$ be the set of all directed paths of $G$ from a vertex $i$ to $j$. Then the family $\{P_{i,j}\}_{i,j \in V(G)}$, denoted by $P = P(G)$, forms a category with concatenation as composition, where the units are trivial paths (paths of length 0) at each vertex.

For each $e \in E(G)$ with certain direction, let $e^{-1}$ denote the same edge but has the opposite direction. Further, for any path $p = e_{1}e_{2}\cdots e_{n} \in P_{i,j}$, define $p^{-1} = e_{n}^{-1}\cdots e_{2}^{-1}e_{1}^{-1} \in P_{j,i}$. Now let $\sim$ denote the equivalence relation on $P_{i,j}$ generated by the relation

$$e_{1}\cdots e_{k-1}e_{k}^{-1}e_{k+1}\cdots e_{n} \sim e_{1}\cdots e_{k-1}e_{k+1}\cdots e_{n},$$

the homotopy equivalence of paths. Then the multiplication of $P$ induces a partial multiplication $[p][q]$ of homotopy classes, such that $[p][q]$ is defined if and only if $pq$ is defined.

Let $\overline{P}_{i,j} = \overline{P}_{i,j}(G)$ denote the set of homotopy classes of all $p \in P_{i,j}$. Then the family $\overline{P} = \{\overline{P}_{i,j}\}_{i,j \in V(G)}$ forms a groupoid with above multiplication, as required. \hfill \Box

A subgroupoid $H$ of a groupoid $G$ is defined similarly to the groups, with the additional condition $V(H) \subset V(G)$. $H$ is called full if $H_{i,j} = G_{i,j}$ for all $i,j \in V(H)$, and called wide if $V(H) = V(G)$. (Similar notions are also defined for categories and graphs.) Further, $H$ is called normal if $H$ is wide and $gxg^{-1} \in H_{j,j}$ for all $x \in H_{i,i}$, $g \in G_{j,i}$.

**Example 2.3.** For a groupoid $G$ with wide subgroupoid $H$, let $\sim_{H}$ be the equivalence relation on $V(G)$ such that $i \sim_{H} j$ if and only if $H_{i,j} \neq \emptyset$. Then any full subgroupoid of $G$ on an equivalence class with respect to $\sim_{G}$ is called a connected component of $G$, and $G$ is called connected if $G$ consists of only one connected component. \hfill \Box

Let $G$ be a groupoid. Then the intersection $\bigcap_{H} H_{i,j}$ of subgroupoids $H_{i,j}$ of $G$ is defined naturally, with $V(\bigcap_{H} H_{i,j}) = \bigcap_{H} V(H_{i,j})$, and forms a subgroupoid of $G$. Note that it becomes normal in $G$ whenever all $H_{i,j}$ are. Further, for a subgraph $X$ of $G$, the (normal) subgroupoid of $G$ generated by $X$ is the intersection of all (normal) subgroupoids of $G$ containing $X$, or equivalently the smallest (normal) subgroupoid of $G$ containing $X$. 
Let $X, X'$ be graphs. A graph homomorphism $f : X \to X'$ sends each $i \in V(X)$ to $f(i) \in V(X')$ and each $x \in X_{i,j}$ to $f(x) \in X'_{f(i),f(j)}$. A graph anti-homomorphism is defined similarly but $f(x) \in X'_{f(j),f(i)}$ instead of $X'_{f(i),f(j)}$. Let $f : G \to G'$ be a graph homomorphism between groupoids. Then $f$ is called a groupoid homomorphism if $f(xy) = f(x)f(y)$ for $x, y \in G$ whenever $xy$ is defined and $f(1_i) = 1_{f(i)}$ for $i \in V(G)$, and groupoid anti-homomorphisms are also defined similarly. Then every property about homomorphisms appearing in this subsection can be translated into the case of anti-homomorphisms. Note that $f(x^{-1}) = f(x)^{-1}$ holds for any groupoid homomorphism $f : G \to G'$ and $x \in G$. Further, an isomorphism of two groupoids is defined similarly to the case of groups. Note that any groupoid isomorphism $f : G \to G'$ induces isomorphisms of vertex groups $G_{i,i} \to G'_{f(i),f(i)}$.

The image of a groupoid homomorphism $f : G \to G'$ is the subgraph $f(G)$ of $G'$ on $f(V(G))$ consisting of all $f(x), x \in G$. Note that $f(G)$ is not a subgroupoid of $G'$ in general, but this becomes a subgroupoid whenever $f$ is injective on $V(G)$. On the other hand, the kernel of $f$ is the wide subgraph $\ker f$ of $G$ consisting of all $x \in G$ such that $f(x)$ is a unit of $G'$, and then $\ker f$ always forms a normal subgroupoid of $G$.

For a groupoid $G$ and its normal subgroupoid $N$, the quotient groupoid $G/N$ is defined as follows. Let $V(G/N)$ be the set of all equivalence classes $[i]$ on $V(G)$ with respect to $\sim_N$. For $x, y \in G$, let $\equiv_N$ be an equivalence relation on $G$ such that $x \equiv_N y$ if and only if $x = gh$ for some $g, h \in N$. Let $[x]$ denote the equivalence class of $x \in G$ with respect to $\equiv_N$. Now define

$$(G/N)_{i',j'} = \{ [x] \mid x \in G_{i',j'} \text{ for some } i' \in [i], j' \in [j] \}$$

and $G/N = \{(G/N)_{i,j} \}_{i,j}$. Then the multiplication of $G/N$ is induced naturally and $G/N$ forms a groupoid in fact.

Now an analogy of "The First Isomorphism Theorem" is given as follows:

**Theorem 2.4 (see [2] or [6]).** If a groupoid homomorphism $f : G \to G'$ is injective on $V(G)$, then the induced map $\overline{f} : G/\ker f \to f(G)$ is an isomorphism. □

Let $G$ be a groupoid with subgraph $X$. We say that $G$ is free on $X$ if any graph homomorphism $f : X \to G'$ to a groupoid $G'$ extends uniquely to a groupoid homomorphism $\overline{f} : G \to G'$. Note that the free groupoid on $X$ is unique (up to isomorphism) if it exists. Conversely, the existence is deduced from the following fact:

**Theorem 2.5 (see [6]).** Let $G$ be an undirected graph. Fix an orientation for $G$, so $G$ is considered as a subgraph of its fundamental groupoid $\overline{P}$. Then $\overline{P}$ is free on $G$. □
3 Decomposition of centralizers

From now on, we assume that $S$ is a finite set. In this section, we state that the centralizer $C_W(W_I)$ of $W_I$ admits a decomposition $W_{[x]} \times (W_x \times G_{x,x})$ as described in Introduction, and define a normal subgroup $H_{x,x}$ of $G_{x,x}$.

We start with some notations. For a nonnegative integer $N$, let
\[ S^{(N)} = \{ x = (x_1, \ldots, x_N) \in S^N | x_i \neq x_j \text{ for all } i \neq j \} \]
and let $[x] = \{x_1, \ldots, x_N\}$ for $x \in S^{(N)}$. Further, for $I \subset S$, let $I^{\text{iso}}$ be the set of all isolated points of $I_I$ and let $I_\perp = \{ \gamma \in \Phi | \langle \gamma, \alpha_s \rangle = 0 \text{ for all } s \in I \}$.

**Definition 3.1.** For $x, y \in S^{(N)}$, let
\[
C_{x,y} = \{ w \in W | wy_i w^{-1} = x_i \text{ for all } 1 \leq i \leq N \} = \{ w \in W | w \cdot \alpha_{y_i} = \pm \alpha_{x_i} \text{ for all } 1 \leq i \leq N \},
C'_{x,y} = \{ w \in C_{x,y} | w \cdot \alpha_{y_i} = \alpha_{x_i} \text{ for all } y_i \in [y]^{\text{iso}} \},
C''_{x,y} = \{ w \in C_{x,y} | w \cdot \alpha_{y_i} = \alpha_{x_i} \text{ for all } 1 \leq i \leq N \}.
\]

Note that $C''_{x,y} \subset C'_{x,y} \subset C_{x,y}$ by this definition. Further, the following lemma also follows from this definition:

**Lemma 3.2.** $C = \{ C_{x,y} \}_{x,y}, C' = \{ C'_{x,y} \}_{x,y}, C'' = \{ C''_{x,y} \}_{x,y}$ are groupoids on $S^{(N)}$.

Since the centralizer of each $W_I$ occurs as $C_{x,x}$ by taking $x \in S^{(#I)}$ such that $[x] = I$, we examine $C_{x,x}$ hereafter. Now we have the decomposition of $C_{x,x}$ as follows:

**Theorem 3.3.** Let $x \in S^{(N)}$. Then $C_{x,x} = W_{[x]} \times C'_{x,x}, W_{[x]} \simeq (\mathbb{Z}/2\mathbb{Z}) \# [x]^{\text{iso}}$.

Secondly, we give a certain decomposition of $C''_{x,x}$. Let $W_x^{\perp}, x \in S^{(N)}$ be the subgroup of $W$ generated by all reflections $s_\gamma$ such that $\gamma \in [x]_\perp^+$, and let
\[ G_{x,y} = \{ w \in C'_{x,y} | \Phi_w^+ \cap [y]_\perp = \emptyset \} \]
for $x, y \in S^{(N)}$. Then it can be shown that $G_{x,y} = \{ w \in C'_{x,y} | w \cdot [y]^+ = [x]^+ \}$, and so $G = \{ G_{x,y} \}_{x,y}$ forms a wide subgroupoid of $C'$. In particular, $G_{x,x}$ is a subgroup of $C'_{x,x}$.

On the other hand, for the subgroup $W_x^{\perp}$, the following lemma holds:

**Lemma 3.4.** For $x \in S^{(N)}$, $W_x^{\perp}$ is a normal subgroup of $C'_{x,x}$, and the set $[x]_\perp$ is $W_x^{\perp}$-invariant.
Deodhar [4] and Dyer [5] proved independently that every reflection subgroup (that is, a subgroup generated by reflections) forms a Coxeter system with certain generating set. Now each $W^+_x$ is a reflection subgroup, so $W^+_x$ also forms a Coxeter group.

Further, they also gave a characterization of the generating set. Now we determine the generating set $\tilde{S}_x$ of $W^+_x$ by using Deodhar’s characterization; let $\tilde{\Pi}_x$ be the set of all $\gamma \in [x]_+^r$ such that $\gamma$ cannot be written as a nonnegative $\mathbb{R}$-linear combination of other elements of $[x]_+^r$, and let $\tilde{S}_x = \{s_\gamma \mid \gamma \in \tilde{\Pi}_x\}$. Then we have the following by [4] since $[x]_+^r$ is $W^+_x$-invariant (cf. Lemma 3.4):

**Theorem 3.5.** $(W^+_x, \tilde{S}_x)$ is a Coxeter system, and its length function $\tilde{\ell}$ satisfies $\tilde{\ell}(w) = \#(\Phi^+_w \cap [x]_+^r)$ for all $w \in W^+_x$. □

We note that Dyer’s characterization gives the same generating set with the set obtained by Deodhar’s.

Now the decomposition of $C'_{x,x}$ is given as follows:

**Theorem 3.6.** $C'_{x,x} = W^+_x \times G_{x,x}$ for all $x \in S^{(N)}$. □

Here we consider the structure of $(W^+_x, \tilde{S}_x)$ and the action of $G_{x,x}$ on $W^+_x$ more. Firstly, the following theorem is a special case of Theorem 4.4 of [5]:

**Theorem 3.7.** Let $\gamma_1, \gamma_2 \in \tilde{\Pi}_x$, $\gamma_1 \neq \gamma_2$. Then either $\langle \gamma_1, \gamma_2 \rangle = -\cos(\pi/m)$ for some $m \in \mathbb{Z}$, $m \geq 2$ or $\langle \gamma_1, \gamma_2 \rangle \leq -1$. □

Then the structure of $(W^+_x, \tilde{S}_x)$ can be determined whenever $\tilde{\Pi}_x$ is well understood, by using the following fact:

**Proposition 3.8.** Let $\gamma_1, \gamma_2 \in \tilde{\Pi}_x$, $\gamma_1 \neq \gamma_2$. Then $s_{\gamma_1}s_{\gamma_2}$ has finite order $m$ if and only if $\langle \gamma_1, \gamma_2 \rangle = -\cos(\pi/m)$. □

Secondly, we examine the action of $G_{x,x}$ on $W^+_x$. Let $\tilde{\Gamma}$ denote the Coxeter graph of $(W^+_x, \tilde{S}_x)$. Note that for arbitrary Coxeter system $(W, S)$ (temporarily we do not assume that $S$ is a finite set) with Coxeter graph $\Gamma$, each $\sigma \in \text{Aut}\Gamma$ induces an automorphism $f_\sigma : W \rightarrow W$, and the map $\text{Aut}\Gamma \rightarrow \text{Aut}W$, $\sigma \mapsto f_\sigma$ is a group homomorphism. Now the following theorem follows from Deodhar’s characterization of $\tilde{S}_x$:

**Theorem 3.9.** There exists a unique group homomorphism $G_{x,x} \rightarrow \text{Aut}\tilde{\Gamma}$, $w \rightarrow \sigma_w$ such that $wuw^{-1} = f_{\sigma_w}(u)$ for all $u \in W^+_x$. □

**Corollary 3.10.** $C'_{x,x} = W^+_x \times G_{x,x}$ whenever $\text{Aut}\tilde{\Gamma} = 1$. □

At last of this section, we define $H = G \cap C''$, so $H$ is a wide subgroupoid of $G$. Then it follows from the definition that each $H_{x,x}$ is a normal subgroup of $G_{x,x}$. The structures of $G_{x,x}$ and $H_{x,x}$ are discussed in the following sections.
4 Transition diagram and the groupoid $H$

Before we consider the structure of $G_{x,x}$, we examine the groupoid $H$ in this section.

To do this, we define a graph (which we call the transition diagram) $G = G^{(N)}(W, S)$ of $(W, S)$ for each nonnegative integer $N$; it is an undirected graph on $S^{(N)}$ which have close relation with the action of the longest elements of finite parabolic subgroups. Then we construct below a certain anti-homomorphism $F$ from the fundamental groupoid $\overline{\mathcal{P}} = \overline{\mathcal{P}}(\mathcal{G})$ of $\mathcal{G}$ to $H$ which is surjective. This implies that $H$ is anti-isomorphic to the quotient groupoid $\overline{\mathcal{P}}/\ker F$. Finally, we give a certain generating set of $\ker F$ (as a normal subgroup) and a method for obtaining a presentation of $H_{x,x}$.

Now we start to define $\mathcal{G}$. For $I \subset S$ and $s \in S$, let $I_{\sim s}$ denote the vertex set of connected component of $\Gamma_{I \cup \{s\}}$ containing $s$, so $I_{\sim s} \subset I \cup \{s\}$. For $x \in S^{(N)}$, we write $x_{\sim s}$ as a shorthand for $[x]_{\sim s}$. Further, recall (cf. Section 2.1) that we call $I \subset S$ of finite type if $W_I$ is a finite group.

**Definition 4.1.** Let $x \in S^{(N)}, s \in S$. Then we say that $s$ reacts on $x$ if $s \not\in [x]$ and $x_{\sim s}$ is of finite type. In this case, the product $\varphi(x, s)$ of this reaction is defined to be the unique element of $S^{(N)}$ such that

$$\varphi(x, s)_i = \begin{cases} \sigma_{x_{\sim s}, \sigma_{x_{\sim s}^{-1}}(s)}(x_i) & \text{if } x_i \in x_{\sim s} \\ x_i & \text{otherwise} \end{cases}$$

and the residue of this reaction is $\psi(x, s) = \sigma_{x_{\sim s}}(s)$ (see Section 2.1 for the definition of $\sigma$). Moreover, we say that this reaction is dynamic if $\varphi(x, s) \neq x$.

The definition of $\mathcal{G}$ is as follows:

**Definition 4.2.** Let $x, y \in S^{(N)}$ and $s, s' \in S$. Then $\mathcal{G}$ is defined to be a graph on the vertex set $S^{(N)}$ such that, it has an undirected edge $\{(x, s), (y, s')\}$ between $x$ and $y$ if and only if $s$ reacts dynamically on $x$ and its product, residue are $y, s'$ respectively.

In addition, when we picture the picture of $\mathcal{G}$, this edge $\{(x, s), (y, s')\}$ is represented, for example, as an edge with labels $s$ close to the vertex $x$ and $s'$ close to $y$; moreover, for the case $s = s'$, the repeated $s$'s may be replaced by a single $s$.

Though the definition of edges of $\mathcal{G}$ in Definition 4.2 seems to be asymmetric about $(x, s)$ and $(y, s')$, $\mathcal{G}$ is well-defined, thanks to the following proposition:

**Proposition 4.3.** If $s$ reacts on $x$, then $\psi(x, s)$ also reacts on $\varphi(x, s)$, and the product, residue of the latter reaction are $x, s$ respectively. In particular, the latter reaction is dynamic if and only if the former is dynamic.
This proposition is deduced from the following characterization of reactions:

Lemma 4.4. Let \( x, y \in S^{(N)} \) and \( s \in S \). Then \( s \) reacts on \( x \) and the product is \( y \) if and only if the following two conditions hold:

(i) \( [y] \subset [x] \cup \{s\} \),
(ii) there exists some \( w \in W_{[x]} \cup \{s\} \cap C_{y,x}^{n} \), \( w \neq 1 \).

Further, \( w = w^{s}_{x} \) whenever these conditions hold, where \( w^{s}_{x} = w_{0}(x_{\sim s})w_{0}(x_{\sim s} \setminus \{s\}) \).

Moreover, this lemma implies also the following proposition:

Proposition 4.5. If \( s \) reacts on \( x \), then \( w_{\varphi(x,s)}^{\psi(x,s)} = (w^{s}_{x})^{-1} \).

Example 4.6. Let \((W, S)\) be a finite Coxeter system of type \( B_{5} \) with numbering on \( S \) in Figure 1, and let \( N = 3 \), \( x = (s_{1}, s_{3}, s_{4}) \in S^{(N)} \). Then we show that the connected component \( G_{x} \) of \( G \) containing \( x \) is

\[
y \quad s_{3} \quad s_{2} \quad x \quad s_{5} \quad x' \quad s_{2} \quad s_{3} \quad y'
\]

where \( x' = (s_{1}, s_{4}, s_{3}) \), \( y = (s_{4}, s_{1}, s_{2}) \) and \( y' = (s_{4}, s_{2}, s_{1}) \).

Firstly, \( s_{5} \) reacts on \( x \) and \( \varphi(x, s_{5}) = x', \psi(x, s_{5}) = s_{5} \). In fact, \( x_{\sim s_{5}} = \{s_{3}, s_{4}, s_{5}\}, x_{\sim s_{5}} \setminus \{s_{5}\} = \{s_{3}, s_{4}\} \) are of type \( B_{3}, A_{2} \) respectively. Then we have \( \varphi(x, s_{5}) = x' \), \( \psi(x, s_{5}) = s_{5} \) since the action of the longest element turns the Coxeter graph for the case of type \( A_{n} \), while it fixes the Coxeter graph for \( B_{n} \) (cf. Figure 1).

Further, \( s_{2} \) also reacts dynamically on \( x, x' \) and \( \varphi(x, s_{2}) = y, \psi(x, s_{2}) = s_{3}, \varphi(x', s_{2}) = y' \) and \( \psi(x', s_{2}) = s_{3} \) by similar argument. Finally, it can be checked that \( s_{5} \) reacts not dynamically on \( y, y' \). Hence the connected component becomes as above.

For each edge \( \{(x, s), (y, s')\} \) of \( G \), let \( e_{x}^{s} \) denote this edge with direction from \( x \) to \( y \) (note that \( y \) and \( s' \) are uniquely determined by \( x \) and \( s \) whenever the edge exists, so this notation is unambiguous), and let \((e_{x}^{s})^{-1}\) denote the same edge but has the opposite direction (namely, from \( y \) to \( x \)). Then \((e_{x}^{s})^{-1} = e_{\varphi(x,s)}^{\psi(x,s)}\) and every directed path \( p \) of \( G \) is written as the form \( p = e_{x_{1}}^{s_{1}}e_{x_{2}}^{s_{2}} \cdots e_{x_{\ell}}^{s_{\ell}} \), with \( \varphi(x_{i}, s_{i}) = x_{i+1} \) for all \( 1 \leq i \leq \ell - 1 \). We write \( p^{-1} = (e_{x_{\ell}}^{s_{\ell}})^{-1} \cdots (e_{x_{2}}^{s_{2}})^{-1}(e_{x_{1}}^{s_{1}})^{-1} \) for such \( p \). As in Section 2.2, let \( \mathcal{P} = \mathcal{P}(G) \), \( \mathcal{P}_{x,y} = \mathcal{P}_{x,y}(G) \) denote the set of all directed paths of \( G \), all directed paths of \( G \) from \( x \) to \( y \) respectively, and let \( [p] \) denote the homotopy class of \( p \in \mathcal{P} \). Note that \([p^{-1}] = [p]^{-1}\) for any \( p \in \mathcal{P} \).

Now we define an anti-homomorphism \( F : \overline{\mathcal{P}} \to H \) as follows. \( F \) is defined to be the identity map on \( S^{(N)} \), and to satisfy \( F(e_{x}^{s}) = w_{x}^{s} \) for each directed edge of \( G \) (here we write \( F(e_{x}^{s}) \) as a shorthand for \( F([e_{x}^{s}]) \)). Then we have \( F((e_{x}^{s})^{-1}) = F(e_{x}^{s})^{-1} \) by Proposition 4.5. Since \( \overline{\mathcal{P}} \) is a free groupoid on \( G \) (cf. Theorem 2.5), this \( F \) extends uniquely to an
anti-homomorphism $F : \overline{P} \to H$ (so $F(e_{x_{1}}^{s_{1}} \cdots e_{x_{l}}^{s_{l}}) = w_{x_{l}}^{s_{l}} \cdots w_{x_{1}}^{s_{1}}$), provided the following proposition holds:

**Proposition 4.7.** If $s$ reacts dynamically on $x$, then $w_{x}^{s} \in H_{\varphi(x,s),x}$. □

For each $p \in P$, we also write $F(p)$ as a shorthand for $F([p])$.

The key to the proof of Proposition 4.7 is the following lemma:

**Lemma 4.8.** Suppose that $s$ reacts on $x$. Then $\Phi_{w_{x}^{s}}^{+} \cap [x]_{\perp} = \emptyset$ if and only if this reaction is dynamic. □

Then for each $x$, $s$ such that $s$ reacts dynamically on $x$, we have $w_{x}^{s} \in C''_{\varphi(x,s),x}$ by definition of $w_{x}^{s}$, while $\Phi_{w_{x}^{s}}^{+} \cap [x]_{\perp} = \emptyset$ by this lemma. Hence $w_{x}^{s} \in H_{\varphi(x,s),x}$, so Proposition 4.7 holds.

Now we state a theorem which implies that $F$ is surjective, by using the following notations and terminology:

**Definition 4.9.** For $p = e_{x_{1}}^{s_{1}} \cdots e_{x_{n}}^{s_{n}} \in P$, define

$$\ell(p) = n, \quad |p| = \sum_{i=1}^{n} \ell(w_{x_{i}}^{s_{i}}), \quad L(p) = \ell(F(p)) = \ell(w_{x_{n}}^{s_{n}} \cdots w_{x_{1}}^{s_{1}}).$$

Further, we say that $p$ is nondegenerate if $L(p) = |p|$ and degenerate if $L(p) < |p|$ (note that $L(p) \leq |p|$ for all $p \in P$). □

The theorem is as follows:

**Theorem 4.10.** For each $w \in H_{y,x}$, there exists a nondegenerate path $p \in P_{x,y}$ such that $F(p) = w$. In particular, $F$ is surjective. Moreover, if $s \in S$ and $w \cdot \alpha_{s} < 0$, then we can choose such $p$ having $e_{x}^{s}$ as the first edge. □

To prove this theorem, we use the following lemma:

**Lemma 4.11.** Let $w \in H_{y,x}$, $s \in S$ and suppose $w \cdot \alpha_{s} < 0$. Then $s$ reacts dynamically on $x$ and $\ell(w) = \ell(w_{x}^{s})^{-1} + \ell(w_{x}^{s})$. □

Then Theorem 4.10 follows from this lemma, by induction on $\ell(w)$.

Thus we conclude the construction of the surjective anti-homomorphism $F$. Now let $F_{z}$ be the restriction of $F$ to the connected component $\overline{P}(G_{z})$ of $\overline{P}$ containing $x \in S^{(N)}$. Then $F_{z}$ is also a surjective anti-homomorphism from $\overline{P}(G_{z})$ to the connected component $H_{z}$ of $H$ containing $x$. Since $F$ is injective on the vertex set of $\overline{P}$, $F$, $F_{z}$ induce an anti-isomorphism $\overline{P}/\ker F \to H$, $\overline{P}(G_{z})/\ker F_{z} \to H_{z}$ respectively, as we remarked before.
Finally, we give a generating set of $\ker F_x$ as a normal subgroupoid, and a presentation of $H_{x,x}$. For each $J \subset S$, let $G_x^{(J)}$ denote the “restriction” of $G_x$ to $J$; that is, the subgraph of $G_x$ consisting of all vertices $y$ of $G_x$ such that $[y] \subset J$, and all edges $\{(x,s),(y,s')\}$ of $G_x$ such that $[x],[y] \subset J$ and $s,s' \in J$. Now define $C_x$ to be the set of all simple closed paths of $G_x^{(J)}$, where $J$ runs over all subset of $S$ such that $\# J = N+2$ and $J$ is of finite type (actually, for each simple closed path $c = e_1^{s_1} \cdots e_{\ell}^{s_{\ell}}$ of such $G_x^{(J)}$, only one of its cyclic permutations $e_1^{s_1} \cdots e_{\ell}^{s_{\ell}} e_1^{s_{\ell}} \cdots e_{\ell-1}^{s_{\ell-1}}$, $1 \leq i \leq \ell$, or their inverses must be contained in $C_x$ and the others may be excluded). Then the following theorem holds, but the proof of this is too long and intricate to write, or even to sketch, in this paper:

**Theorem 4.12.** $\ker F_x$ is generated by all $[c]$, $c \in C_x$ as a normal subgroupoid.

Further, we consider the presentation of $H_{x,x}$. Let $E_x$ denote the set of all directed edges of $G_x$. Then every path of $G_x$ can be regarded as an element of the free group with basis $E_x$. Now the following theorem is a special case of Theorem 5.17 of [3]:

**Theorem 4.13.** Let $T$ be a maximal tree in $G_x$. Then the vertex group $(\overline{P}/\ker F)_{x,x}$ is isomorphic to the group presented by $(E_x | C_x \cup \{ee^{-1} | e \in E_x \} \cup \{e | e \in T\})$.

Moreover, the corresponding anti-isomorphism sends each $e_{y}^{s} \in E_x$ to $F(p_{\varphi(y,s)})^{-1}w_{p}^{s}F(p_{y})$, where $p_{z}$ denotes the unique reduced path in $T$ from $x$ to $z$.

## 5 Representatives of $G_{x,x}/H_{x,x}$ and their product

In this section, we examine the quotient group $G_{x,x}/H_{x,x}$. We show below that $G_{x,x}/H_{x,x}$ is a finite elementary abelian 2-group (Corollary 5.5), and that we can choose its coset representatives in the form $w_{0}(I)F(p)$, where $I \subset [x]$ is of finite type and $p$ is a path of $G_x$ (Theorem 5.7). Moreover, the multiplication in $G_{x,x}$ is described only by the structure of $G_x$ (Corollary 5.11); for this description, certain automorphisms on $G_x$ are defined and used.

We start with some notations. For $x \in S^{(N)}$, define

- $CO(x) = \{ A \subset \{1,2,\ldots,N\} | x_{A}$ is a connected component of $x\}$
- $CO_{<\infty}^{1}(x) = \{ A \in CO(x) | x_{A}$ is of finite type, $\#A > 1\}$

where $x_{A} = \{ x_{i} | i \in A \}$. Note that the power set $P(CO(x))$ of $CO(x)$ forms a finite elementary abelian 2-group with symmetric difference as multiplication denoted by $\cdot$.

**Example 5.1.** Let $(W,S)$ be a Coxeter system with Coxeter graph below and let $x = (7,1,3,6,4) \in S^{(5)}$. Then we have $CO(x) = \{ \{1,4\}, \{2\}, \{3,5\} \}$, $CO_{<\infty}^{1}(x) = \{ \{3,5\} \}$. $\square$
The following basic lemma is used many times:

**Lemma 5.2.** Let $x, y \in S^{(N)}$, $w \in G_{x,y}$.
1. $\text{CO}(x) = \text{CO}(y)$ and $\text{CO}_{>1}(x) = \text{CO}_{>1}(y)$.
2. For $i, j \in A \in \text{CO}(x)$, $w \cdot \alpha_{y_i} = -\alpha_{x_i}$ if and only if $w \cdot \alpha_{y_j} = -\alpha_{x_j}$.
3. If $i \in A \in \text{CO}(x) \setminus \text{CO}_{>1}(x)$, then $w \cdot \alpha_{y_i} = -\alpha_{x_i}$.

For $x, y \in S^{(N)}$ and $A \subset \text{CO}_{>1}(y)$, define

$$G^A_{x,y} = \{w \in G_{x,y} | w \cdot \alpha_{y_i} = -\alpha_{x_i} \text{ if and only if } i \in \bigcup A\}.$$  

Then Lemma 5.2 yields the following decomposition of $G_{x,y}$:

**Lemma 5.3.** $G_{x,y} = \bigcup_{A \subset \text{CO}_{>1}(y)} G^A_{x,y}$ for all $x, y \in S^{(N)}$.  

Further, the following lemma is deduced immediately from the definition:

**Lemma 5.4.** Let $x, y, z \in S^{(N)}$, $A, A' \subset \text{CO}_{>1}(x)$ and suppose $G_{x,y} \neq \emptyset$, $G_{y,z} \neq \emptyset$ (so all $\text{CO}_{>1}(x), \text{CO}_{>1}(y), \text{CO}_{>1}(z)$ coincide by Lemma 5.2 (i)). Then

$$G^A_{x,y} \cdot G^{A'}_{y,z} \subset G^A_{x,z}, (G^A_{x,y})^{-1} = G^A_{y,x}, G^{A'}_{x,y} = H_{x,y}.$$  

For $x \in S^{(N)}$, define $E_x = \{A \subset \text{CO}_{>1}(x) | G^A_{x,x} \neq \emptyset\}$. Then the preceding lemmas imply the structure of $G_{x,x}/H_{x,x}$ as follows:

**Corollary 5.5.** Let $x \in S^{(N)}$. Then $E_x$ is a subgroup of $\mathcal{P}(\text{CO}_{>1}(x))$ and isomorphic to $G_{x,x}/H_{x,x}$, so it is also a finite elementary abelian 2-group. Further, this isomorphism sends each coset $wH_{x,x}$ to the unique $A_w \subset \text{CO}_{>1}(x)$ satisfying $w \in G^A_{x,x}$.  

Now we give certain coset representatives of $G_{x,x}/H_{x,x}$. For $x \in S^{(N)}$ and $A \subset \text{CO}_{>1}(x)$, define

$$w_0(A; x) = \prod_{A \in A} w_0(x_A) = w_0(x_{\cup A})$$

(note that all $w_0(x_A)$ in the above product commute). Further, let $y \in S^{(N)}$, $y \in G_x$. Then we have $G_{y,x} \neq \emptyset$ since $G_x$ is connected and $F(\overline{P}_{x,y}) \subset G_{y,x}$, and so $A \subset \text{CO}_{>1}(y)$ by Lemma 5.2 (i). Now define $y^A \in S^{(N)}$ by

$$(y^A)_i = w_0(A; y)_i w_0(A; y) = \begin{cases} \sigma_{y_A}(y_i) & \text{if } i \in A \text{ for some } A \in A \\ y_i & \text{otherwise.} \end{cases}$$

Then we have the following lemma:
Lemma 5.6. Let \( x \in S^{(N)} \), \( A \subset CO_{<\infty}^{>1}(x) \). Then \( w_0(A; x) \in G_x^A \). Hence \( G_x^A = w_0(A; x)H_{x^A,x} \) and so \( E_x = \{ A \subset CO_{<\infty}^{>1}(x) \mid H_{x^A,x} \neq \emptyset \} \).

The coset representatives of \( G_{x,x}/H_{x,x} \) are given as follows (recall that the map \( F \) is surjective):

Theorem 5.7. \( G_{x,x} = \bigcup_{A \in E_x} w_0(A; x)F(p_A)H_{x,x} \), where \( p_A \) is an arbitrarily chosen element of \( P(x,x^A) \) for each \( A \in E_x \).

Since \( H_{x,x} \) is generated by certain elements \( F(p) \), \( p \in P(x,x) \) (cf. Section 4), this theorem implies that \( G_{x,x} \) is generated by such \( F(p) \) and these coset representatives. In the rest of this section, we describe the multiplication of these generators of \( G_{x,x} \), using certain automorphisms on \( G_x \) defined in Theorem 5.10 below.

We use the following two lemmas:

Lemma 5.8. Let \( x \in S^{(N)} \), \( A, A' \subset CO_{<\infty}^{>1}(x) \) (so \( CO_{<\infty}^{>1}(x^A) = CO_{<\infty}^{>1}(x) \) by Lemmas 5.2 (i) and 5.6). Then

\[
w_0(A; x)w_0(A'; x) = w_0(A \cdot A'; x), w_0(A'; x^A) = w_0(A'; x), (x^A)^{A'} = x^{A \cdot A'}.
\]

Lemma 5.9. Let \( x \in S^{(N)} \), \( y, z \in G_x \), \( A \subset CO_{<\infty}^{>1}(x) \) and \( s \in S \). Then \( s \) reacts dynamically on \( y \) and \( \varphi(y, s) = z \) if and only if \( s \) reacts dynamically on \( y^A \) and \( \varphi(y^A, s) = z^A \). Moreover, if above conditions hold, then \( \psi(y, s) \) and \( \psi(y^A, s) \) coincide, and \( w_{y^A}^s = w_0(A; z)w_y^s w_0(A; y) \).

The automorphisms on \( G_x \) are given as follows:

Theorem 5.10. For each \( A \in E_x \), define \( \rho_A : G_x \to G_x \) by

\[
\rho_A(y) = y^A \quad (y \in V(G_x)), \quad \rho_A(e_y^s) = e_{y^A}^s \quad (e_y^s \in E(G_x)).
\]

(i) \( \rho_A \) is an involutive graph automorphism on \( G_x \).

(ii) \( \rho_A \rho_A' = \rho_{A \cdot A'} \) holds for all \( A, A' \in E_x \).

(iii) If \( \rho_A \) also denotes the extension of \( \rho_A \) to \( P(G_x) \) (the directed paths of \( G_x \)), then it is an involutive automorphism and satisfies \( \rho_A(p^{-1}) = \rho_A(p)^{-1}, \rho_A \rho_A' = \rho_{A \cdot A'} \) and

\[
F(\rho_A(p)) = w_0(A; z)F(p)w_0(A; y)
\]

for all \( p \in P(G_x)_{y,z} \).

Now the multiplication (in \( G_{x,x} \)) of the representatives of \( G_{x,x}/H_{x,x} \) and the action of these to the generators \( F(p) \) of \( H_{x,x} \) are described as follows, by using the automorphisms...
Corollary 5.11. (i) Let $A \in E_x$, $p_A \in P_{x,x^A}$ and $q \in P_{x,x}$. Then
\[ w_0(A;x)F(p_A)F(q)(w_0(A;x)F(p_A))^{-1} = F(\rho_A((p_A)^{-1}qp_A)). \]
(ii) Let $A, A' \in E_x$, $p_A \in P_{x,x^A}$ and $p_{A'} \in P_{x,x^{A'}}$. Then
\[ w_0(A;x)F(p_A)w_0(A';x)F(p_{A'}) = w_0(A \cdot A';x)F(p_{A'}\rho_{A'}(p_{A})). \]
\[ \square \]
Note that to obtain $\rho_A(p)$ for each $A \in E_x$ and $p = e_{x_1}^{s_1} \cdots e_{x_n}^{s_n} \in P(G_x)$, we need not compute $(x_i)^A$ for any $i \geq 2$; indeed, we have only to compute $(x_1)^A$, and then start at $(x_1)^A$ and trace each (unique) directed edge labeled $s_i$ step by step.

6 Examples

Example 6.1. $(W, S)$ is of type $\overline{B_7}$ and $x = (1, 2, 4, 5, 8) \in S^{(N)} (N = 5)$, as in Figure 2 (in this section we write $i$ as a shorthand for $s_i$). Then we compute the centralizer $C_{x,x}$ of $W_{\{1,2,4,5,8\}}$.

![Coxeter graph of type $\overline{B_7}$](image)

Figure 2: Coxeter graph of type $\overline{B_7}$

1. Figure 2 implies $[x]^{iso} = \{1, 2, 8\}$, so by Theorem 3.3,
\[ C_{x,x} = W_{\{1,2,8\}} \times C'_{x,x} \simeq (\mathbb{Z}/2\mathbb{Z})^3 \times C'_{x,x}. \]

2. We determine the structure of the Coxeter system $(W_x^{\perp}, S_x)$. Let
\[ \delta = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \sqrt{2}\alpha_8, \]
which is called the null root of $(W, S)$. So $\langle \delta, \alpha_i \rangle = 0$ for all $1 \leq i \leq 8$. Now $\Phi$ is the (disjoint) union of following two sets
\[ \Phi' = \{ n\delta \pm \gamma \mid n \in \mathbb{Z}, \gamma \in \Phi_+^{\perp}_{S \setminus \{8\}} \}, \]
\[ \Phi'' = \{ n\sqrt{2}\delta \pm \sum_{i=k}^{7} \sqrt{2}\alpha_i + \alpha_8 \mid n \in \mathbb{Z}, 2 \leq k \leq 8 \}. \]
Moreover, $\Phi' = W \cdot \alpha_i$ for each $1 \leq i \leq 7$ and $\Phi'' = W \cdot \alpha_8$. To show these, we have only to check that every element of $\Phi', \Phi''$ is indeed a root of $(W, S)$ (this can be proved by the induction on $|n|$), both $\Phi', \Phi''$ are $W$-invariant (this follows from that $W \cdot \Phi^+_S \subset \Phi'$ and $W \cdot (\sum_{i=k}^7 \sqrt{2} \alpha_i + \alpha_8) \subset \Phi''$ for all $2 \leq k \leq 8$), and $\Pi_{S \setminus \{8\}} \subset \Phi'$, $\alpha_8 \in \Phi''$ (these are trivial).

By the above result, we have $[x]_\perp = \{n\sqrt{2} \delta \pm \beta \mid n \in \mathbb{Z}\}$ and so $\widetilde{\Pi}_x = \{\beta, \beta'\}$, where $\beta = \sqrt{2} \alpha_i + \alpha_8$, $\beta' = \sqrt{2} \delta - \beta$. Further, since $\langle \beta, \beta' \rangle = -1$, Proposition 3.8 implies that $s_\beta s_{\beta'}$ has infinite order. Hence $(W^\perp_x, \widetilde{S}_x)$ is of type $\widetilde{A}_1$ (the infinite dihedral group), and by Theorem 3.6, we have $C'_{x,x} = W^\perp_x \rtimes G_{x,x} \simeq \widetilde{A}_1 \rtimes G_{x,x}$.

3. The connected component $\mathcal{G}_x$ of $\mathcal{G}$ containing $x$ is as in Figure 3. In this case, let $e(y, z)$ denote the unique directed edge of $\mathcal{G}_x$ from $y$ to $z$. Now we determine the structure of the groupoid $H$, by using Theorems 4.12 and 4.13. Firstly, we examine the generating set of $\ker F_x$. Since $N + 2 = 7 = \#S - 1$, we have only to consider $\mathcal{G}_x^{(J)}$ for $J = S \setminus \{s\}$, $s \in S$. For example, if $s = 4$, then we obtain $\mathcal{G}_x^{(J)}$ from $\mathcal{G}_x$ by deleting four vertices II, III, VI, VII and six edges $e(I, II), e(II, III), e(III, IV), e(V, VI), e(VI, VII), e(VII, VIII)$. By similar argument, $\mathcal{G}_x^{(J)}$ is nonempty for $s = 3, 4, 6, 7$, as in Figure 4, while this is empty for $s = 1, 2, 5, 8$. Now by Theorem 4.12, $\ker F_x$ is generated (as a normal subgroupoid) by $[c_1]$ and $[c_2]$, where

\[
c_1 = e(I, VIII)e(VIII, IV)e(IV, V)e(V, I),
\]

\[
c_2 = e(I, II)e(II, III)e(III, IV)e(IV, VIII)e(V, VII)e(VII, VI)e(IV, V)e(V, I)
\]

(note that in this case, every proper subset of $S$ is of finite type).

Secondly, we give a presentation of $H_{x,x}$ by Theorem 4.13. Recall that $\mathcal{E}_x$ denotes the set of all directed edges of $\mathcal{G}_x$. Now we choose a maximal tree $T$ in $\mathcal{G}_x$ as in Figure 5, then
we have:

\[ H_{x,x} \cong^{o} \langle \mathcal{E}_{x} \mid c_{1} = 1, c_{2} = 1, e e^{-1} = 1 (e \in \mathcal{E}_{x}), e = 1 (e \in T) \rangle \]
\[ \cong \langle e(I, VIII), e(IV, V), e(VIII, VII) \mid e(I, VIII)e(IV, V) = 1, e(VIII, VII) = 1 \rangle \]
\[ \cong \langle e(IV, V) \rangle \cong \mathbb{Z}. \]

The corresponding anti-isomorphism sends \( e(IV, V) \) to \( F(q) \in H_{x,x} \), where

\[ q = e(II, III)e(III, IV)e(IV, V)e(V, I)e(I, II) = e_{II}^{3}e_{III}^{6}e_{IV}^{7}e_{V}^{3}e_{I}^{4}, \]

so \( H_{x,x} \) is the free group generated by \( F(q) \).
4. We describe the structure of \( G_{x,x} \) as in Section 5. Firstly, it follows from Figure 2 that \( \text{CO}(x) = \{\{1\}, \{2\}, \{3, 4\}, \{5\}\} \), \( \text{CO}_{\text{tr}}(x) = \{3, 4\} \). Put \( \mathcal{A}_0 = \{\{3, 4\}\} \), then \( x^{\mathcal{A}_0} = (1, 2, 5, 4, 8) = \text{VII} \) and so \( E_x = \emptyset, \mathcal{A}_0 \). Let \( p_0 \in \mathcal{P}_{x,x} \) be the trivial path and let

\[
p_{\mathcal{A}_0} = e(\text{II}, \text{I})e(\text{I}, \text{V})e(\text{V}, \text{VI})e(\text{VI}, \text{VII}) = e_\text{II}^6 e_\text{I}^6 e_\text{V}^6 e_{\text{VI}}^6 \in \mathcal{P}_{x,x}^{x^{\mathcal{A}_0}}.
\]

Then Theorem 5.7 implies \( G_{x,x} = H_{x,x} \cup aH_{x,x} \), where \( a = w_0(\{4, 5\})F(p_{\mathcal{A}_0}) \). Hence \( G_{x,x} \)

is generated by \( a \) and \( F(q) \).

As remarked in the last of Section 5, \( \rho_{\mathcal{A}_0}(p_{\mathcal{A}_0}) \) is the path which starts at \( x^{\mathcal{A}_0} = \text{VII} \) and traces the directed edges labeled as \( 6, 4, 3, 1 \) one by one; that is,

\[
\rho_{\mathcal{A}_0}(p_{\mathcal{A}_0}) = e(\text{VII}, \text{VIII})e(\text{VIII}, \text{IV})e(\text{IV}, \text{III})e(\text{III}, \text{II}).
\]

We write \( p \sim_F p' \) for two paths \( p, p' \) if \( F(p) = F(p') \). Then we have

\[
p_{\mathcal{A}_0} \rho_{\mathcal{A}_0}(p_{\mathcal{A}_0}) = e(\text{II}, \text{I})e(\text{I}, \text{V})e(\text{V}, \text{VI})e(\text{VI}, \text{VII})e(\text{VII}, \text{VIII})e(\text{VIII}, \text{IV})e(\text{IV}, \text{III})e(\text{III}, \text{II})
\sim e(\text{II}, \text{I})c_2^{-1}e(\text{II}, \text{I})^{-1} \sim_F 1
\]

since \( F(c_2) = 1 \). So we have \( a^2 = F(p_{\mathcal{A}_0} \rho_{\mathcal{A}_0}(p_{\mathcal{A}_0})) = 1 \) by Corollary 5.11 (ii). Similarly, we have \( \rho_{\mathcal{A}_0}(q) = e(\text{VII}, \text{VI})e(\text{VI}, \text{V})e(\text{V}, \text{IV})e(\text{IV}, \text{VII})e(\text{VIII}, \text{VII}) \) and so

\[
\rho_{\mathcal{A}_0}((p_{\mathcal{A}_0})^{-1})q_{\mathcal{A}_0}
= e(\text{II}, \text{II})e(\text{II}, \text{IV})e(\text{IV}, \text{VII})e(\text{VII}, \text{VIII})
\cdot e(\text{VII}, \text{VI})e(\text{VI}, \text{V})e(\text{V}, \text{IV})e(\text{IV}, \text{VII})e(\text{VII}, \text{VIII})
\cdot e(\text{VII}, \text{VIII})e(\text{VIII}, \text{IV})e(\text{IV}, \text{III})e(\text{III}, \text{II})
\sim (e(\text{II}, \text{II})e(\text{II}, \text{IV})e(\text{IV}, \text{VII})e(\text{VII}, \text{VIII})e(\text{VIII}, \text{IV})e(\text{IV}, \text{III})e(\text{III}, \text{II}))
\sim_F (e(\text{II}, \text{I})e(\text{I}, \text{V}))e(\text{V}, \text{IV})e(\text{IV}, \text{III})e(\text{III}, \text{II})
(F(c_2) = 1)
= q^{-1}.
\]

Hence we have \( aF(q)a^{-1} = F(\rho_{\mathcal{A}_0}(p_{\mathcal{A}_0}^{-1}q_{\mathcal{A}_0})) = F(q)^{-1} \) by Corollary 5.11 (i).

By these calculation, \( \{1, a\} \) forms a subgroup of \( G_{x,x} \) isomorphic to \( E_x \), and we have

\[
G_{x,x} \simeq H_{x,x} \rtimes E_x \simeq \mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z}),
\]

where \( 1 \in \mathbb{Z}/2\mathbb{Z} \) acts on \( \mathbb{Z} \) as multiplication by \( -1 \). Moreover, put \( a' = a \) and \( b' = aF(q) \), then we have \( G_{x,x} = \langle a', b' \mid a'^2 = 1, b'^2 = 1 \rangle \simeq \mathbb{A}_1 \).
5. Finally, we describe the action of $G_{x,x}$ on $W_{x}^{\perp}$. By direct computation, we have

$$a' \cdot \beta = \beta', \quad b' \cdot \beta = \beta.$$ 

Now recall (Theorem 3.9) that both $a'$, $b'$ act on $W_{x}^{\perp}$ as automorphisms of the Coxeter graph $\tilde{\Gamma}$ of $(W_{x}^{\perp}, \tilde{S}_{x})$; so we have $a' \cdot \beta' = \beta$, $b' \cdot \beta' = \beta'$.

Summarizing, we have $C_{x,x} \simeq (\mathbb{Z}/2\mathbb{Z})^{3} \times (\tilde{A}_{1} \rtimes \tilde{A}_{1})$, where each of two generators of the right $\tilde{A}_{1}$ acts on the left $\tilde{A}_{1}$ trivially, as the unique involution of $\tilde{\Gamma}$ respectively.

In this example, $G_{x,x}$ is isomorphic to the semidirect product of $H_{x,x}$ by $E_{x}$, and $H_{x,x}$ forms a free group. But these properties may fail in general.

Let $(W, S)$ be as in Figure 6 and let $x = (1, 2, 4, 5, 7, 8)$. Then it can be proved that

![Figure 6: Coxeter graph of another example](image)

$$W_{x}^{\infty} = 1, \quad W_{x}^{\perp} = 1, \quad G_{x,x} \simeq \mathbb{Z}^{2}, \quad H_{x,x} \simeq (2\mathbb{Z})^{2}.$$ 

Thus $H_{x,x}$ is not a free group, and $G_{x,x}$ is not isomorphic to a semidirect product of $H_{x,x}$ by any group, since $G_{x,x}$ has no subgroup isomorphic to $G_{x,x}/H_{x,x} \simeq (\mathbb{Z}/2\mathbb{Z})^{2}$.

Finally, we consider the centralizers of maximal parabolic subgroups (that is, parabolic subgroups generated by maximal proper subsets of $S$). Note that for $I \subset S$, the centralizer $C_{W}(W_{I})$ of $W_{I}$ is the direct product of $C_{W_{S_{i}}}(W_{I\cap S_{i}})$, where $S_{i}$ runs over all connected components of $S$. Thus we assume that $S$ is (finite and) connected.

Let $I$ be a maximal proper subset of $S$, with connected components $I_{1}, \ldots, I_{k}$. For $J \subset S$ such that $J$ is of finite type and $\sigma_{J} = \text{id}_{J}$, let $-1_{J} = w_{0}(J)$. Then it is obvious that each $-1_{I_{j}}, -1_{S}$ is contained in $C_{W}(W_{I})$ whenever it exists. Conversely, it can be deduced, by using the result of this paper, that $C_{W}(W_{I})$ is generated by these elements for almost all (possibly infinite) $W$ and $I$, except only two cases.

One of the exception is the case $W = D_{2n+1}$, $n \geq 2$ and $I = S \setminus \{s_{2i}\}$, $1 \leq i \leq n - 1$ (we use the numbering on $S$ in Figure 1); in this case, $C_{W}(W_{I})$ is generated by the involution $w_{0}(S)w_{0}(I')$, where $I' = \{s_{2i+1}, s_{2i+2}, \ldots, s_{2n+1}\}$ is a connected component of $I$. The other is the case $W = E_{6}$ and $I = S \setminus \{s_{2}\}$; now $C_{W}(W_{I})$ is generated by the involution $w_{0}(S)w_{0}(I)$. 
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