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On Centralizers of Parabolic Subgroups in Coxeter Groups

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Abstract

We describe the structure of the centralizer of an arbitrary parabolic subgroup in any finitely generated Coxeter group.

1 Introduction

Let \((W, S)\) be a Coxeter system such that \(S\) is a finite set. A parabolic subgroup \(W_I\) of \(W\) is the subgroup generated by a subset \(I\) of \(S\). In this paper, we determine the structure of the centralizer \(C_W(W_I)\) of \(W_I\) in \(W\) for arbitrary finite \(S\) and \(I\). In particular, we do not assume that \(W\) is finite.

The structure of \(C_W(W_I)\) has been known in certain cases, such as \(I = S\) (in this case, \(C_W(W_I)\) is the center of \(W\)) and \(#I = 1\) (this case is examined by Brink [1]). However, no corresponding general result for an arbitrary \(I\) has been known. Our result generalizes these results to the general case.

\(W\) has a well known faithful reflection representation in a real vector space with a symmetric bilinear form (which may not be positive definite in general), and the notion of root system in this vector space (see Section 2.1). Using this terminology, we decompose \(C_W(W_I)\) (in Section 3) as \(W_{iso} \times (W_I^\perp \times G_I)\), where \(W_{iso}\) is the parabolic subgroup generated by the elements of \(I\) which are isolated in the Coxeter graph of \(I\), \(W_I^\perp\) is the subgroup of \(W\) generated by reflections fixing all simple roots in \(W_I\), and \(G_I\) is a certain subgroup defined in that section. We have \(W_{iso} \simeq (\mathbb{Z}/2\mathbb{Z})^{#I_{iso}}\), and by a result by Deodhar [4] or by Dyer [5], \(W_I^\perp\) is a Coxeter group, whose Coxeter generators and Coxeter relations can be determined if the root system of \(W\) is well understood. Our main objective in this paper is to determine the structure of \(G_I\).

To describe \(G_I\), we define a groupoid (see Section 2.2 for the notion of groupoids) \(H\) on the set \(S^{(N)}, \, N = #I\), where \(S^{(N)} = \{x = (x_1, x_2, \ldots x_N) \in S^N \, | \, x_i \neq x_j \text{ for all } i \neq j\},\)
such that its vertex group \( H_{x,x} \) is a normal subgroup of \( G_{I} \) whenever \( I = \{x_{1}, x_{2}, \ldots, x_{N}\} \), and introduce a graph \( \mathcal{G} \), which we call the transition diagram, based on information on the subsets of \( S \) of finite type. Note that \( W_{I}^{\infty}, W_{I}^{\perp}, G_{I} \) are denoted by \( W_{[x]}^{\infty}, W_{x}^{\perp}, G_{x,x} \) respectively in the text, by taking such \( x \in S^{(N)} \). Then we present \( H \) as a quotient groupoid of the fundamental groupoid of \( \mathcal{G} \) (which is a free groupoid), and give a method to specify the generators of the kernel of the quotient map in terms of the directed paths of \( \mathcal{G} \). As a consequence, a presentation of \( H_{x,x} \) is also obtained. (These are done in Section 4.) Moreover, in Section 5, coset representatives of \( G_{x,x}/H_{x,x} \) and their products in \( G_{x,x} \) are described using the graph \( \mathcal{G} \).

Section 6 deals with certain examples; we compute \( C_{W}(W_{I}) \) for an affine Coxeter group according to the result of previous sections. Further, in that section, we also consider the case of maximal parabolic subgroups; that is, \( I \) is a maximal proper subset of \( S \).

Finally, note that all the proofs of the results are omitted in this paper, by reason of space. The detailed proofs will be given in [8].

2 Background material

2.1 Coxeter groups

In this paper, the basic notations and well-known facts about Coxeter groups are based on Humphreys [7]. Here we do not yet assume that \( S \) is a finite set.

A pair \((W,S)\) is called a Coxeter system (or simply, \( W \) is called a Coxeter group) if \( W \) is a group presented as

\[ W = \langle S | s^{2} = 1 \text{ for all } s \in S, (ss')^{m_{s,s'}} = 1 \text{ for some } s \neq s' \rangle, \]

where \( m_{s,s'} \) denotes the (possibly infinite) order of \( ss' \) in \( W \). (In this paper \((W,S)\) always denotes a Coxeter system.) Then the structure of \( W \) is described also by its Coxeter graph \( \Gamma \); this is a simple, undirected graph on \( S \) which has an edge between \( s \) and \( s' \) labeled \( m_{s,s'} \) if and only if \( m_{s,s'} \geq 3 \). Note that labels \( m_{s,s'} \) are usually omitted if \( m_{s,s'} = 3 \).

\((W,S)\) has a well-known geometric representation (over \( \mathbb{R} \)) as follows. Let \( V \) be a real vector space with basis \( \Pi = \{\alpha_{s}\}_{s \in S} \) having a symmetric bilinear form \( \langle , \rangle \) determined by \( \langle \alpha_{s}, \alpha_{s'} \rangle = -\cos(\pi/m_{s,s'}) \), where \( \pi/\infty \) is interpreted as 0. Then \( W \) acts on \( V \) by \( s \cdot v = v - 2 \langle v, \alpha_{s} \rangle \alpha_{s} \) for all \( s \in S \), and this action is faithful and preserves the bilinear form. The orbit \( \Phi = W \cdot \Pi \) is called the root system of \((W,S)\), and its elements are called roots of \((W,S)\). Obviously, each element of \( \Pi \) is a root; this is called a simple root. Further, a root \( \gamma \) is called positive, negative, denoted by \( \gamma > 0, \gamma < 0 \), if \( \gamma \) is a linear
combination of simple roots with all coefficients nonnegative, nonpositive respectively. For $\Psi \subset \Phi$, let $\Psi^+$, $\Psi^-$ denote the set of all positive, negative roots of $\Psi$ respectively. Then it is well known (see [7]) that $\Phi^- = -\Phi^+$, $\Phi = \Phi^+ \cup \Phi^-$ (disjoint union).

Let $\gamma = w \cdot \alpha_s$ be a root with $s \in S$, $w \in W$. Then the reflection $s_\gamma = ws w^{-1}$ about $\gamma$ is determined independently on the choice of $w, s$, and acts on $V$ by $s_\gamma \cdot v = v - 2(v, \gamma) \gamma$.

For $w \in W$, the length $\ell(w)$ of $w$ is the minimal number $k$ such that $w = s_1 s_2 \ldots s_k$ for some $s_i \in S$. Then it is also well known that $\ell(w) = \#\Phi_w^+$, where $\Phi_w^+$ denotes the set of all positive roots $\gamma \in \Phi$ such that $w \cdot \gamma < 0$.

For $I \subset S$, let $W_I = \langle I \rangle$ denote the parabolic subgroup of $W$ generated by $I$ and let $\Pi_I = \{\alpha_s\}_{s \in I}$, $V_I = \text{span}_\mathbb{R} \Pi_I$ and $\Phi_I = W_I \cdot \Pi_I$. Then $(W_I, I)$ is also a Coxeter system with geometric representation $V_I$, root system $\Phi_I$ and simple roots $\Pi_I$. Let $\Gamma_I$ denote the Coxeter graph of $(W_I, I)$. We often say that $I$ is connected instead of that $\Gamma_I$ is connected. For example, $(W, S)$ is called irreducible if $S$ is connected.

For $I \subset S$, we say that $I$ is of finite type if $W_I$ is a finite group. Then $W_I$ has the (unique) element $w_0(I)$ with maximal length, called the longest element of $W_I$, if and only if $I$ is of finite type. Further, the following theorem holds:

**Theorem 2.1** (see [9]). If $I \subset S$ is of finite type, then $w_0(I) \cdot \Pi_I = -\Pi_I$. \hfill \Box

Note that $w_0(I)$ is involutive. Owing to Theorem 2.1, we define a permutation $\sigma_I$ on $I$ by $w_0(I) \cdot \alpha_s = -\alpha_{\sigma_I(s)}$ for each $s \in I$. Then $\sigma_I(s) = w_0(I)sw_0(I)$ and so $\sigma_I$ is an involutive automorphism of $\Gamma_I$. Figure 1 shows the table of all finite irreducible Coxeter systems (see, for example, [7] for the classification of finite irreducible Coxeter systems) and the action of $\sigma$ for each Coxeter system.

### 2.2 Groupoids

In this paper, the basic notations and well-known facts about groupoids are based on Higgins [6] or Brown [2].

A family $X = \{X_{i,j}\}_{i,j \in V(X)}$ of sets $X_{i,j}$ is called a graph on vertex set $V(X)$. We write $x \in X$ when $x \in X_{i,j}$ for some $i, j$. Now a groupoid, certain generalization of a group, $G$ is a graph (in the above sense) satisfying the following axioms:

- **(G1)** For $x \in G_{i,j}$ and $y \in G_{j,k}$, the composition (or multiplication) $xy \in G_{i,k}$ is defined.
- **(G2)** $(xy)z = x(yz)$ holds for all $x \in G_{i,j}$, $y \in G_{j,k}$, $z \in G_{k,l}$.
- **(G3)** For $i \in V(G)$, there exists a unit $1_i \in G_{i,i}$ such that $1_i x = x$ for all $x \in G_{i,j}$, $j \in V(G)$ and $y 1_i = y$ for all $y \in G_{k,i}$, $k \in V(G)$.
- **(G4)** For $x \in G_{i,j}$, there exists an inverse $x^{-1} \in G_{j,i}$ of $x$ such that $xx^{-1} = 1_i$ and
Figure 1: Finite irreducible Coxeter systems

Each $s_i$ is the numbering on $S$, and $\overline{s_i} = \sigma_S(s_i)$.

$A_n \ (n \geq 1)\quad \overline{s_n} \quad \overline{s_{n-1}} \quad \ldots \quad \overline{s_2} \quad \overline{s_1}$

$B_n \ (n \geq 2)\quad \overline{s_1} \quad \overline{s_2} \quad \ldots \quad \overline{s_{n-1}} \quad 4 \quad \overline{s_n}$

$D_n \ (n \geq 4, \ n \text{ is even})\quad \overline{s_{n-1}} \quad \overline{s_{n-2}} \quad \overline{s_{n-1}} \quad 4 \quad \overline{s_n}$

$D_n \ (n \geq 4, \ n \text{ is odd})\quad \overline{s_{n-1}} \quad \overline{s_{n-2}} \quad \overline{s_{n-1}} \quad 4 \quad \overline{s_n}$

$E_6\quad \overline{s_6} \quad \overline{s_5} \quad \overline{s_3} \quad \overline{s_1}$

$E_7\quad \overline{s_2} \quad \overline{s_1} \quad \overline{s_3} \quad \overline{s_4} \quad \overline{s_5} \quad \overline{s_6} \quad \overline{s_7}$

$E_8\quad \overline{s_2} \quad \overline{s_1} \quad \overline{s_3} \quad \overline{s_4} \quad \overline{s_5} \quad \overline{s_6} \quad \overline{s_7} \quad \overline{s_8}$

$F_4\quad \overline{s_1} \quad \overline{s_2} \quad 4 \quad \overline{s_3} \quad \overline{s_4}$

$H_3\quad \overline{s_1} \quad \overline{s_2} \quad \overline{s_3}$

$I_2(m) \ (m \geq 5, \ m \text{ even})\quad \overline{s_1} \quad \overline{m} \quad \overline{s_2}$

$I_2(m) \ (m \geq 5, \ m \text{ odd})\quad \overline{s_1} \quad \overline{s_2}$
The unit is unique for each \( i \in V(G) \) and \( x^{-1} \) is unique for each \( x \), similarly to the case of groups. Note that each \( G_{i,j}, i \in V(G) \) is a group, called a vertex group of \( G \). On the other hand, for a generalization of semigroups, any graph satisfying all axioms above except (G4) is called a category. (In the context of the usual category theory, a 'category' defined above is indeed a small category, with objects \( V(G) \) and morphisms \( G_{i,j} \). Further, a groupoid is now a small category such that all morphisms are invertible.)

**Example 2.2.** Now we define the fundamental groupoid \( \overline{P} = \overline{P}(G) \) of any undirected graph \( G \), which is one of the important examples of groupoids.

Let \( P_{i,j} = P_{i,j}(G) \) be the set of all directed paths of \( G \) from a vertex \( i \) to \( j \). Then the family \( \{P_{i,j}\}_{i,j \in V(G)} \), denoted by \( P = P(G) \), forms a category with concatenation as composition, where the units are trivial paths (paths of length 0) at each vertex.

For each \( e \in E(G) \) with certain direction, let \( e^{-1} \) denote the same edge but has the opposite direction. Further, for any path \( p = e_{1}e_{2}\cdots e_{n} \in P_{i,j} \), define \( p^{-1} = e_{n}^{-1}\cdots e_{2}^{-1}e_{1}^{-1} \in P_{j,i} \). Now let \( \sim \) denote the equivalence relation on \( P_{i,j} \) generated by the relation

\[
e_{1}\cdots e_{k-1}e_{k}^{-1}e_{k+1}\cdots e_{n} \sim e_{1}\cdots e_{k-1}e_{k+1}\cdots e_{n},
\]

the homotopy equivalence of paths. Then the multiplication of \( P \) induces a partial multiplication \([p][q]\) of homotopy classes, such that \([p][q]\) is defined if and only if \( pq \) is defined. Let \( \overline{P}_{i,j} = \overline{P}_{i,j}(G) \) denote the set of homotopy classes of all \( p \in P_{i,j} \). Then the family \( \overline{P} = \{\overline{P}_{i,j}\}_{i,j \in V(G)} \) forms a groupoid with above multiplication, as required. \( \square \)

A subgroupoid \( H \) of a groupoid \( G \) is defined similarly to the groups, with the additional condition \( V(H) \subset V(G) \). \( H \) is called full if \( H_{i,j} = G_{i,j} \) for all \( i,j \in V(H) \), and called wide if \( V(H) = V(G) \). (Similar notions are also defined for categories and graphs.) Further, \( H \) is called normal if \( H \) is wide and \( gxg^{-1} \in H_{j,j} \) for all \( x \in H_{i,i}, g \in G_{j,j} \).

**Example 2.3.** For a groupoid \( G \) with wide subgroupoid \( H \), let \( \sim_{H} \) be the equivalence relation on \( V(G) \) such that \( i \sim_{H} j \) if and only if \( H_{i,j} \neq \emptyset \). Then any full subgroupoid of \( G \) on an equivalence class with respect to \( \sim_{G} \) is called a connected component of \( G \), and \( G \) is called connected if \( G \) consists of only one connected component. \( \square \)

Let \( G \) be a groupoid. Then the intersection \( \bigcap_{\lambda} H_{\lambda} \) of subgroupoids \( H_{\lambda} \) of \( G \) is defined naturally, with \( V(\bigcap_{\lambda} H_{\lambda}) = \bigcap_{\lambda} V(H_{\lambda}) \), and forms a subgroupoid of \( G \). Note that it becomes normal in \( G \) whenever all \( H_{\lambda} \) are. Further, for a subgraph \( X \) of \( G \), the (normal) subgroupoid of \( G \) generated by \( X \) is the intersection of all (normal) subgroupoids of \( G \) containing \( X \), or equivalently the smallest (normal) subgroupoid of \( G \) containing \( X \).
Let $X, X'$ be graphs. A graph homomorphism $f : X \to X'$ sends each $i \in V(X)$ to $f(i) \in V(X')$ and each $x \in X_{i,j}$ to $f(x) \in X'_{f(i),f(j)}$. A graph anti-homomorphism is defined similarly but $f(x) \in X'_{f(j),f(i)}$ instead of $X'_{f(i),f(j)}$. Let $f : G \to G'$ be a graph homomorphism between groupoids. Then $f$ is called a groupoid homomorphism if $f(xy) = f(x)f(y)$ for $x, y \in G$ whenever $xy$ is defined and $f(1_i) = 1_{f(i)}$ for $i \in V(G)$, and groupoid anti-homomorphisms are also defined similarly. Then every property about homomorphisms appearing in this subsection can be translated into the case of anti-homomorphisms. Note that $f(x^{-1}) = f(x)^{-1}$ holds for any groupoid homomorphism $f : G \to G'$ and $x \in G$. Further, an isomorphism of two groupoids is defined similarly to the case of groups. Note that any groupoid isomorphism $f : G \to G'$ induces isomorphisms of vertex groups $G_{i,i} \to G'_{f(i),f(i)}$.

The image of a groupoid homomorphism $f : G \to G'$ is the subgraph $f(G)$ of $G'$ on $f(V(G))$ consisting of all $f(x)$, $x \in G$. Note that $f(G)$ is not a subgroupoid of $G'$ in general, but this becomes a subgroupoid whenever $f$ is injective on $V(G)$. On the other hand, the kernel of $f$ is the wide subgraph $\ker f$ of $G$ consisting of all $x \in G$ such that $f(x)$ is a unit of $G'$, and then $\ker f$ always forms a normal subgroupoid of $G$.

For a groupoid $G$ and its normal subgroupoid $N$, the quotient groupoid $G/N$ is defined as follows. Let $V(G/N)$ be the set of all equivalence classes $[i]$ on $V(G)$ with respect to $\sim_N$. For $x, y \in G$, let $\equiv_N$ be an equivalence relation on $G$ such that $x \equiv_N y$ if and only if $x = gh$ for some $g, h \in N$. Let $[x]$ denote the equivalence class of $x \in G$ with respect to $\equiv_N$. Now define

$$(G/N)_{[i],[j]} = \{[x] \in G_{i',j'}' \text{ for some } i', j' \in [i], j' \in [j]\}$$

and $G/N = \{(G/N)_{[i],[j]}\}_{[i],[j]}$. Then the multiplication of $G/N$ is induced naturally and $G/N$ forms a groupoid in fact.

Now an analogy of "The First Isomorphism Theorem" is given as follows:

**Theorem 2.4 (see [2] or [6]).** If a groupoid homomorphism $f : G \to G'$ is injective on $V(G)$, then the induced map $\overline{f} : G/\ker f \to f(G)$ is an isomorphism. □

Let $G$ be a groupoid with subgraph $X$. We say that $G$ is free on $X$ if any graph homomorphism $f : X \to G'$ to a groupoid $G'$ extends uniquely to a groupoid homomorphism $\overline{f} : G \to G'$. Note that the free groupoid on $X$ is unique (up to isomorphism) if it exists. Conversely, the existence is deduced from the following fact:

**Theorem 2.5 (see [6]).** Let $G$ be an undirected graph. Fix an orientation for $G$, so $G$ is considered as a subgraph of its fundamental groupoid $\mathcal{G}$. Then $\overline{\mathcal{G}}$ is free on $\mathcal{G}$. □
3 Decomposition of centralizers

From now on, we assume that $S$ is a finite set. In this section, we state that the centralizer $C_W(W_I)$ of $W_I$ admits a decomposition $W_{[x]}^{iso} \times (W_x^{\perp} \rtimes G_{x,x})$ as described in Introduction, and define a normal subgroup $H_{x,x}$ of $G_{x,x}$.

We start with some notations. For a nonnegative integer $N$, let

$$S^{(N)} = \{ x = (x_1, \ldots, x_N) \in S^N \mid x_i \neq x_j \text{ for all } i \neq j \}$$

and let $[x] = \{x_1, \ldots, x_N\}$ for $x \in S^{(N)}$. Further, for $I \subset S$, let $I^{iso}$ be the set of all isolated points of $\Gamma_I$ and let $I_\perp = \{ \gamma \in \Phi \mid \langle \gamma, \alpha_s \rangle = 0 \text{ for all } s \in I \}$.

**Definition 3.1.** For $x, y \in S^{(N)}$, let

$$C_{x,y} = \{ w \in W \mid wy_iw^{-1} = x_i \text{ for all } 1 \leq i \leq N \}$$

$$= \{ w \in W \mid w \cdot \alpha_{y_i} = \pm \alpha_{x_i} \text{ for all } 1 \leq i \leq N \},$$

$$C'_{x,y} = \{ w \in C_{x,y} \mid w \cdot \alpha_{y_i} = \alpha_{x_i} \text{ for all } y_i \in [y]^{iso} \},$$

$$C''_{x,y} = \{ w \in C_{x,y} \mid w \cdot \alpha_{y_i} = \alpha_{x_i} \text{ for all } 1 \leq i \leq N \}. \square$$

Note that $C''_{x,y} \subset C'_{x,y} \subset C_{x,y}$ by this definition. Further, the following lemma also follows from this definition:

**Lemma 3.2.** $C = \{ C_{x,y} \}_{x,y}$, $C' = \{ C'_{x,y} \}_{x,y}$, $C'' = \{ C''_{x,y} \}_{x,y}$ are groupoids on $S^{(N)}$. \square

Since the centralizer of each $W_I$ occurs as $C_{x,x}$ by taking $x \in S^{(#I)}$ such that $[x] = I$, we examine $C_{x,x}$ hereafter. Now we have the decomposition of $C_{x,x}$ as follows:

**Theorem 3.3.** Let $x \in S^{(N)}$. Then $C_{x,x} = W_{[x]}^{iso} \times C'_{x,x}$, $W_{[x]}^{iso} \simeq (\mathbb{Z}/2\mathbb{Z}) \# [x]^{iso}$. \square

Secondly, we give a certain decomposition of $C'_{x,x}$. Let $W_x^{\perp}$, $x \in S^{(N)}$ be the subgroup of $W$ generated by all reflections $s_\gamma$ such that $\gamma \in [x]_1^{+}$, and let

$$G_{x,y} = \{ w \in C'_{x,y} \mid \Phi^+_w \cap [y]_\perp = \emptyset \}$$

for $x, y \in S^{(N)}$. Then it can be shown that $G_{x,y} = \{ w \in C'_{x,y} \mid w \cdot [y]_\perp^+ = [x]_1^+ \}$, and so $G = \{ G_{x,y} \}_{x,y}$ forms a wide subgroupoid of $C'$. In particular, $G_{x,x}$ is a subgroup of $C'_{x,x}$.

On the other hand, for the subgroup $W_x^{\perp}$, the following lemma holds:

**Lemma 3.4.** For $x \in S^{(N)}$, $W_x^{\perp}$ is a normal subgroup of $C'_{x,x}$, and the set $[x]_\perp$ is $W_x^{\perp}$-invariant. \square
Deodhar [4] and Dyer [5] proved independently that every reflection subgroup (that is, a subgroup generated by reflections) forms a Coxeter system with certain generating set. Now each \( W_x^+ \) is a reflection subgroup, so \( W_x^+ \) also forms a Coxeter group.

Further, they also gave a characterization of the generating set. Now we determine the generating set \( \tilde{S}_x \) of \( W_x^+ \) by using Deodhar's characterization; let \( \tilde{\Pi}_x \) be the set of all \( \gamma \in [x]_1^+ \) such that \( \gamma \) cannot be written as a nonnegative \( \mathbb{R} \)-linear combination of other elements of \( [x]_1^+ \), and let \( \tilde{S}_x = \{ s_\gamma \mid \gamma \in \tilde{\Pi}_x \} \). Then we have the following by [4] since \( [x]_1 \) is \( W_x^+ \)-invariant (cf. Lemma 3.4):

**Theorem 3.5.** \((W_x^+, \tilde{S}_x)\) is a Coxeter system, and its length function \( \tilde{\ell} \) satisfies \( \tilde{\ell}(w) = \#(\Phi^+_w \cap [x]_1) \) for all \( w \in W_x^+ \).

We note that Dyer's characterization gives the same generating set with the set obtained by Deodhar's.

Now the decomposition of \( C'_{x,x} \) is given as follows:

**Theorem 3.6.** \( C'_{x,x} = W_x^+ \times G_{x,x} \) for all \( x \in S^{(N)} \).

Here we consider the structure of \((W_x^+, \tilde{S}_x)\) and the action of \( G_{x,x} \) on \( W_x^+ \) more. Firstly, the following theorem is a special case of Theorem 4.4 of [5]:

**Theorem 3.7.** Let \( \gamma_1, \gamma_2 \in \tilde{\Pi}_x \), \( \gamma_1 \neq \gamma_2 \). Then either \( \langle \gamma_1, \gamma_2 \rangle = -\cos(\pi/m) \) for some \( m \in \mathbb{Z}, m \geq 2 \) or \( \langle \gamma_1, \gamma_2 \rangle \leq -1 \).

Then the structure of \((W_x^+, \tilde{S}_x)\) can be determined whenever \( \tilde{\Pi}_x \) is well understood, by using the following fact:

**Proposition 3.8.** Let \( \gamma_1, \gamma_2 \in \tilde{\Pi}_x \), \( \gamma_1 \neq \gamma_2 \). Then \( s_{\gamma_1} s_{\gamma_2} \) has finite order \( m \) if and only if \( \langle \gamma_1, \gamma_2 \rangle = -\cos(\pi/m) \).

Secondly, we examine the action of \( G_{x,x} \) on \( W_x^+ \). Let \( \tilde{\Gamma} \) denote the Coxeter graph of \((W_x^+, \tilde{S}_x)\). Note that for arbitrary Coxeter system \((W, S)\) (temporarily we do not assume that \( S \) is a finite set) with Coxeter graph \( \Gamma \), each \( \sigma \in \text{Aut} \Gamma \) induces an automorphism \( f_{\sigma} : W \rightarrow W \), and the map \( \text{Aut} \Gamma \rightarrow \text{Aut}W, \sigma \mapsto f_{\sigma} \) is a group homomorphism. Now the following theorem follows from Deodhar's characterization of \( \tilde{S}_x \):

**Theorem 3.9.** There exists a unique group homomorphism \( G_{x,x} \rightarrow \text{Aut} \tilde{\Gamma}, w \mapsto \sigma_w \) such that \( wuw^{-1} = f_{\sigma_w}(u) \) for all \( u \in W_x^+ \).

**Corollary 3.10.** \( C'_{x,x} = W_x^+ \times G_{x,x} \) whenever \( \text{Aut} \tilde{\Gamma} = 1 \).

At last of this section, we define \( H = G \cap C'' \), so \( H \) is a wide subgroupoid of \( G \). Then it follows from the definition that each \( H_{x,x} \) is a normal subgroup of \( G_{x,x} \). The structures of \( G_{x,x} \) and \( H_{x,x} \) are discussed in the following sections.
4 Transition diagram and the groupoid $H$

Before we consider the structure of $G_{x,x}$, we examine the groupoid $H$ in this section.

To do this, we define a graph (which we call the transition diagram) $G = G^{(N)}(W, S)$ of $(W, S)$ for each nonnegative integer $N$; it is an undirected graph on $S^{(N)}$ which have close relation with the action of the longest elements of finite parabolic subgroups. Then we construct below a certain anti-homomorphism $F$ from the fundamental groupoid $\overline{\mathcal{P}} = \overline{\mathcal{P}}(\mathcal{G})$ of $\mathcal{G}$ to $H$ which is surjective. This implies that $H$ is anti-isomorphic to the quotient groupoid $\overline{\mathcal{P}}/\ker F$. Finally, we give a certain generating set of $\ker F$ (as a normal subgroupoid) and a method for obtaining a presentation of $H_{x,x}$.

Now we start to define $\mathcal{G}$. For $I \subseteq S$ and $s \in S$, let $I \sim s$ denote the vertex set of connected component of $\Gamma_{I \cup \{s\}}$ containing $s$, so $I \sim s \subseteq I \cup \{s\}$. For $x \in S^{(N)}$, we write $x_{\sim s}$ as a shorthand for $[x]_{\sim s}$. Further, recall (cf. Section 2.1) that we call $I \subseteq S$ of finite type if $W_I$ is a finite group.

**Definition 4.1.** Let $x \in S^{(N)}$, $s \in S$. Then we say that $s$ reacts on $x$ if $s \not\in [x]$ and $x_{\sim s}$ is of finite type. In this case, the product $\varphi(x, s)$ of this reaction is defined to be the unique element of $S^{(N)}$ such that

$$\varphi(x, s)_i = \begin{cases} 
\sigma_{x_{\sim s} \setminus \{s\}}(x_i) & \text{if } x_i \in x_{\sim s} \\
x_i & \text{otherwise}
\end{cases}$$

and the residue of this reaction is $\psi(x, s) = \sigma_{x_{\sim s}}(s)$ (see Section 2.1 for the definition of $\sigma$). Moreover, we say that this reaction is dynamic if $\varphi(x, s) \neq x$. \hfill \Box

The definition of $\mathcal{G}$ is as follows:

**Definition 4.2.** Let $x, y \in S^{(N)}$ and $s, s' \in S$. Then $\mathcal{G}$ is defined to be a graph on the vertex set $S^{(N)}$ such that, it has an undirected edge $\{(x, s), (y, s')\}$ between $x$ and $y$ if and only if $s$ reacts dynamically on $x$ and its product, residue are $y, s'$ respectively. \hfill \Box

In addition, when we picture the picture of $\mathcal{G}$, this edge $\{(x, s), (y, s')\}$ is represented, for example, as an edge with labels $s$ close to the vertex $x$ and $s'$ close to $y$; moreover, for the case $s = s'$, the repeated $s$'s may be replaced by a single $s$.

Though the definition of edges of $\mathcal{G}$ in Definition 4.2 seems to be asymmetric about $(x, s)$ and $(y, s')$, $\mathcal{G}$ is well-defined, thanks to the following proposition:

**Proposition 4.3.** If $s$ reacts on $x$, then $\psi(x, s)$ also reacts on $\varphi(x, s)$, and the product, residue of the latter reaction are $x, s$ respectively. In particular, the latter reaction is dynamic if and only if the former is dynamic. \hfill \Box
This proposition is deduced from the following characterization of reactions:

**Lemma 4.4.** Let $x, y \in S^{(N)}$ and $s \in S$. Then $s$ reacts on $x$ and the product is $y$ if and only if the following two conditions hold:

(i) $[y] \subset [x] \cup \{s\}$,

(ii) there exists some $w \in W_{[x] \cup \{s\}} \cap C^w_{y,x}$, $w \neq 1$.

Further, $w = w^s_2$ whenever these conditions hold, where $w^s_2 = w_0(x_{\sim s})w_0(x_{\sim s} \setminus \{s\})$. \hfill \Box

Moreover, this lemma implies also the following proposition:

**Proposition 4.5.** If $s$ reacts on $x$, then $w^{\psi(x,s)}_{\varphi(x,s)} = (w^s_2)^{-1}$. \hfill \Box

**Example 4.6.** Let $(W, S)$ be a finite Coxeter system of type $B_5$ with numbering on $S$ in Figure 1, and let $N = 3$, $x = (s_1, s_3, s_4) \in S^{(N)}$. Then we show that the connected component $G_x$ of $G$ containing $x$ is

$$y \overset{s_3}{\rightarrow} \overset{s_2}{\rightarrow} x \overset{s_5}{\rightarrow} x' \overset{s_2}{\rightarrow} s_3 \quad y'$$

where $x' = (s_1, s_4, s_3)$, $y = (s_4, s_1, s_2)$ and $y' = (s_4, s_2, s_1)$.

Firstly, $s_5$ reacts on $x$ and $\varphi(x, s_5) = x'$, $\psi(x, s_5) = s_5$. In fact, $x_{\sim s_5} = \{s_3, s_4, s_5\}$, $x_{\sim s_5} \setminus \{s_5\} = \{s_3, s_4\}$ are of type $B_3$, $A_2$ respectively. Then we have $\varphi(x, s_5) = x'$, $\psi(x, s_5) = s_5$ since the action of the longest element turns the Coxeter graph for the case of type $A_n$, while it fixes the Coxeter graph for $B_n$ (cf. Figure 1).

Further, $s_2$ also reacts dynamically on $x, x'$ and $\varphi(x, s_2) = y$, $\psi(x, s_2) = s_3$, $\varphi(x', s_2) = y'$ and $\psi(x', s_2) = s_3$ by similar argument. Finally, it can be checked that $s_5$ reacts not dynamically on $y, y'$. Hence the connected component becomes as above. \hfill \Box

For each edge $\{(x, s), (y, s')\}$ of $G$, let $e^s_{x,y}$ denote this edge with direction from $x$ to $y$ (note that $y$ and $s'$ are uniquely determined by $x$ and $s$ whenever the edge exists, so this notation is unambiguous), and let $(e^s_{x,y})^{-1}$ denote the same edge but has the opposite direction (namely, from $y$ to $x$). Then $(e^s_{x,y})^{-1} = e^\varphi(x,s)_{\psi(x,s)}$ and every directed path $p$ of $G$ is written as the form $p = e^s_{x_1}e^s_{x_2} \cdots e^s_{x_\ell}$, with $\varphi(x_i, s_i) = x_{i+1}$ for all $1 \leq i \leq \ell - 1$. We write $p^{-1} = (e^s_{x_\ell})^{-1} \cdots (e^s_{x_2})^{-1}(e^s_{x_1})^{-1}$ for such $p$. As in Section 2.2, let $P = P(G)$, $P_{x,y} = P_{x,y}(G)$ denote the set of all directed paths of $G$, all directed paths of $G$ from $x$ to $y$ respectively, and let $[p]$ denote the homotopy class of $p \in P$. Note that $[p^{-1}] = [p]^{-1}$ for any $p \in P$.

Now we define an anti-homomorphism $F : \overline{P} \rightarrow H$ as follows. $F$ is defined to be the identity map on $S^{(N)}$, and to satisfy $F(e^s_{x,y}) = w^s_2$ for each directed edge of $G$ (here we write $F(e^s_{x,y})$ as a shorthand for $F((e^s_{x,y})^{-1})$). Then we have $F((e^s_{x,y})^{-1}) = F(e^s_{x,y})^{-1}$ by Proposition 4.5. Since $\overline{P}$ is a free groupoid on $G$ (cf. Theorem 2.5), this $F$ extends uniquely to an
anti-homomorphism $F : \overline{P} \to H$ (so $F(e_{x_{1}}^{s_{1}} \cdots e_{x_{l}}^{s_{l}}) = w_{x_{l}}^{s_{l}} \cdots w_{x_{1}}^{s_{1}}$), provided the following proposition holds:

**Proposition 4.7.** If $s$ reacts dynamically on $x$, then $w_{x}^{s} \in H_{\varphi(x,s),x}$. □

For each $p \in P$, we also write $F(p)$ as a shorthand for $F([p])$.

The key to the proof of Proposition 4.7 is the following lemma:

**Lemma 4.8.** Suppose that $s$ reacts on $x$. Then $\Phi_{w_{x}^{s}}^{+} \cap [x]_{\perp} = \emptyset$ if and only if this reaction is dynamic. □

Then for each $x$, $s$ such that $s$ reacts dynamically on $x$, we have $w_{x}^{s} \in C''_{\varphi(x,s),x}$ by definition of $w_{x}^{s}$, while $\Phi_{w_{x}^{s}}^{+} \cap [x]_{\perp} = \emptyset$ by this lemma. Hence $w_{x}^{s} \in H_{\varphi(x,s),x}$, so Proposition 4.7 holds.

Now we state a theorem which implies that $F$ is surjective, by using the following notations and terminology:

**Definition 4.9.** For $p = e_{x_{1}}^{s_{1}} \cdots e_{x_{n}}^{s_{n}} \in P$, define

\[ \ell(p) = n, \quad |p| = \sum_{i=1}^{n} \ell(w_{x_{i}}^{s_{i}}), \quad L(p) = \ell(F(p)) = \ell(w_{x_{n}}^{s_{n}} \cdots w_{x_{1}}^{s_{1}}). \]

Further, we say that $p$ is nondegenerate if $L(p) = |p|$ and degenerate if $L(p) < |p|$ (note that $L(p) \leq |p|$ for all $p \in P$). □

The theorem is as follows:

**Theorem 4.10.** For each $w \in H_{y,x}$, there exists a nondegenerate path $p \in P_{x,y}$ such that $F(p) = w$. In particular, $F$ is surjective. Moreover, if $s \in S$ and $w \cdot \alpha_{s} < 0$, then we can choose such $p$ having $e_{x}^{s}$ as the first edge. □

To prove this theorem, we use the following lemma:

**Lemma 4.11.** Let $w \in H_{y,x}$, $s \in S$ and suppose $w \cdot \alpha_{s} < 0$. Then $s$ reacts dynamically on $x$ and $\ell(w) = \ell(w_{x}^{s})^{-1} + \ell(w_{x}^{s})$. □

Then Theorem 4.10 follows from this lemma, by induction on $\ell(w)$.

Thus we conclude the construction of the surjective anti-homomorphism $F$. Now let $F_{x}$ be the restriction of $F$ to the connected component $\overline{P}(G_{x})$ of $\overline{P}$ containing $x \in S^{(N)}$. Then $F_{x}$ is also a surjective anti-homomorphism from $\overline{P}(G_{x})$ to the connected component $H_{x}$ of $H$ containing $x$. Since $F$ is injective on the vertex set of $\overline{P}$, $F$, $F_{x}$ induce an anti-isomorphism $\overline{P}/\ker F \to H, \overline{P}(G_{x})/\ker F_{x} \to H_{x}$ respectively, as we remarked before.
Finally, we give a generating set of $\ker F_x$ as a normal subgroupoid, and a presentation of $H_{x,x}$. For each $J \subset S$, let $\mathcal{G}^{(J)}_x$ denote the "restriction" of $\mathcal{G}_x$ to $J$; that is, the subgraph of $\mathcal{G}_x$ consisting of all vertices $y$ of $\mathcal{G}_x$ such that $[y] \subset J$, and all edges $\{(x,s),(y,s')\}$ of $\mathcal{G}_x$ such that $[x],[y] \subset J$ and $s,s' \in J$. Now define $\mathcal{C}_x$ to be the set of all simple closed paths of $\mathcal{G}^{(J)}_x$, where $J$ runs over all subset of $S$ such that $\#J = N + 2$ and $J$ is of finite type (actually, for each simple closed path $c = e_{x_1}^{s_1} \cdots e_{x_{\ell}}^{s_{\ell}}$ of such $\mathcal{G}^{(J)}_x$, only one of its cyclic permutations $e_{x_1}^{s_1} \cdots e_{x_{\ell}}^{s_{\ell}} e_{x_1}^{s_{\ell}} \cdots e_{x_{\ell-1}}$, $1 \leq i \leq \ell$, or their inverses must be contained in $\mathcal{C}_x$ and the others may be excluded). Then the following theorem holds, but the proof of this is too long and intricate to write, or even to sketch, in this paper:

**Theorem 4.12.** $\ker F_x$ is generated by all $[c]$, $c \in \mathcal{C}_x$ as a normal subgroupoid.

Further, we consider the presentation of $H_{x,x}$. Let $\mathcal{E}_x$ denote the set of all directed edges of $\mathcal{G}_x$. Then every path of $\mathcal{G}_x$ can be regarded as an element of the free group with basis $\mathcal{E}_x$. Now the following theorem is a special case of Theorem 5.17 of [3]:

**Theorem 4.13.** Let $T$ be a maximal tree in $\mathcal{G}_x$. Then the vertex group $(\overline{P}/\ker F)_{x,x}$ is isomorphic to the group presented by $\langle \mathcal{E}_x \mid \mathcal{C}_x \cup \{ee^{-1} \mid e \in \mathcal{E}_x\} \cup \{e \mid e \in T\} \rangle$.

Moreover, the corresponding anti-isomorphism sends each $e_y^s \in \mathcal{E}_x$ to $F(p_y)^{-1}w_y^sF(p_y)$, where $p_z$ denotes the unique reduced path in $T$ from $x$ to $z$.

## 5 Representatives of $G_{x,x}/H_{x,x}$ and their product

In this section, we examine the quotient group $G_{x,x}/H_{x,x}$. We show below that $G_{x,x}/H_{x,x}$ is a finite elementary abelian 2-group (Corollary 5.5), and that we can choose its coset representatives in the form $u_0(I)F(p)$, where $I \subset [x]$ is of finite type and $p$ is a path of $\mathcal{G}_x$ (Theorem 5.7). Moreover, the multiplication in $G_{x,x}$ is described only by the structure of $\mathcal{G}_x$ (Corollary 5.11); for this description, certain automorphisms on $\mathcal{G}_x$ are defined and used.

We start with some notations. For $x \in S^{(N)}$, define

\[ \text{CO}(x) = \{ A \subset \{1,2,\ldots,N\} \mid x_A \text{ is a connected component of } x \}, \]
\[ \text{CO}^{>1}_{<\infty}(x) = \{ A \in \text{CO}(x) \mid x_A \text{ is of finite type, } \#A > 1 \} \]

where $x_A = \{ x_i \mid i \in A \}$. Note that the power set $\mathcal{P}($CO$(x))$ of CO$(x)$ forms a finite elementary abelian 2-group with symmetric difference as multiplication denoted by $\Delta$.

**Example 5.1.** Let $(W,S)$ be a Coxeter system with Coxeter graph below and let $x = (7,1,3,6,4) \in S^{(5)}$. Then we have $\text{CO}(x) = \{\{1,4\}, \{2\}, \{3,5\}\}$, $\text{CO}^{>1}_{<\infty}(x) = \{\{3,5\}\}$. \[\square\]
The following basic lemma is used many times:

**Lemma 5.2.** Let \( x, y \in S^{(N)}, w \in G_{x,y} \).

(i) \( \text{CO}(x) = \text{CO}(y) \) and \( \text{CO}_{>1}^{>1}(x) = \text{CO}_{>1}^{>1}(y) \).

(ii) For \( i, j \in A \subseteq \text{CO}(x) \), \( w \cdot \alpha_{y_i} = -\alpha_{x_i} \) if and only if \( w \cdot \alpha_{y_j} = -\alpha_{x_j} \).

(iii) If \( i \in A \subseteq \text{CO}(x) \setminus \text{CO}_{>1}^{>1}(x) \), then \( w \cdot \alpha_{y_i} = \alpha_{x_i} \).

For \( x, y \in S^{(N)} \) and \( A \subseteq \text{CO}_{>1}^{>1}(y) \), define

\[
G_{x,y}^{A} = \{ w \in G_{x,y} | w \cdot \alpha_{y_i} = -\alpha_{x_i} \text{ if and only if } i \in \bigcup A \}.
\]

Then Lemma 5.2 yields the following decomposition of \( G_{x,y} \):

**Lemma 5.3.** \( G_{x,y} = \bigsqcup_{A \subseteq \text{CO}_{>1}^{>1}(x)} G_{x,y}^{A} \) for all \( x, y \in S^{(N)} \).

Further, the following lemma is deduced immediately from the definition:

**Lemma 5.4.** Let \( x, y, z \in S^{(N)}, A, A' \subseteq \text{CO}_{>1}^{>1}(x) \) and suppose \( G_{x,y} \neq \emptyset, G_{y,z} \neq \emptyset \) (so all \( \text{CO}_{>1}^{>1}(x), \text{CO}_{>1}^{>1}(y), \text{CO}_{>1}^{>1}(z) \) coincide by Lemma 5.2 (i)). Then

\[
G_{x,y}^{A} \cdot G_{y,z}^{A'} \subseteq G_{x,z}^{A \cdot A'}, (G_{x,y}^{A})^{-1} = G_{y,x}^{A}, G_{x,y}^{0} = H_{x,y}.
\]

For \( x \in S^{(N)} \), define \( E_{x} = \{ A \subseteq \text{CO}_{>1}^{>1}(x) | G_{x,x}^{A} \neq \emptyset \} \). Then the preceding lemmas imply the structure of \( G_{x,x}/H_{x,x} \) as follows:

**Corollary 5.5.** Let \( x \in S^{(N)} \). Then \( E_{x} \) is a subgroup of \( \mathcal{P}(\text{CO}_{>1}^{>1}(x)) \) and isomorphic to \( G_{x,x}/H_{x,x} \), so it is also a finite elementary abelian 2-group. Further, this isomorphism sends each coset \( wH_{x,x} \) to the unique \( A \subseteq \text{CO}_{>1}^{>1}(x) \) satisfying \( w \in G_{x,x}^{A} \).

Now we give certain coset representatives of \( G_{x,x}/H_{x,x} \). For \( x \in S^{(N)} \) and \( A \subseteq \text{CO}_{>1}^{>1}(x) \), define

\[
w_{0}(A; x) = \prod_{A \in A} w_{0}(x_{A}) = w_{0}(x_{\bigcup A})
\]

(note that all \( w_{0}(x_{A}) \) in the above product commute). Further, let \( y \in S^{(N)}, y \in G_{x} \). Then we have \( G_{y,x} \neq \emptyset \) since \( G_{x} \) is connected and \( F(\overline{P}_{x,y}) \subseteq G_{y,x} \), and so \( A \subseteq \text{CO}_{>1}^{>1}(y) \) by Lemma 5.2 (i). Now define \( y^{A} \in S^{(N)} \) by

\[
(y^{A})_{i} = w_{0}(A; y)y_{i}w_{0}(A; y) = \begin{cases} \sigma_{yA}(y_{i}) & \text{if } i \in A \text{ for some } A \in A \\ y_{i} & \text{otherwise.} \end{cases}
\]

Then we have the following lemma:
Lemma 5.6. Let \( x \in S^{(N)} \), \( \mathcal{A} \subset \text{CO}_{<\infty}^{>1}(x) \). Then \( w_{0}(\mathcal{A}; x) \in G^{\mathcal{A}}_{x,x} \). Hence \( G^{\mathcal{A}}_{x,x} = w_{0}(\mathcal{A}; x)H_{x,\mathcal{A},x} \) and so \( E_{x} = \{ \mathcal{A} \subset \text{CO}_{<\infty}^{>1}(x) \mid H_{x,\mathcal{A},x} \neq \emptyset \} \).

The coset representatives of \( G_{x,x}/H_{x,x} \) are given as follows (recall that the map \( F \) is surjective):

Theorem 5.7. \( G_{x,x} = \bigcup_{\mathcal{A} \in E_{x}} w_{0}(\mathcal{A}; x)F(p_{\mathcal{A}})H_{x,x} \), where \( p_{\mathcal{A}} \) is an arbitrarily chosen element of \( P_{x,x}^{\mathcal{A}} \) for each \( \mathcal{A} \in E_{x} \).

Since \( H_{x,x} \) is generated by certain elements \( F(p) \), \( p \in P_{x,x} \) (cf. Section 4), this theorem implies that \( G_{x,x} \) is generated by such \( F(p) \) and these coset representatives. In the rest of this section, we describe the multiplication of these generators of \( G_{x,x} \), using certain automorphisms on \( \mathcal{G}_{x} \) defined in Theorem 5.10 below.

We use the following two lemmas:

Lemma 5.8. Let \( x \in S^{(N)} \), \( \mathcal{A}, \mathcal{A}' \subset \text{CO}_{<\infty}^{>1}(x) \) (so \( \text{CO}_{<\infty}^{>1}(x^{\mathcal{A}}) = \text{CO}_{<\infty}^{>1}(x) \) by Lemmas 5.2 (i) and 5.6). Then
\[
w_{0}(\mathcal{A};x)w_{0}(\mathcal{A}';x) = w_{0}(\mathcal{A} \cdot \mathcal{A}';x), w_{0}(\mathcal{A};x)w_{0}(\mathcal{A}'^{x^{\mathcal{A}}}) = w_{0}(\mathcal{A}'^{x^{\mathcal{A}}}); (x^{\mathcal{A}})^{\mathcal{A}'} = x^{\mathcal{A} \cdot \mathcal{A}'}.
\]

Lemma 5.9. Let \( x \in S^{(N)} \), \( y, z \in \mathcal{G}_{x} \), \( \mathcal{A} \subset \text{CO}_{<\infty}^{>1}(x) \) and \( s \in S \). Then \( s \) reacts dynamically on \( y \) and \( \varphi(y, s) = z \) if and only if \( s \) reacts dynamically on \( y^{\mathcal{A}} \) and \( \varphi(y^{\mathcal{A}}, s) = z^{\mathcal{A}} \). Moreover, if above conditions hold, then \( \psi(y, s) \) and \( \psi(y^{\mathcal{A}}, s) \) coincide, and \( w_{y^{\mathcal{A}}} = w_{0}(\mathcal{A}; z)w_{y}^{s}w_{0}(\mathcal{A}; y) \).

The automorphisms on \( \mathcal{G}_{x} \) are given as follows:

Theorem 5.10. For each \( \mathcal{A} \in E_{x} \), define \( \rho_{\mathcal{A}} : \mathcal{G}_{x} \to \mathcal{G}_{x} \) by
\[
\rho_{\mathcal{A}}(y) = y^{\mathcal{A}} (y \in V(\mathcal{G}_{x})), \rho_{\mathcal{A}}(e_{y}^{s}) = e_{y^{\mathcal{A}}}^{s} (e_{y}^{s} \in E(\mathcal{G}_{x})).
\]

(i) \( \rho_{\mathcal{A}} \) is an involutive graph automorphism on \( \mathcal{G}_{x} \).
(ii) \( \rho_{\mathcal{A}}\rho_{\mathcal{A}'} = \rho_{\mathcal{A} \cdot \mathcal{A}'} \) holds for all \( \mathcal{A}, \mathcal{A}' \in E_{x} \).
(iii) If \( \rho_{\mathcal{A}} \) also denotes the extension of \( \rho_{\mathcal{A}} \) to \( \mathcal{P}(\mathcal{G}_{x}) \) (the directed paths of \( \mathcal{G}_{x} \)), then it is an involutive automorphism and satisfies \( \rho_{\mathcal{A}}(p^{-1}) = \rho_{\mathcal{A}}(p)^{-1}, \rho_{\mathcal{A}}\rho_{\mathcal{A}'} = \rho_{\mathcal{A} \cdot \mathcal{A}'} \) and
\[
F(\rho_{\mathcal{A}}(p)) = w_{0}(\mathcal{A}; z)F(p)w_{0}(\mathcal{A}; y)
\]
for all \( p \in \mathcal{P}(\mathcal{G}_{x})_{y,z} \).

Now the multiplication (in \( G_{x,x} \)) of the representatives of \( G_{x,x}/H_{x,x} \) and the action of these to the generators \( F(p) \) of \( H_{x,x} \) are described as follows, by using the automorphisms
Corollary 5.11. (i) Let $A \in E_x$, $p_A \in P_{x,x^A}$ and $q \in P_{x,x}$. Then
\[
w_0(A;x)F(p_A)F(q)(w_0(A;x)F(p_A))^{-1} = F(\rho_A((p_A)^{-1}qp_A)).
\]
(ii) Let $A, A' \in E_x$, $p_A \in P_{x,x^A}$ and $p_{A'} \in P_{x,x^{A'}}$. Then
\[
w_0(A;x)F(p_A)w_0(A';x)F(p_{A'}) = w_0(A \cdot A';x)F(p_{A'}\rho_{A'}(p_A)). \quad \square
\]

Note that to obtain $\rho_A(p)$ for each $A \in E_x$ and $p = e_{x_1}^{s_1} \cdots e_{x_n}^{s_n} \in P(G_x)$, we need not compute $(x_i)^A$ for any $i \geq 2$; indeed, we have only to compute $(x_1)^A$, and then start at $(x_1)^A$ and trace each (unique) directed edge labeled $s_i$ step by step.

6 Examples

Example 6.1. $(W, S)$ is of type $\widetilde{B}_7$ and $x = (1, 2, 4, 5, 8) \in S^{(N)}$ ($N = 5$), as in Figure 2 (in this section we write $i$ as a shorthand for $s_i$). Then we compute the centralizer $C_{x,x}$ of $W_{\{1,2,4,5,8\}}$.

\begin{itemize}
  \item 1. Figure 2 implies $[x]^\text{iso} = \{1, 2, 8\}$, so by Theorem 3.3,
    \[C_{x,x} = W_{\{1,2,8\}} \times C'_{x,x} \simeq (\mathbb{Z}/2\mathbb{Z})^3 \times C'_{x,x}.\]
  \item 2. We determine the structure of the Coxeter system $(W_x^\perp, \overline{S}_x)$. Let
    \[\delta = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \sqrt{2}\alpha_8,
    \]
    which is called the null root of $(W, S)$. So $\langle \delta, \alpha_i \rangle = 0$ for all $1 \leq i \leq 8$. Now $\Phi$ is the (disjoint) union of following two sets
    \[\Phi' = \{n\delta \pm \gamma \mid n \in \mathbb{Z}, \gamma \in \Phi^+_{S\setminus\{8\}}\},\]
    \[\Phi'' = \{n\sqrt{2}\delta \pm \left(\sum_{i=k}^{7}\sqrt{2}\alpha_i + \alpha_8\right) \mid n \in \mathbb{Z}, 2 \leq k \leq 8\}.
    \]
\end{itemize}
Moreover, $\Phi' = W \cdot \alpha_i$ for each $1 \leq i \leq 7$ and $\Phi'' = W \cdot \alpha_8$. To show these, we have only to check that every element of $\Phi', \Phi''$ is indeed a root of $(W, S)$ (this can be proved by the induction on $|n|$), both $\Phi', \Phi''$ are $W$-invariant (this follows from that $W \cdot \Phi_S^{\perp, \{8\}} \subset \Phi'$ and $W \cdot \left(\sum_{i=k}^{7} \sqrt{2} \alpha_i + \alpha_8\right) \subset \Phi''$ for all $2 \leq k \leq 8$), and $\Pi_{S \setminus \{8\}} \subset \Phi'$, $\alpha_8 \in \Phi''$ (these are trivial).

By the above result, we have $[x]_\perp = \{n \sqrt{2} \delta \pm \beta \mid n \in \mathbb{Z}\}$ and so $\Pi_{x} = \{\beta, \beta'\}$, where $\beta = \sqrt{2} \alpha_8 + \alpha_8$, $\beta' = \sqrt{2} \delta - \beta$. Further, since $\langle \beta, \beta' \rangle = -1$, Proposition 3.8 implies that $s_{\beta} s_{\beta'}$ has infinite order. Hence $(W_x^\perp, \Sigma_x)$ is of type $A_1$ (the infinite dihedral group), and by Theorem 3.6, we have $C'_{x,x} = W_x^{\perp} \rtimes G_{x,x} \simeq A_1 \rtimes G_{x,x}$.

3. The connected component $\mathcal{G}_x$ of $\mathcal{G}$ containing $x$ is as in Figure 3. In this case, let $e(y, z)$ denote the unique directed edge of $\mathcal{G}_x$ from $y$ to $z$. Now we determine the structure of the groupoid $H$, by using Theorems 4.12 and 4.13. Firstly, we examine the generating set of $\ker F_x$. Since $N + 2 = 7 = \#S - 1$, we have only to consider $\mathcal{G}_x^{(J)}$ for $J = S \setminus \{s\}$, $s \in S$. For example, if $s = 4$, then we obtain $\mathcal{G}_x^{(J)}$ from $\mathcal{G}_x$ by deleting four vertices II, III, VI, VII and six edges $e(I, II)$, $e(II, III)$, $e(III, IV)$, $e(V, VI)$, $e(VI, VII)$, $e(VII, VIII)$. By similar argument, $\mathcal{G}_x^{(J)}$ is nonempty for $s = 3, 4, 6, 7$, as in Figure 4, while this is empty for $s = 1, 2, 5, 8$. Now by Theorem 4.12, $\ker F_x$ is generated (as a normal subgroupoid) by $[c_1]$ and $[c_2]$, where

$$c_1 = e(I, VIII)e(VIII, IV)e(IV, V)e(V, I),$$

$$c_2 = e(I, II)e(II, III)e(III, IV)e(IV, VIII)e(VIII, VII)e(VII, VI)e(VI, V)e(V, I)$$

(note that in this case, every proper subset of $S$ is of finite type).

Secondly, we give a presentation of $H_{x,x}$ by Theorem 4.13. Recall that $\mathcal{E}_x$ denotes the set of all directed edges of $\mathcal{G}_x$. Now we choose a maximal tree $T$ in $\mathcal{G}_x$ as in Figure 5, then
we have:

\[ H_{x,x} \cong \langle \mathcal{E}_x \mid c_1 = 1, c_2 = 1, ee^{-1} = 1 \ (e \in \mathcal{E}_x), e = 1 \ (e \in T) \rangle \]
\[ \cong \langle e(I, VIII), e(IV, V), e(VIII, VII) \mid e(I, VIII)e(IV, V) = 1, e(VIII, VII) = 1 \rangle \]
\[ \cong \langle e(IV, V) \mid \rangle \cong \mathbb{Z}. \]

The corresponding anti-isomorphism sends \( e(IV, V) \) to \( F(q) \in H_{x,x} \), where

\[ q = e(II, III)e(III, IV)e(IV, V)e(V, I)e(I, II) = e_{III}^3 e_{IV}^6 e_{V}^7 e_{I}^3 e_{IV}^4, \]

so \( H_{x,x} \) is the free group generated by \( F(q) \).
4. We describe the structure of $G_{x,x}$ as in Section 5. Firstly, it follows from Figure 2 that $CO(x) = \{1, 2, 3, 4, 5\}$, $CO_{<\alpha}(x) = \{3, 4\}$. Let $A_0 = \{3, 4\}$, then $x^{A_0} = (1, 2, 5, 4, 8) = \text{VII}$ and so $E_x = \emptyset, A_0$. Let $p_0 \in P_{x,x}$ be the trivial path and let
\[
p_{A_0} = e(II, I)e(I, V)e(V, VI)e(VI, VII) = e^6_1 e^3_2 e^3_1 \in P_{x,x^{A_0}}.
\]
Then Theorem 5.7 implies $G_{x,x} = H_{x,x} \cup aH_{x,x}$, where $a = w_0(\{4, 5\}) F(p_{A_0})$. Hence $G_{x,x}$ is generated by $a$ and $F(q)$.

As remarked in the last of Section 5, $\rho_{A_0}(p_{A_0})$ is the path which starts at $x^{A_0} = \text{VII}$ and traces the directed edges labeled as 6, 3, 4, 3 one by one; that is,
\[
\rho_{A_0}(p_{A_0}) = e(\text{VII, VIII})e(\text{VIII, IV})e(\text{IV, III})e(\text{III, II}).
\]
We write $p \sim_F p'$ for two paths $p, p'$ if $F(p) = F(p')$. Then we have
\[
p_{A_0} \rho_{A_0}(p_{A_0}) = e(II, I)e(I, V)e(V, VI)e(VI, VII)e(VII, VIII)e(VIII, IV)e(IV, III)e(III, II)
\]
\[
\sim e(II, I)c_2^{-1}e(II, I)^{-1} \sim_F 1
\]
since $F(c_2) = 1$. So we have $a^2 = F(p_{A_0} \rho_{A_0}(p_{A_0})) = 1$ by Corollary 5.11 (ii). Similarly, we have $\rho_{A_0}(q) = e(\text{VII, VI})e(\text{VI, V})e(\text{V, IV})e(\text{IV, VIII})e(\text{VIII, VII})$ and so
\[
\rho_{A_0}((p_{A_0})^{-1}q p_{A_0}) = e(II, III)e(\text{III, IV})e(\text{IV, VIII})e(\text{VIII, VII})
\]
\[
\cdot e(\text{VII, VI})e(\text{VI, V})e(\text{V, IV})e(\text{IV, VIII})e(\text{VIII, VII})
\]
\[
\cdot e(\text{VII, VIII})e(\text{VIII, IV})e(\text{IV, III})e(\text{III, II})
\]
\[
\sim (e(II, III)e(\text{III, IV})e(\text{IV, VIII})e(\text{VIII, VII})e(\text{VII, VI})e(\text{VI, V}))e(\text{V, IV})e(\text{IV, III})e(\text{III, II})
\]
\[
\sim_F (e(II, I)e(I, V)e(\text{V, IV})e(\text{IV, III})e(\text{III, II}) \quad (F(c_2) = 1)
\]
\[
= q^{-1}.
\]
Hence we have $aF(q)a^{-1} = F(\rho_{A_0}(p_{A_0}^{-1}q p_{A_0})) = F(q)^{-1}$ by Corollary 5.11 (i).

By these calculation, $\{1, a\}$ forms a subgroup of $G_{x,x}$ isomorphic to $E_x$, and we have
\[
G_{x,x} \simeq H_{x,x} \rtimes E_x \simeq \mathbb{Z} \rtimes (\mathbb{Z}/2\mathbb{Z}),
\]
where $1 \in \mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Z}$ as multiplication by $-1$. Moreover, put $a' = a$ and $b' = aF(q)$, then we have $G_{x,x} = \langle a', b' \mid a'^2 = 1, b'^2 = 1 \rangle \simeq \overline{A}_1$. 

5. Finally, we describe the action of $G_{x,x}$ on $W_{x}^{\perp}$. By direct computation, we have

$$a' \cdot \beta = \beta', \ b' \cdot \beta = \beta.$$

Now recall (Theorem 3.9) that both $a'$, $b'$ act on $W_{x}^{\perp}$ as automorphisms of the Coxeter graph $\tilde{\Gamma}$ of $(W_{x}^{\perp}, \tilde{S}_{x})$; so we have $a' \cdot \beta' = \beta$, $b' \cdot \beta' = \beta'$.

Summarizing, we have $C_{x,x} \simeq (\mathbb{Z}/2\mathbb{Z})^{3} \times (\tilde{A}_{1} \rtimes \tilde{A}_{1})$, where each of two generators of the right $\tilde{A}_{1}$ acts on the left $\tilde{A}_{1}$ trivially, as the unique involution of $\tilde{\Gamma}$ respectively.

In this example, $G_{x,x}$ is isomorphic to the semidirect product of $H_{x,x}$ by $E_{x}$, and $H_{x,x}$ forms a free group. But these properties may fail in general.

Let $(W, S)$ be as in Figure 6 and let $x = (1, 2, 4, 5, 7, 8)$. Then it can be proved that

![Coxeter graph of another example](image)

Figure 6: Coxeter graph of another example

$W_{x}^{\perp} = 1$, $W_{x}^{\perp} = 1$, $G_{x,x} \simeq \mathbb{Z}^{2}$, $H_{x,x} \simeq (2\mathbb{Z})^{2}$.

Thus $H_{x,x}$ is not a free group, and $G_{x,x}$ is not isomorphic to a semidirect product of $H_{x,x}$ by any group, since $G_{x,x}$ has no subgroup isomorphic to $G_{x,x} / H_{x,x} \simeq (\mathbb{Z}/2\mathbb{Z})^{2}$.

Finally, we consider the centralizers of maximal parabolic subgroups (that is, parabolic subgroups generated by maximal proper subsets of $S$). Note that for $I \subset S$, the centralizer $C_{W}(W_{I})$ of $W_{I}$ is the direct product of $C_{W_{S_{i}}}(W_{I \cap S_{i}})$, where $S_{i}$ runs over all connected components of $S$. Thus we assume that $S$ is (finite and) connected.

Let $I$ be a maximal proper subset of $S$, with connected components $I_{1}, \ldots, I_{k}$. For $J \subset S$ such that $J$ is of finite type and $\sigma_{J} = \text{id}_{J}$, let $-1_{J} = w_{0}(J)$. Then it is obvious that each $-1_{I_{j}}, -1_{S}$ is contained in $C_{W}(W_{I})$ whenever it exists. Conversely, it can be deduced, by using the result of this paper, that $C_{W}(W_{I})$ is generated by these elements for almost all (possibly infinite) $W$ and $I$, except only two cases.

One of the exceptions is the case $W = D_{2n+1}$, $n \geq 2$ and $I = S \setminus \{s_{2i}\}, 1 \leq i \leq n - 1$ (we use the numbering on $S$ in Figure 1); in this case, $C_{W}(W_{I})$ is generated by the involution $w_{0}(S)w_{0}(I')$, where $I' = \{s_{2i+1}, s_{2i+2}, \ldots, s_{2n+1}\}$ is a connected component of $I$. The other is the case $W = E_{6}$ and $I = S \setminus \{s_{2}\}$; now $C_{W}(W_{I})$ is generated by the involution $w_{0}(S)w_{0}(I)$.
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References


