数理解析研究所講究録 1310 巻 2003 年 169-177

ON THE AUTOMORPHISM GROUP OF THE SUBGROUP LATTICE OF A FINITE ABELIAN p-GROUP; SOME GENERALIZATIONS

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ABSTRACT. The automorphism group $\operatorname{Aut}\mathcal{L}(M)$ of the submodule lattice $\mathcal{L}(M)$ of a finite-length module M over complete discrete valuation ring $\mathfrak{o}$ is studied. Let $\lambda=(\lambda_1,\cdots,\lambda_l)$ be the type of M. We show that for those M with $\lambda_1\geq\lambda_2\geq\lambda_3\geq 1$, $\operatorname{Aut}\mathcal{L}(M)$ can be analyzed by computing a certain subgroup of the bijections on a quotient of the scalar ring $\mathfrak{o}$. In particular, when the residue field $k=\mathfrak{o}/\mathfrak{p}$ is a finite fieled $\mathbb{F}_q$, we compute the order of the group.

1. OBJECTIVE

Let $\mathfrak{o}$ be a discrete valuation ring with the maximal ideal $\mathfrak{p}$, a prime element π (i.e., $\mathfrak{o}\pi=\mathfrak{p}$) and the valuation function $v:\mathfrak{o}\setminus\{0\}\longrightarrow\mathbb{Z}_{\geq 0}$. Let $k\cong\mathfrak{o}/\mathfrak{p}$ denote the residue field. Let M be an $\mathfrak{o}$-module of finite length. Then, since $\mathfrak{o}$ is a principal ideal domain, M can be written as a sum of cyclic $\mathfrak{o}$-submodules:

$$M\cong\mathfrak{o}/\mathfrak{p}^{\lambda_1}\oplus\cdots\oplus\mathfrak{o}/\mathfrak{p}^{\lambda_l},$$

with $\lambda=(\lambda_1,\cdots,\lambda_l)$ being some partition of a non-negative integer (That is, we have $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_l$). λ is called the *type* of M. Now since we have $\mathfrak{o}/\mathfrak{p}^i\cong\bar{\mathfrak{o}}/\bar{\mathfrak{p}}^i$ where $\bar{\mathfrak{o}}$ is the completion of $\mathfrak{o}$ and $\bar{\mathfrak{p}}$ its maximal ideal, without loss of generality we can assume $\mathfrak{o}$ to be complete. Let $\mathcal{L}(M)$ denote the set of $\mathfrak{o}$-submodules of M. $\mathcal{L}(M)$ inherits a lattice structure by inclusion relation. Our main objective is to compute $\operatorname{Aut}\mathcal{L}(M)$, the automorphism group of the lattice $\mathcal{L}(M)$, for such λ as $\lambda_1\geq\lambda_2\geq\lambda_3\geq 1$.

When $\mathfrak{o}=\mathbb{Z}_p$, the ring of p-adic integers, M becomes nothing but a finite abelian p-group and $\mathcal{L}(M)$ the subgroup lattice of M. This can be generalized by considering the case $\mathfrak{o}=W[\mathbb{F}_q]$, the ring of Witt vectors over the finite field $\mathbb{F}_q$, for $W[\mathbb{F}_p]\cong\mathbb{Z}_p$. Another example of $\mathfrak{o}$ is the ring $k[[t]]$ of formal power series in one variable t.

We call $e=(e_1,\cdots,e_l)\in M^l$ an *ordered basis* for M if $M=\bigoplus_{i=1}^l\mathfrak{o}e_i$ and $\mathfrak{o}e_i\cong\mathfrak{o}/\mathfrak{p}^{\lambda_i}$. Let e be fixed. We denote by $R(e)$ the set of $\varphi\in\operatorname{Aut}\mathcal{L}(M)$ satisfying

$$\begin{cases}\varphi(\mathfrak{o}e_i)=\mathfrak{o}e_i & {}^\forall i\in[1,l]\\ \varphi(\mathfrak{o}(e_1+e_i))=\mathfrak{o}(e_1+e_i) & {}^\forall i\in[2,l]\end{cases}$$

In most cases it boils down to computing $R(e)$ in order to analyze $\operatorname{Aut}\mathcal{L}(M)$, in the sense we describe as follows.

Since an autormophism of $\mathfrak{o}$-module M induces an automorphism of the lattice $\mathcal{L}(M)$, we have the natural group homomorphism

$$\Theta:\operatorname{Aut}M\longrightarrow\operatorname{Aut}\mathcal{L}(M).$$

It can be directly checked that $\operatorname{Ker}\Theta\cong(\dot{\mathfrak{o}}/\mathfrak{p}^{\lambda_1})^\times$ and that $\operatorname{Aut}M$ can be expressed in matrix form, as described in the sequel. Naturally $\operatorname{Aut}\mathcal{L}(M)$ contains a subgroup isomorphic to $\operatorname{Aut}M/\operatorname{Ker}\Theta$, and we let $\operatorname{PAut}M$ denote this subgroup.

It turns out that $\operatorname{Aut}\mathcal{L}(M)$ is a product of these two subgroups $R(e)$ and $\operatorname{PAut}M$. Namely, we have

Lemma 1.

$$R(e)\cdot\operatorname{PAut}M=\operatorname{Aut}\mathcal{L}(M)$$
$$R(e)\cap\operatorname{PAut}M=1.$$

Also, we remark that if e and e' are ordered base for M, then it is easily checked that $\varphi R(e)\varphi^{-1} = R(e')$, where $\varphi \in \mathrm{PAut}\, M$ is the lattice automorphism induced by the module automorphism of M defined by $e_i \mapsto e'_i$ $(1 \leq i \leq l)$. Hence the isomorphism type of $R(e)$ does not depend on the choice of e. We content ourselves with computing $R(e)$ instead of computing $\mathrm{Aut}\, \mathcal{L}(M)$ for our purpose.

2. Historical Background

Let us mention the relation with earlier results. The structure of $\mathrm{Aut}\, \mathcal{L}(M)$ is well-known for the case $\lambda_1 = \lambda_2 = \lambda_3$, which is essentially the result of Baer [2]. In this case, we have $\mathrm{Aut}\, \mathcal{L}(M) \cong R(e) \ltimes \mathrm{PAut}\, M$, and

$$R(e) \cong \mathrm{Aut}\, \mathfrak{o}/\mathfrak{p}^{\lambda_3},$$

where $\mathrm{Aut}\, \mathfrak{o}/\mathfrak{p}^{\lambda_3}$ is the group of automorphisms of ring $\mathfrak{o}/\mathfrak{p}^{\lambda_3}$. In particular, when $\lambda_1 = \cdots = \lambda_l = 1$ $(l \geq 3)$, M becomes a vector space over the residue filed k of $\mathfrak{o}$, and $\mathrm{Aut}\, \mathcal{L}(M)$ is isomorphic to $P\Gamma L(l, k)$, the group of projective semi-linear automorphisms. This result is a variation of so called *the Fundamental Theorem of Finite Projective Geometry.*

We next consider the case when the residue field of $\mathfrak{o}$ is the finite field $\mathbb{F}_p$. Let $M = \mathfrak{o}/\mathfrak{p} \oplus \mathfrak{o}/\mathfrak{p} \cong \mathbb{F}_p \oplus \mathbb{F}_p$. Then $\mathrm{Aut}\, \mathcal{L}(M)$ is isomorphic to the symmetric group $\mathfrak{S}_{p+1}$ and $\mathrm{PAut}\, M$ isomorphic to the projective general linear group $PGL(2,p)$ (Note that $|PGL(2,p)| = (p+1)p(p-1)$). In this case, $R(e)$ is a subgroup that fixes three points and isomorphic to $\mathfrak{S}_{p-2}$. More generally, for $M = \mathbb{Z}_p/p^{\lambda_2}\mathbb{Z}_p \oplus \mathbb{Z}_p/p^{\lambda_2}\mathbb{Z}_p$ ($\mathfrak{o} = \mathbb{Z}_p$ is the ring of p-adic integers), Holmes' result [5] states that $\mathrm{Aut}\, \mathcal{L}(M)$ is isormorphic to $\mathfrak{S}_p^{\wr(\lambda_2 - 1)} \wr \mathfrak{S}_{p+1}$, where $\mathfrak{S}_p^{\wr n}$ means $\mathfrak{S}_p \wr \cdots \wr \mathfrak{S}_p$ (n times) and $\wr$ denotes the standard wreath product. In this case, $\mathrm{PAut}\, M$ is nothing but $PGL_2(\mathbb{Z}_p/p^{\lambda_2}\mathbb{Z}_p)$, and we note that $|PGL_2(\mathbb{Z}_p/p^{\lambda_2}\mathbb{Z}_p)| = (p+1)p(p-1) \cdot (p^{\lambda_2 - 1})^3$. $R(e)$ is the subgroup that fixes three points $\mathbb{Z}_p(1,0)$, $\mathbb{Z}_p(0,1)$ and $\mathbb{Z}_p(1,1)$; in fact, we have

$$R(e) \cong (\mathfrak{S}_p^{\wr(\lambda_2 - 1)} \wr \mathfrak{S}_{p-2}) \times \left\{ \prod_{i=0}^{\lambda_2 - 2} (\mathfrak{S}_p^{\wr i} \wr \mathfrak{S}_{p-1}) \right\}^3 .$$

Holmes [5] also obtains a result for the case $\lambda_1 > \lambda_2 > \lambda_3 = 0$: $\mathrm{Aut}\, \mathcal{L}(M) \cong G^2 \times H^{\lambda_1 - \lambda_2 - 1}$, where $G = \mathfrak{S}_p^{\wr \lambda_2}$ and $H = \mathfrak{S}_p^{\wr(\lambda_2 - 1)} \wr \mathfrak{S}_{p-1}$.

There have been works to bridge the gap between Baer's result and Holmes'. Costantini-Holmes-Zacher[3] and Costantini-Zacher[4] treated the case of abelian p-groups in a rather general framework. Yasuda[11] studied the case of finite abelian p-groups for $\lambda_1 > \lambda_2 = \lambda_3$ with explicit computation of $R(e)$ and $\mathrm{Aut}\, \mathcal{L}(M)$. In this work, we shall treat the case $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 1$, in the general setting of finite-length modules over (complete) discrete valuation ring.

3. Notations and Notions

Here we give some supplementary definitions and notations. Put $\mathfrak{q}_i = \mathfrak{p}^i \setminus \mathfrak{p}^{i+1}$ for $i \geq 1$; i.e., $\mathfrak{q}_i = \{a \in \mathfrak{o} \mid v(a) = i\}$. We define $\mathfrak{q}_0 = \mathfrak{o} \setminus \mathfrak{p} = \mathfrak{o}^\times$, the set of invertible elements. For $a, b \in \mathfrak{o}$ such that $v(a) \geq v(b)$ $(b \neq 0)$, there exists an element $x \in \mathfrak{o}$ such that $a = xb$. As $\mathfrak{o}$ is a domain, x must be unique. We use the notation $\frac{a}{b} = x$.

Given a set X, $\mathrm{Map}(X)$ denotes the set of maps $f : X \longrightarrow X$. $\mathrm{Sym}(X)$ denotes the set of bijections $f : X \longrightarrow X$. $\mathrm{Map}(X)$ forms a monoid with respect to function composition, whereas $\mathrm{Sym}(X)$ forms a group. Given two sets X and Y, we define Y^X to be the set of maps $f : X \longrightarrow Y$.

Let G be a group, and H a group acting on a set X. Let $f, g : X \longrightarrow G$ be two maps, and define a map $f \circ g : X \longrightarrow G$ by $f \circ g(x) = f(x) \cdot g(x)$ where $\cdot$ is the product in G. Then G^X becomes a group with respect to this $\circ$. Let $h \in H$ and $f \in G^X$. We define a semidirect product $G^X \rtimes H$ with respect to the group homomorphism $H \longrightarrow \mathrm{Aut}\, G^X$ $(h \mapsto (f \mapsto fh^{-1}))$. We write $G \wr H$ to denote this semidirect product, and call it the *wreath product* of G and H.

We now give the description of the automorphism group $\mathrm{Aut}\, M$ of an $\mathfrak{o}$-module M in matrix form, as promised. Let e be fixed. The action of $f \in \mathrm{Aut}\, M$ is then determined by its action on $e = (e_1, \cdots, e_l)$. Write

$$f(e_j) = \sum_{i=1}^{l} a_{ij} e_i$$

and express f as the matrix $(a_{ij})_{i,j=1}^{l}$. Rewriting $\lambda = (\lambda_1, \dots, \lambda_l) = \langle {d_1}^{m_1}, \dots, {d_r}^{m_r} \rangle$ $(d_1 > \dots > d_r)$ (this means that λ contains m_r-many components equal to d_r), $\operatorname{Aut} M$ can be expressed in matrix form as

$$\begin{pmatrix} GL_{m_1}(\mathfrak{o}/\mathfrak{p}^{d_1}) & \cdots & \operatorname{Hom}((\mathfrak{o}/\mathfrak{p}^{d_r})^{\oplus m_r}, (\mathfrak{o}/\mathfrak{p}^{d_1})^{\oplus m_1}) \\ \vdots & \ddots & \vdots \\ \operatorname{Hom}((\mathfrak{o}/\mathfrak{p}^{d_1})^{\oplus m_1}, (\mathfrak{o}/\mathfrak{p}^{d_r})^{\oplus m_r}) & \cdots & GL_{m_r}(\mathfrak{o}/\mathfrak{p}^{d_r}) \end{pmatrix},$$

with respect to the ordered basis e. Here, the block matrix in the diagonal

$$A \in GL_{m_i}(\mathfrak{o}/\mathfrak{p}^{d_i})$$

is of size $m_i \times m_i$ and has elements of $\mathfrak{o}/\mathfrak{p}^{d_i}$ in its components, satisfying $\pi \nmid \det A$. Also, the block matrix at (i,j)-position $(i \neq j)$

$$A \in \operatorname{Hom}((\mathfrak{o}/\mathfrak{p}^{d_j})^{\oplus m_j}, (\mathfrak{o}/\mathfrak{p}^{d_i})^{\oplus m_i})$$

is of size $m_i \times m_j$ and in its components has elements of $\mathfrak{p}^{d_i - \min(d_j, d_i)}(\mathfrak{o}/\mathfrak{p}^{d_i})$, that is, for $i < j$ $(\Longrightarrow d_i > d_j)$ elements of $\mathfrak{p}^{d_i - d_j}(\mathfrak{o}/\mathfrak{p}^{d_i})$, and for $i > j$ $(\Longrightarrow d_i < d_j)$ elements of $\mathfrak{o}/\mathfrak{p}^{d_i}$.

4. Main Resutls

For the case $\lambda_1 > \lambda_2 = \lambda_3$, we can state our main result as follows:

Theorem 2. *Assume $\lambda_2 = \lambda_3$. Then $R(e)$ contains a normal subgroup N such that*

$$R(e)/N \cong \operatorname{Aut} \mathfrak{o}/\mathfrak{p}^{\lambda_3},$$

$$N \cong \begin{cases} k^{\lambda_1 - \lambda_2} & \lambda_2 = \lambda_3 > 2, \\ (k^{\times})^{\lambda_1 - \lambda_2} & \lambda_2 = \lambda_3 = 1. \end{cases}$$

The case $\lambda_1 \geq \lambda_2 > \lambda_3$ turns out to be rather complicated. The rest of this section is dedicated to explain our main result for this case.

Let $i \geq 1$. For $a, b \in \mathfrak{o}$, we write

$$a \equiv b \bmod \mathfrak{p}^i$$

to mean $a - b \in \mathfrak{p}^i$. With abuse of notation, we write $\mathfrak{p}^i$ also to denote this equivalence relation. Then obviously we have $\mathfrak{p} \succ \mathfrak{p}^2 \succ \mathfrak{p}^3 \succ \cdots$. On the other hand, put $\mathfrak{u}_i = 1 + \mathfrak{p}^i \subset \mathfrak{o}$ $(i \geq 1)$. For $a, b \in \mathfrak{o}$, write

$$a \sim b \bmod \mathfrak{u}_i$$

if $a \in \mathfrak{u}_i b$. Clearly this defines an equivalence relation on $\mathfrak{o}$. Again with abuse of notation, we just write $\mathfrak{u}_i$ to denote this relation. Then note that we have $\mathfrak{u}_1 \succ \mathfrak{u}_2 \succ \mathfrak{u}_3 \succ \cdots$. Also note that $\mathfrak{p}^i \succ \mathfrak{u}_i$ holds for all $i \geq 1$.

Lemma 3. *The union of relations $\mathfrak{p}^i \cup \mathfrak{u}_j$ is an equivalence relation for all $i, j \geq 1$.*

Hence we have $\mathfrak{p}^i \vee \mathfrak{u}_j = \mathfrak{p}^i \cup \mathfrak{u}_j$, and it makes sense to denote the quotient set by $\mathfrak{o}/\mathfrak{p}^i/\mathfrak{u}_j = \mathfrak{o}/\mathfrak{u}_j/\mathfrak{p}^i = \mathfrak{o}/\mathfrak{p}^i \vee \mathfrak{u}_j$ for all $i, j \geq 1$.

Now we proceed to the following lemma:

Lemma 4. *Let $\varphi \in R(e)$ be given. There exist bijective maps $\tau : \mathfrak{o} \longrightarrow \mathfrak{o}$ and $\sigma : \mathfrak{o} \longrightarrow \mathfrak{o}$ such that $\varphi\mathfrak{o}(ae_1 + e_2) = \mathfrak{o}(\tau(a)e_1 + e_2)$ and $\varphi\mathfrak{o}(e_1 + ae_2) = \mathfrak{o}(e_1 + \sigma(a)e_2)$ for all $a \in \mathfrak{o}$. τ and σ induce bijections $\tau : \mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2} \longrightarrow \mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}$ and $\sigma : \mathfrak{o}/\mathfrak{p}^{\lambda_2} \longrightarrow \mathfrak{o}/\mathfrak{p}^{\lambda_2}$, respectively, which are uniquely determined by φ.*

Let $\varphi \in R(e)$ be given and τ, σ as in the preceding lemma. We list in the following lemma some of the properties satisfied by τ and σ.

Lemma 5. *We have*

- **(1):** $\tau(1) \sim 1 \bmod \mathfrak{p}^{\lambda_1} \vee \mathfrak{u}_{\lambda_2}$, $\sigma(1) \equiv 1 \bmod \mathfrak{p}^{\lambda_2}$,
- **(2):** $\tau(\mathfrak{p}) \subset \mathfrak{p}$, $\sigma(\mathfrak{p}) \subset \mathfrak{p}$,
- **(3):** $\tau(ab) \sim \tau(a)\tau(b) \bmod \mathfrak{p}^{\lambda_1} \vee \mathfrak{u}_{\lambda_3}$ *for all* $a, b \in \mathfrak{o}$,
- **(4):** $\sigma(ab) \sim \sigma(a)\sigma(b) \bmod \mathfrak{p}^{\lambda_2} \vee \mathfrak{u}_{\lambda_3}$ *for all* $a, b \in \mathfrak{o}$,
- **(5):** $\tau(a - b) \sim \tau(a) - \tau(b) \bmod \mathfrak{p}^{\lambda_1} \vee \mathfrak{p}^{\lambda_2 + v(b)} \vee \mathfrak{u}_{\lambda_3}$ *for all* $a, b \in \mathfrak{o}$,

(6): $\sigma(a-b) \sim \sigma(a) - \sigma(b) \bmod \mathfrak{p}^{\lambda_2} \vee \mathfrak{u}_{\lambda_3}$ *for all* $a, b \in \mathfrak{o}$,
(7): $\tau(a) \equiv \sigma(a^{-1})^{-1} \bmod \mathfrak{p}^{\lambda_2}$ *for all* $a \in \mathfrak{o}^{\times}$,
(8): $\tau(a) \sim \sigma(a) \bmod \mathfrak{p}^{\lambda_2} \vee \mathfrak{u}_{\lambda_3}$ *for all* $a \in \mathfrak{o}$.

Given three positive integers $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 1$, let

$$\mathrm{Aut}_{\lambda_1,\lambda_2,\lambda_3}(\mathfrak{o})$$

denote the set of bijections $\tau : \mathfrak{o} \longrightarrow \mathfrak{o}$ that satisfy the following three conditions:

Valuation law: $\tau(\mathfrak{p}) \subset \mathfrak{p}$,
Strict product law: $\tau(ab) \sim \tau(a)\tau(b) \bmod \mathfrak{p}^{\lambda_1} \vee \mathfrak{u}_{\lambda_3}$ for all $a, b \in \mathfrak{o}$,
Difference law: $\tau(a-b) \sim \tau(a) - \tau(b) \bmod \mathfrak{p}^{\lambda_1} \vee \mathfrak{p}^{\lambda_2+v(b)} \vee \mathfrak{u}_{\lambda_3}$ for all $a, b \in \mathfrak{o}$.

In this section we shall prove that this set forms a group and a $\tau \in \mathrm{Aut}_{\lambda_1,\lambda_2,\lambda_3}(\mathfrak{o})$ induces a bijection $\tau : \mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2} \longrightarrow \mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}$. It turns out that $R(e)$ can be described by using this group.

Lemma 6. *Let $\tau : \mathfrak{o} \longrightarrow \mathfrak{o}$ be a bijective map that satisfies the valuation law and the strict product law. Then we have*

$$\tau(\mathfrak{p}^i) = \mathfrak{p}^i$$

for all $i \in [0, \lambda_1]$; that is, we have

$$v(\tau(a)) = v(a)$$

for all $a \in \mathfrak{o} \setminus \mathfrak{p}^{\lambda_1}$.

Lemma 7. *Let $i \leq \lambda_1$ and $j \leq \lambda_2$. Let $\tau : \mathfrak{o} \longrightarrow \mathfrak{o}$ be a bijective map that satisfies the difference law and the condition $v(\tau(a)) = v(a)$ for all $a \in \mathfrak{o} \setminus \mathfrak{p}^{\lambda_1}$. For $a, b \in \mathfrak{o}$, we have $a \sim b \bmod \mathfrak{p}^i \vee \mathfrak{u}_j$ if and only if $\tau(a) \sim \tau(b) \bmod \mathfrak{p}^i \vee \mathfrak{u}_j$. That is, there exists a unique bijective map $\overline{\tau}$ that makes the diagram below commutative:*

$$\begin{array}{ccc} \mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2} & \xrightarrow{\ \tau\ } & \mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2} \\ \downarrow & & \downarrow \\ \mathfrak{o}/\mathfrak{p}^{i}/\mathfrak{u}_{j} & \xrightarrow[\overline{\tau}]{} & \mathfrak{o}/\mathfrak{p}^{i}/\mathfrak{u}_{j} \end{array}$$

Proposition 8. $\mathrm{Aut}_{\lambda_1,\lambda_2,\lambda_3}(\mathfrak{o})$ *forms a subgroup of* $\mathrm{Sym}(\mathfrak{o})$.

Now denote by $\mathrm{Aut}_{\lambda_1,\lambda_2,\lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}}$ the stabilizer of $\mathfrak{u}_{\lambda_2}$ in $\mathrm{Aut}_{\lambda_1,\lambda_2,\lambda_3}$. That is,

$$\mathrm{Aut}_{\lambda_1,\lambda_2,\lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}} = \{\tau \in \mathrm{Aut}_{\lambda_1,\lambda_2,\lambda_3}(\mathfrak{o}) \mid \tau(\mathfrak{u}_{\lambda_2}) = \mathfrak{u}_{\lambda_2}\}.$$

Then define

$$\Delta_{\lambda_3}^{-1}\left(\mathrm{Aut}_{\lambda_1,\lambda_2,\lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}} \times \mathrm{Aut}_{\lambda_2,\lambda_2,\lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}}\right)$$

to be the set of $(\tau, \sigma) \in \mathrm{Aut}_{\lambda_1,\lambda_2,\lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}} \times \mathrm{Aut}_{\lambda_2,\lambda_2,\lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}}$ satisfying the conditions

$$\begin{cases} \tau(a)^{-1} \equiv \sigma(a^{-1}) \bmod \mathfrak{p}^{\lambda_2} & {}^{\forall}a \in \mathfrak{o}^{\times}, \\ \tau(a) \sim \sigma(a) \bmod \mathfrak{p}^{\lambda_2} \vee \mathfrak{u}_{\lambda_3} & {}^{\forall}a \in \mathfrak{o}. \end{cases}$$

Note that since $\tau(a)\tau(a^{-1}) \sim 1 \bmod \mathfrak{p}^{\lambda_1} \vee \mathfrak{u}_{\lambda_3}$ whence $\tau(a)^{-1} \sim \tau(a^{-1}) \bmod \mathfrak{p}^{\lambda_1} \vee \mathfrak{u}_{\lambda_3}$ for all $a \in \mathfrak{o}^{\times}$, the first condition $\tau(a)^{-1} \equiv \sigma(a^{-1}) \bmod \mathfrak{p}^{\lambda_2}$ implies the second conditon $\tau(a) \sim \sigma(a) \bmod \mathfrak{p}^{\lambda_1} \vee \mathfrak{u}_{\lambda_3}$.

Lemma 9. *The set*

$$\Delta_{\lambda_3}^{-1}\left(\mathrm{Aut}_{\lambda_1,\lambda_2,\lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}} \times \mathrm{Aut}_{\lambda_2,\lambda_2,\lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}}\right)$$

forms a subgroup of $\mathrm{Aut}_{\lambda_1,\lambda_2,\lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}} \times \mathrm{Aut}_{\lambda_2,\lambda_2,\lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}}$.

We have observed that $\tau \in \mathrm{Aut}_{\lambda_1,\lambda_2,\lambda_3}(\mathfrak{o})$ induces a bijective map $\tau : \mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2} \longrightarrow \mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}$. That is to say, there exists a natural group homomorphism $\mathrm{Aut}_{\lambda_1,\lambda_2,\lambda_3}(\mathfrak{o}) \longrightarrow \mathrm{Sym}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)$. Let us define

$$\mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)$$
$$\mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_2}\right)$$

to be the images of $\mathrm{Aut}_{\lambda_1,\lambda_2,\lambda_3}(\mathfrak{o}) \longrightarrow \mathrm{Sym}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)$ and $\mathrm{Aut}_{\lambda_2,\lambda_2,\lambda_3}(\mathfrak{o}) \longrightarrow \mathrm{Sym}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_2}\right)$, respectively. Furthermore, let

$$\mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)_1$$
$$\mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_2}\right)_1$$

be the subgroups of the above two, corresponding to $\mathrm{Aut}_{\lambda_1,\lambda_2,\lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}}$ and $\mathrm{Aut}_{\lambda_2,\lambda_2,\lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}}$, respectively. Lastly, we denote by

$$\Delta_{\lambda_3}^{-1}\left(\mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)_1 \times \mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_2}\right)_1\right)$$

the subgroup of $\mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}\mathfrak{u}_{\lambda_2}\right)_1 \times \mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_2}\right)_1$ that corresponds to $\Delta_{\lambda_3}^{-1}\left(\mathrm{Aut}_{\lambda_1,\lambda_2,\lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}} \times \mathrm{Aut}_{\lambda_2,\lambda_2,\lambda_3}(\mathfrak{o})_{\mathfrak{u}_{\lambda_2}}\right)$. Now we can state our:

Theorem 10 (Main Isomorphism Theorem). *We have*

$$R(e) \cong \Delta_{\lambda_3}^{-1}\left(\mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)_1 \times \mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_2}\right)_1\right)$$

if $\lambda_2 > \lambda_3$.

Note that one way of isomorphism

$$\Phi : R(e) \longrightarrow \Delta_{\lambda_3}^{-1}\left(\mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)_1 \times \mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_2}\right)_1\right)$$

is already given, by sending $\Phi : \varphi \mapsto (\tau, \sigma)$. In order to compute $R(e)$, this theorem allows us to compute $\Delta_{\lambda_3}^{-1}\left(\mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)_1 \times \mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_2}\right)_1\right)$ instead. Let

$$\Lambda : \Delta_{\lambda_3}^{-1}(\mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)_1 \times \mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_2}\right)_1) \longrightarrow \mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)_1$$

be the "projection" map to the first component; i.e., $\Lambda : (\tau, \sigma) \mapsto \tau$. Then $\mathrm{Ker}\,\Lambda$ is the set of $(1, \sigma)$ satisfying

$$\begin{cases} \sigma(a) \equiv a \bmod \mathfrak{p}^{\lambda_2} & a \in \mathfrak{o}^{\times}/\mathfrak{p}^{\lambda_2}, \\ \sigma(a) \sim a \bmod \mathfrak{p}^{\lambda_2} \vee \mathfrak{u}_{\lambda_3} & a \in \mathfrak{p}/\mathfrak{p}^{\lambda_2}. \end{cases}$$

Let K be the kernel of the natural map $\mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_2}\right)_1 \longrightarrow \mathfrak{o}^{\times}/\mathfrak{p}^{\lambda_2}$, that is,

$$K = \{\sigma \in \mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_2}\right)_1 \mid \sigma(a) = a \text{ for all } a \in \mathfrak{o}^{\times}/\mathfrak{p}^{\lambda_2}\}.$$

Lemma 11. *We have*

$$\mathrm{Ker}\,\Lambda \cong K.$$

We shall show that the group in question, $\Delta_{\lambda_3}^{-1}(\mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)_1 \times \mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_2}\right)_1)$, is isomorphic to a semidirect product of K and the first component $\mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)_1$:

Proposition 12. *The sequence*

$$1 \longrightarrow K \longrightarrow \Delta_{\lambda_3}^{-1}(\mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)_1 \times \mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_2}\right)_1) \longrightarrow \mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)_1 \longrightarrow 1$$

is exact and splitting. In other words, we have

$$\Delta_{\lambda_3}^{-1}(\mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)_1 \times \mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_2}\right)_1) \cong \mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)_1 \ltimes K.$$

This result divides our investigation into two parts: the analysis of the structure of K and that of $\mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)_1$. We begin with the former.

Recall that

$$\begin{aligned} K &= \left\{\sigma \in \mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_2}\right)_1 \mid \sigma(a) = a \ {}^{\forall}a \in \left(\mathfrak{o}^{\times}/\mathfrak{p}^{\lambda_2}\right)\right\} \\ &= \left\{\sigma \in \mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_2}\right)_1 \mid \sigma(a) = a \ {}^{\forall}a \in \left(\mathfrak{o}^{\times}/\mathfrak{p}^{\lambda_2}\right) \text{ and } \sigma(a) \sim a \bmod \mathfrak{p}^{\lambda_2} \vee \mathfrak{u}_{\lambda_3} \ {}^{\forall}a \in \mathfrak{p}/\mathfrak{p}^{\lambda_2}\right\}. \end{aligned}$$

For the sake of convenience, we shall analyze groups slightly larger than K; namely,

$$\tilde{K} = \left\{\sigma \in \mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_2}\right) \mid \sigma(a) \sim a \bmod \mathfrak{p}^{\lambda_2} \vee \mathfrak{u}_{\lambda_3} \ {}^{\forall}a \in \mathfrak{o}/\mathfrak{p}^{\lambda_2}\right\},$$

and

$$\tilde{K}_1 = \left\{\sigma \in \mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_2}\right)_1 \mid \sigma(a) \sim a \bmod \mathfrak{p}^{\lambda_2} \vee \mathfrak{u}_{\lambda_3} \ {}^{\forall}a \in \mathfrak{o}/\mathfrak{p}^{\lambda_2}\right\}.$$

Of course we have $\tilde{K}_1 = \{\sigma \in \tilde{K} \mid \sigma(1) = 1\}$.

Proposition 13. *$\tilde{K}$ decomposes into a direct product as*

$$\tilde{K} \cong Q_0 \times Q_1 \times \cdots \times Q_{\lambda_2-\lambda_3-1}$$

where each factor Q_i (defined for $0 \le i \le \lambda_2 - 1$) is given by

$$Q_i = \left\{ \sigma \in \tilde{K} \mid \sigma(a) = a \text{ for all } a \in (\mathfrak{o} \setminus \mathfrak{q}_i)/\mathfrak{p}^{\lambda_2} \right\}.$$

Corollary 14. *$\tilde{K}_1$ and K decompose into direct products as*

$$\tilde{K}_1 \cong \overline{Q}_0 \times Q_1 \times \cdots \times Q_{\lambda_2-\lambda_3-1} \text{ and}$$
$$K \cong Q_1 \times Q_2 \times \cdots \times Q_{\lambda_2-\lambda_3-1},$$

respectively, where

$$\overline{Q}_0 = \left\{ \sigma \in \tilde{K}_1 \mid \sigma(a) = a \ {}^{\forall} a \in \mathfrak{p}/\mathfrak{p}^{\lambda_2} \right\} = \{\sigma \in Q_0 \mid \sigma(1) = 1\}.$$

We now focus on the calculation of each Q_i. First, we give a description of generators of Q_i. We begin with the following lemma.

Lemma 15. *Let $\sigma \in \tilde{K}$ and $0 \le j \le \lambda_2 - \lambda_3$. If $a \equiv b \bmod \mathfrak{p}^j$, then $\sigma(a) - a \equiv \sigma(b) - b \bmod \mathfrak{p}^{j+\lambda_3}$.*

We apply this lemma particularly to Q_i ($0 \le i \le \lambda_2 - \lambda_3 - 1$). For $j \in [i+1, \lambda_2 - \lambda_3]$, let $\left(\mathfrak{p}^{j+\lambda_3-1}/\mathfrak{p}^{\lambda_2}\right)^{\mathfrak{q}_i/\mathfrak{p}^j}$ denote the set of maps $z : \mathfrak{q}_i/\mathfrak{p}^j \longrightarrow \mathfrak{p}^{j+\lambda_3-1}/\mathfrak{p}^{\lambda_2}$. Note that since $\mathfrak{p}^{j+\lambda_3-1}/\mathfrak{p}^{\lambda_2}$ is an abelian group, so becomes $\left(\mathfrak{p}^{j+\lambda_3-1}/\mathfrak{p}^{\lambda_2}\right)^{\mathfrak{q}_i/\mathfrak{p}^j}$ naturally. Given $j \in [i+1, \lambda_2 - \lambda_3]$ and $z : \mathfrak{q}_i/\mathfrak{p}^j \longrightarrow \mathfrak{p}^{j+\lambda_3-1}/\mathfrak{p}^{\lambda_2}$, define a map $g_{j,z} : \mathfrak{o}/\mathfrak{p}^{\lambda_2} \longrightarrow \mathfrak{o}/\mathfrak{p}^{\lambda_2}$ by

$$g_{j,z}(a) = \begin{cases} a + z(a \bmod \mathfrak{p}^j) & a \in \mathfrak{q}_i, \\ a & \text{otherwise.} \end{cases}$$

We shall show that $g_{j,z}$ is in Q_i; more precisely,

Proposition 16. *We have*

$$Q_i = \langle g_{j,z} \rangle_{\substack{j \in [i+1, \lambda_2-\lambda_3] \\ z \in \left(\mathfrak{p}^{j+\lambda_3-1}/\mathfrak{p}^{\lambda_2}\right)^{\mathfrak{q}_i/\mathfrak{p}^j}}} = \langle g_{j,z} \rangle_{\substack{j \in [i+1, \lambda_2-\lambda_3] \\ z \in \left(S\pi^{j+\lambda_3-1}\right)^{\mathfrak{q}_i/\mathfrak{p}^j}}},$$

where $(S\pi^{j+\lambda_3-1})^{\mathfrak{q}_i/\mathfrak{p}^j}$ denotes the set of maps $z : \mathfrak{q}_i/\mathfrak{p}^j \longrightarrow S\pi^{j+\lambda_3-1} \subset \mathfrak{p}^{j+\lambda_3-1}/\mathfrak{p}^{\lambda_2}$.

Using these generators, we give two sorts of descriptions of Q_i. The former turns out to be useful particularly for the case $\lambda_3 = 1$, whereas the latter being useful for the case $\lambda_3 \ge \frac{1}{2}\lambda_2$.

For $j \in [i+1, \lambda_2 - \lambda_3 + 1]$, define

$$L_i^{(j)} = \left\{ \sigma \in Q_i \mid \sigma(a) \equiv a \bmod \mathfrak{p}^{j+\lambda_3-1} \ {}^{\forall} a \in \mathfrak{o}/\mathfrak{p}^{\lambda_2} \right\}.$$

Then clearly we have $Q_i \triangleright L_i^{(j)}$ for each j whence obtain a chain of normal subgroups $L_i^{(j)}$.

Proposition 17. *We have a normal series of Q_i:*

$$Q_i = L_i^{(i+1)} \triangleright L_i^{(i+2)} \triangleright \cdots \triangleright L_i^{(\lambda_2-\lambda_3+1)} = 1,$$

where the factors of the series are given by

$$L_i^{(j)}/L_i^{(j+1)} \cong (\mathfrak{o}/\mathfrak{p})^{\mathfrak{q}_i/\mathfrak{p}^j},$$

for all $j \in [i+1, \lambda_2 - \lambda_3]$.

Proposition 18. *We have*

$$L_i^{(j)} = \langle g_{n,z} \rangle_{\substack{n \in [j, \lambda_2-\lambda_3] \\ z \in \left(\mathfrak{p}^{n+\lambda_3-1}/\mathfrak{p}^{\lambda_2}\right)^{\mathfrak{q}_i/\mathfrak{p}^n}}} = \langle g_{n,z} \rangle_{\substack{n \in [j, \lambda_2-\lambda_3] \\ z \in \left(S\pi^{n+\lambda_3-1}\right)^{\mathfrak{q}_i/\mathfrak{p}^n}}}.$$

Lemma 19. *We have the exact sequence of groups*

$$1 \longrightarrow L_i^{(\lambda_2-\lambda_3)} \longrightarrow Q_i \longrightarrow Q_{i+1} \longrightarrow 1.$$

This sequence splits if $\lambda_3 = 1$.

Proposition 20. *If $\lambda_3 = 1$, then Q_i decomposes into a wreath product as*

$$Q_i \simeq \underbrace{k \wr k \wr \cdots \wr k}_{\lambda_2 - i - 1} \wr 1$$

where 1 *is to act on $k^\times$ trivially and k on k by addition. To put it more concisely,*

$$Q_i \simeq (k^{\wr(\lambda_2 - i - 1)})^{k^\times}.$$

We present another way of describing the structure of Q_i. In order to do this, let us define

$$U_i^{(j)} = \left\{\sigma \in Q_i \mid \sigma(a) - a \equiv \sigma(b) - b \bmod \mathfrak{p}^{\lambda_2} \text{ if } a \equiv b \bmod \mathfrak{p}^j\right\},$$

where $i \leq j \leq \lambda_2 - \lambda_3$. This gives us a filtration of Q_i:

$$Q_i = U_i^{(\lambda_2 - \lambda_3)} \supset U_i^{(\lambda_2 - \lambda_3 - 1)} \supset \cdots \supset U_i^{(i+1)} \supset U_i^{(i)} = T_i.$$

Here, T_i is the group of translations, i.e.,

$$T_i = \left\{\sigma \in Q_i \mid \sigma(a) - a = \sigma(b) - b \text{ for all } a, b \in \mathfrak{o}/\mathfrak{p}^{\lambda_2}\right\}.$$

Lemma 21. *We have*

$$U_i^{(j)} = \langle g_{n,z} \rangle_{\substack{n \in [i+1, j] \\ z \in \left(\mathfrak{p}^{n + \lambda_3 - 1}/\mathfrak{p}^{\lambda_2}\right)^{\mathfrak{q}_i/\mathfrak{p}^n}}}.$$

Now for each $j \in [i+1, \lambda_2 - \lambda_3]$ define

$$H_i^{(j)} = \{\sigma \in U_i^{(j)} \mid \sigma(a) = a \text{ if } a_{j-1} = 0 \in S\},$$

with $a \in \mathfrak{o}/\mathfrak{p}^{\lambda_2}$ being written as $a = \sum_{n=1}^{\lambda_2 - 1} a_n \pi^n$ with $a_n \in S$. Obviously the definition of $H_i^{(j)}$ depends on the choice of S.

Proposition 22. *The subgroups $H_i^{(j)}$ are abelian; more precisely, we have*

$$\begin{aligned} H_i^{(j)} &\cong \left(\mathfrak{p}^{j + \lambda_3 - 1}/\mathfrak{p}^{\lambda_2}\right)^{(\mathfrak{q}_i/\mathfrak{p}^{j-1}) \times k^\times} \\ &\cong \left(\mathfrak{o}/\mathfrak{p}^{\lambda_2 - \lambda_3 - j + 1}\right)^{(\mathfrak{o}/\mathfrak{p}^{j-i-1})^\times \times k^\times}. \end{aligned}$$

Proposition 23. *Q_i decomposes into a product of abelian subgroups $H_i^{(j)} \subset Q_i$ $(i+1 \leq j \leq \lambda_2 - \lambda_3)$ as*

$$Q_i = H_i^{(i+1)} H_i^{(i+2)} \cdots H_i^{(\lambda_2 - \lambda_3)},$$

with the properties

$$\begin{cases} \left(H_i^{(i+1)} H_i^{(i+2)} \cdots H_i^{(j)}\right) \cap H_i^{(j+1)} = 1, \\ \left(H_i^{(i+1)} H_i^{(i+2)} \cdots H_i^{(j)}\right) H_i^{(j+1)} = H_i^{(j+1)} \left(H_i^{(i+1)} H_i^{(i+2)} \cdots H_i^{(j)}\right). \end{cases}$$

Lemma 24. *If $i + \lambda_3 \geq \lambda_2 - \lambda_3$, then we have*

$$Q_i \cong \bigoplus_{j=i}^{\lambda_2 - \lambda_3 - 1} \left(\mathfrak{o}/\mathfrak{p}^{\lambda_2 - \lambda_3 - j}\right)^{(\mathfrak{o}/\mathfrak{p}^{j-i})^\times \times k^\times}.$$

Proposition 25. *If $\lambda_3 \geq \frac{1}{2}(\lambda_2 - 1)$, then K is abelian and*

$$K \cong \bigoplus_{i=1}^{\lambda_2 - \lambda_3 - 1} \bigoplus_{j=i}^{\lambda_2 - \lambda_3 - 1} \left(\mathfrak{o}/\mathfrak{p}^{\lambda_2 - \lambda_3 - j}\right)^{(\mathfrak{o}/\mathfrak{p}^{j-i})^\times \times k^\times}.$$

Also, if $\lambda_3 \geq \frac{1}{2}\lambda_2$, then Q_0 is abelian and

$$Q_0 \cong \bigoplus_{j=0}^{\lambda_2 - \lambda_3 - 1} \left(\mathfrak{o}/\mathfrak{p}^{\lambda_2 - \lambda_3 - j}\right)^{(\mathfrak{o}/\mathfrak{p}^{j})^\times \times k^\times}.$$

Now we describe the structure of $\overline{Q}_0$, which we shall need later in computing $\mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)_1$. So for each $j \in [1, \lambda_2 - \lambda_3]$, put

$$\overline{L}_0^{(j)} = \left\{\sigma \in \overline{Q}_0 \mid \sigma(a) \equiv a \bmod \mathfrak{p}^{j+\lambda_3-1}\ \forall a \in \mathfrak{o}/\mathfrak{p}^{\lambda_2}\right\}.$$

Evidently we have $\overline{Q}_0 \rhd \overline{L}_0^{(j)}$ for each j.

Proposition 26. *We have a normal series of $\overline{Q}_0$:*

$$\overline{Q}_0 = \overline{L}_0^{(1)} \rhd \overline{L}_0^{(2)} \rhd \cdots \rhd \overline{L}_0^{(\lambda_2-\lambda_3+1)} = 1,$$

where the factors of the series are given by

$$\overline{L}_0^{(j)}/\overline{L}_0^{(j+1)} \cong (\mathfrak{o}/\mathfrak{p})^{(\mathfrak{o}^\times \setminus \{1\})/\mathfrak{p}^j}$$

for each $j \in [1, \lambda_2 - \lambda_3]$.

Proposition 27. *Assume $\lambda_3 = 1$. Then $\overline{Q}_0$ decomposes into a semidirect product as*

$$\overline{Q}_0 \simeq k^{(\mathfrak{o}^\times \setminus \{1\})/\mathfrak{p}^{\lambda_2-1}} \rtimes k^{(\mathfrak{o}^\times \setminus \{1\})/\mathfrak{p}^{\lambda_2-2}} \rtimes \cdots \rtimes k^{(\mathfrak{o}^\times \setminus \{1\})/\mathfrak{p}}.$$

Proposition 28. *$\overline{Q}_0$ decomposes into a product of abelian subgroups as*

$$\overline{Q}_0 \cong \overline{H}_0^{(1)} H_0^{(2)} \cdots H_0^{(\lambda_2-\lambda_3)},$$

where

$$\begin{aligned}\overline{H}_0^{(1)} &= \{\sigma \in H_0^{(1)} \mid \sigma(1) = 1\} \\ &\cong \left(\mathfrak{o}/\mathfrak{p}^{\lambda_2-\lambda_3}\right)^{k^\times \setminus \{1\}}.\end{aligned}$$

Proposition 29. *If $\lambda_3 \geq \frac{1}{2}\lambda_2$, then we have*

$$\overline{Q}_0 \cong \left(\mathfrak{o}/\mathfrak{p}^{\lambda_2-\lambda_3}\right)^{k^\times \setminus \{1\}} \oplus \bigoplus_{j=1}^{\lambda_2-\lambda_3-1} \left(\mathfrak{o}/\mathfrak{p}^{\lambda_2-\lambda_3-j}\right)^{(\mathfrak{o}/\mathfrak{p}^j)^\times \times k^\times}.$$

Now we shift our attention to calculating the structure of $\mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)_1$. Let N and $\overline{N}$ be the kernels of the natural homomorphisms

$$\mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)_1 \longrightarrow \mathrm{Aut}\,\mathfrak{o}/\mathfrak{p}^{\lambda_3}$$

and

$$\mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)_1 \longrightarrow \mathrm{Aut}(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_3}),$$

respectively. That is,

$$\begin{aligned} N &= \left\{\tau \in \mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)_1 \mid \tau(a) \equiv a \bmod \mathfrak{p}^{\lambda_3}\ \forall a \in \mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right\}, \text{ and} \\ \overline{N} &= \left\{\tau \in \mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)_1 \mid \tau(a) \sim a \bmod \mathfrak{p}^{\lambda_1} \vee \mathfrak{u}_{\lambda_3}\ \forall a \in \mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right\}.\end{aligned}$$

Thus we have a normal series

$$\mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)_1 \rhd N \rhd \overline{N} \rhd 1.$$

Theorem 30. *The following holds.*

(1): *We have*

$$\mathrm{Aut}_{\lambda_3}\left(\mathfrak{o}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\right)_1/N \cong \mathrm{Aut}\,\mathfrak{o}/\mathfrak{p}^{\lambda_3}.$$

(2): *We have*

$$N/\overline{N} \cong \begin{cases} k & \lambda_3 \geq 2, \\ k^\times & \lambda_3 = 1. \end{cases}$$

(3): *We have*

$$\overline{N} \cong \overline{Q}_0 \times Q_0^{\lambda_1-\lambda_2} \times K.$$

Hence in particular $\overline{N}$ is abelian if $\lambda_3 \geq \frac{1}{2}\lambda_2$.

We can show that N is abelian for certain types of λ:

Proposition 31. *If $\lambda_3 > \frac{1}{2}\lambda_1$, then N is abelian.*

The structure of $\overline{N}$ resembles that of K; We obtain a decomposition of $\overline{N}$ similar to that of K:

Proposition 32. *$\overline{N}$ decomposes into a direct product as*

$$\overline{N} \cong V_0 \times V_1 \times \cdots \times V_{\lambda_1-\lambda_3-1},$$

where each factor V_i (defined for $0 \leq i \leq \lambda_1 - 1$) is given by

$$V_i = \{\tau \in N \mid \tau(a) = a \; {}^{\forall}a \in (\mathfrak{o} \setminus \mathfrak{q}_i)/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2}\}.$$

Lemma 33. *We have the following.*

(1): $V_i \cong Q_{i-\lambda_1+\lambda_2}$ *for* $i \in [\lambda_1 - \lambda_2 + 1, \lambda_1 - \lambda_3 - 1]$,
(2): $V_1 \cong V_2 \cong \cdots \cong V_{\lambda_1-\lambda_2} \cong Q_0$ *where* $\lambda_1 > \lambda_2$,
(3): $V_0 \cong \overline{Q}_0$.

Lastly, we consider the situation in which the residue field k is the finite field $\mathbb{F}_q$. Then Aut $\mathcal{L}(M)$ is evidently finite, and by the structual theorem we can compute the order of the group. There is not much to do for case $\lambda_2 = \lambda_3$, so assume $\lambda_2 > \lambda_3$. We start with computing the order $|Q_i|$. We can use either the L-sequence of Q_i or H-decomposition. Let us choose the former this time:

$$\begin{aligned} |Q_i| &= q^{(q-1)+(q-1)q+(q-1)q^2+\cdots+(q-1)q^{\lambda_2-\lambda_3-i-1}} \\ &= q^{-1+q^{\lambda_2-\lambda_3-i}}. \end{aligned}$$

In particular, we get $|Q_0| = q^{-1+q^{\lambda_2-\lambda_3}}$. Since $K = \prod_{i=1}^{\lambda_2-\lambda_3-1} Q_i$, we see that

$$|K| = \prod_{i=1}^{\lambda_2-\lambda_3-1} |Q_i| = q^{\sum_{i=1}^{\lambda_2-\lambda_3-1}(-1+q^i)} = q^{-\lambda_2+\lambda_3+1+\sum_{i=1}^{\lambda_2-\lambda_3-1} q^i}.$$

Also, by L-sequence or H-decomposition of $\overline{Q}_0$, we see that $|\overline{Q}_0|q^{\lambda_2-\lambda_3} = |Q_0|$. So we compute:

$$\begin{aligned} |\overline{N}| &= |\overline{Q}_0| \cdot |Q_0|^{\lambda_1-\lambda_2} \cdot |K| \\ &= q^{-\lambda_2+\lambda_3} q^{-1+q^{\lambda_2-\lambda_3}} \left(q^{-1+q^{\lambda_2-\lambda_3}}\right)^{\lambda_1-\lambda_2} \cdot q^{\sum_{i=1}^{\lambda_2-\lambda_3-1}(-1+q^i)} \\ &= q^{-\lambda_2+\lambda_3} q^{-1+q^{\lambda_2-\lambda_3}} q^{-\lambda_1+\lambda_2+(\lambda_1-\lambda_2)q^{\lambda_2-\lambda_3}} q^{-\lambda_2+\lambda_3+1+\sum_{i=1}^{\lambda_2-\lambda_3-1} q^i} \\ &= q^{-\lambda_2-\lambda_1+2\lambda_3+(\lambda_1-\lambda_2)q^{\lambda_1-\lambda_2}+\sum_{i=1}^{\lambda_2-\lambda_3} q^i}. \end{aligned}$$

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