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ON THE AUTOMORPHISM GROUP OF THE SUBGROUP LATTICE OF A FINITE ABELIAN $p$-GROUP; SOME GENERALIZATIONS

KAN YASUDA

ABSTRACT. The automorphism group $\text{Aut}\mathcal{L}(M)$ of the submodule lattice $\mathcal{L}(M)$ of a finite-length module $M$ over complete discrete valuation ring $\mathfrak{o}$ is studied. Let $\lambda = (\lambda_1, \ldots, \lambda_l)$ be the type of $M$. We show that for those $M$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 1$, $\text{Aut}\mathcal{L}(M)$ can be analyzed by computing a certain subgroup of the bijections on a quotient of the scalar ring $\mathfrak{o}$. In particular, when the residue field $k = \mathfrak{o}/p$ is a finite field $\mathbb{F}_q$, we compute the order of the group.

1. OBJECTIVE

Let $\mathfrak{o}$ be a discrete valuation ring with the maximal ideal $\mathfrak{p}$, a prime element $\pi$ (i.e., $\mathfrak{o}\pi = \mathfrak{p}$) and the valuation function $v : \mathfrak{o} \setminus \{0\} \to \mathbb{Z}_{\geq 0}$. Let $k \cong \mathfrak{o}/\mathfrak{p}$ denote the residue field. Let $M$ be an $\mathfrak{o}$-module of finite length. Then, since $\mathfrak{o}$ is a principal ideal domain, $M$ can be written as a sum of cyclic $\mathfrak{o}$-submodules:

$$M \cong \mathfrak{o}/\mathfrak{p}^{\lambda_1} \oplus \cdots \oplus \mathfrak{o}/\mathfrak{p}^{\lambda_l},$$

with $\lambda = (\lambda_1, \ldots, \lambda_l)$ being some partition of a non-negative integer (That is, we have $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$). $\lambda$ is called the type of $M$. Now since we have $\mathfrak{o}/\mathfrak{p}^i \cong \overline{\mathfrak{o}}/\overline{\mathfrak{p}}^i$ where $\overline{\mathfrak{o}}$ is the completion of $\mathfrak{o}$ and $\overline{\mathfrak{p}}$ its maximal ideal, without loss of generality we can assume $\mathfrak{o}$ to be complete. Let $\mathcal{L}(M)$ denote the set of $\mathfrak{o}$-submodules of $M$. $\mathcal{L}(M)$ inherits a lattice structure by inclusion relation. Our main objective is to compute $\text{Aut}\mathcal{L}(M)$, the automorphism group of the lattice $\mathcal{L}(M)$, for such $\lambda$ as $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 1$.

When $\mathfrak{o} = \mathbb{Z}_p$, the ring of $p$-adic integers, $M$ becomes nothing but a finite abelian $p$-group and $\mathcal{L}(M)$ the subgroup lattice of $M$. This can be generalized by considering the case $\mathfrak{o} = W[\mathbb{F}_q]$, the ring of Witt vectors over the finite field $\mathbb{F}_q$, for $W[\mathbb{F}_p] \cong \mathbb{Z}_p$. Another example of $\mathfrak{o}$ is the ring $k[[t]]$ of formal power series in one variable $t$.

We call $e = (e_1, \ldots, e_l) \in M^{l}$ an ordered basis for $M$ if $M = \bigoplus_{i=1}^{l} \mathfrak{o}e_i$ and $\mathfrak{o}e_i \cong \mathfrak{o}/\mathfrak{p}^{\lambda_i}$. Let $e$ be fixed. We denote by $R(e)$ the set of $\varphi \in \text{Aut}\mathcal{L}(M)$ satisfying

$$\begin{cases} 
\varphi(\mathfrak{o}e_i) = \mathfrak{o}e_i & \forall i \in [1, l] \\
\varphi(\mathfrak{o}(e_1 + e_i)) = \mathfrak{o}(e_1 + e_i) & \forall i \in [2, l]
\end{cases}$$

In most cases it boils down to computing $R(e)$ in order to analyze $\text{Aut}\mathcal{L}(M)$, in the sense we describe as follows.

Since an automorphism of $\mathfrak{o}$-module $M$ induces an automorphism of the lattice $\mathcal{L}(M)$, we have the natural group homomorphism

$$\Theta : \text{Aut} M \to \text{Aut}\mathcal{L}(M).$$

It can be directly checked that $\text{Ker} \Theta \cong (\mathfrak{o}/\mathfrak{p}^{\lambda_1})^\times$ and that $\text{Aut} M$ can be expressed in matrix form, as described in the sequel. Naturally $\text{Aut}\mathcal{L}(M)$ contains a subgroup isomorphic to $\text{Aut} M/\text{Ker} \Theta$, and we let $\text{PAut} M$ denote this subgroup.

It turns out that $\text{Aut}\mathcal{L}(M)$ is a product of these two subgroups $R(e)$ and $\text{PAut} M$. Namely, we have

Lemma 1. $R(e) \cdot \text{PAut} M = \text{Aut}\mathcal{L}(M)$

$R(e) \cap \text{PAut} M = 1$. 

Also, we remark that if $e$ and $e'$ are ordered base for $M$, then it is easily checked that $\varphi R(e)\varphi^{-1} = R(e')$, where $\varphi \in \text{PAut } M$ is the lattice automorphism induced by the module automorphism of $M$ defined by $e_i \mapsto e'_i$ $(1 \leq i \leq l)$. Hence the isomorphism type of $R(e)$ does not depend on the choice of $e$. We content ourselves with computing $R(e)$ instead of computing $\text{Aut } L(M)$ for our purpose.

2. HISTORICAL BACKGROUND

Let us mention the relation with earlier results. The structure of $\text{Aut } L(M)$ is well-known for the case $\lambda_1 = \lambda_2 = \lambda_3$, which is essentially the result of Baer [2]. In this case, we have $\text{Aut } L(M) \cong R(e) \ltimes \text{PAut } M$, and

$$R(e) \cong \text{Aut } o/p^{\lambda_3},$$

where $\text{Aut } o/p^{\lambda_3}$ is the group of automorphisms of ring $o/p^{\lambda_3}$. In particular, when $\lambda_1 = \cdots = \lambda_l = 1$ $(l \geq 3)$, $M$ becomes a vector space over the residue field $k$ of $o$, and $\text{Aut } L(M)$ is isomorphic to $\text{PTL}(l,k)$, the group of projective semi-linear automorphisms. This result is a variation of so called the Fundamental Theorem of Finite Projective Geometry.

We next consider the case when the residue field of $o$ is the finite field $F_p$. Let $M = o/p \otimes o/p \cong F_p \oplus F_p$. Then $\text{Aut } L(M)$ is isomorphic to the symmetric group $S_{p+1}$ and $\text{PAut } M$ isomorphic to the projective general linear group $\text{PGL}(2, p)$ (Note that $|\text{PGL}(2, p)| = (p+1)(p-1)$). In this case, $R(e)$ is a subgroup that fixes three points and isomorphic to $S_{p-2}$. More generally, for $M = \mathbb{Z}_p/p^{\lambda_3} \mathbb{Z}_p \oplus \mathbb{Z}_p/p^{\lambda_2} \mathbb{Z}_p$ ($o = \mathbb{Z}_p$ is the ring of $p$-adic integers), Holmes' result [5] states that $\text{Aut } L(M)$ is isomorphic to $S_{p}^{\lambda_3-1} \ltimes S_{p+1}$, where $S_{p}^{\lambda_3}$ means $S_{p} \cdots S_{p}$ (in times) and $\lambda_3$ denotes the standard wreath product. In this case, $\text{PAut } M$ is nothing but $\text{PGL}_2(\mathbb{Z}_p/p^{\lambda_2} \mathbb{Z}_p)$, and we note that $|\text{PGL}_2(\mathbb{Z}_p/p^{\lambda_2} \mathbb{Z}_p)| = (p+1)(p-1) \cdot (p^{\lambda_3-1})^3$. $R(e)$ is the subgroup that fixes three points $\mathbb{Z}_p(1,0)$, $\mathbb{Z}_p(0,1)$ and $\mathbb{Z}_p(1,1)$; in fact, we have

$$R(e) \cong (S_{p}^{\lambda_3-1} \ltimes S_{p+1}) \times \left\{ \prod_{i=0}^{\lambda_3-2} (S_{p}^i \ltimes S_{p-1}) \right\}^3.$$

Holmes [5] also obtains a result for the case $\lambda_1 > \lambda_2 > \lambda_3 = 0$: $\text{Aut } L(M) \cong G^2 \ltimes H^{\lambda_1-\lambda_2-1}$, where $G = S_{p}^{\lambda_3}$ and $H = S_{p}^{\lambda_3} \ltimes S_{p-1}$.

There have been works to bridge the gap between Baer’s result and Holmes’. Costantini-Holmes-Zacher[3] and Costantini-Zacher[4] treated the case of abelian $p$-groups in a rather general framework. Yasuda[11] studied the case of finite abelian $p$-groups for $\lambda_1 > \lambda_2 = \lambda_3$ with explicit computation of $R(e)$ and $\text{Aut } L(M)$. In this work, we shall treat the case $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 1$, in the general setting of finite-length modules over (complete) discrete valuation ring.

3. NOTATIONS AND NOTIONS

Here we give some supplementary definitions and notations. Put $q_i = p^i \setminus p^{i+1}$ for $i \geq 1$; i.e., $q_i = \{a \in o \mid v(a) = i\}$. We define $q_0 = o \setminus p = o^x$, the set of invertible elements. For $a, b \in o$ such that $v(a) \geq v(b)$ ($b \neq 0$), there exists an element $x \in o$ such that $a = xb$. As $o$ is a domain, $x$ must be unique. We use the notation $\frac{a}{b} = x$.

Given a set $X$, $\text{Map}(X)$ denotes the set of maps $f : X \to X$. $\text{Sym}(X)$ denotes the set of bijections $f : X \to X$. $\text{Map}(X)$ forms a monoid with respect to function composition, whereas $\text{Sym}(X)$ forms a group. Given two sets $X$ and $Y$, we define $Y^X$ to be the set of maps $f : X \to Y$.

Let $G$ be a group, and $H$ a group acting on a set $X$. Let $f, g : X \to G$ be two maps, and define a map $f \circ g : X \to G$ by $f \circ g(x) = f(x) \cdot g(x)$ where $\cdot$ is the product in $G$. Then $G^X$ becomes a group with respect to this $\circ$. Let $h \in H$ and $f \in G^X$. We define a semidirect product $G^X \ltimes H$ with respect to the group homomorphism $H \to \text{Aut } G^X (h \mapsto (f \mapsto fh^{-1}))$. We write $G \ltimes H$ to denote this semidirect product, and call it the wreath product of $G$ and $H$.

We now give the description of the automorphism group $\text{Aut } M$ of an $o$-module $M$ in matrix form, as promised. Let $e$ be fixed. The action of $f \in \text{Aut } M$ is then determined by its action on $e = (e_1, \ldots, e_l)$. Write

$$f(e_j) = \sum_{i=1}^{l} a_{ij} e_i,$$
and express $f$ as the matrix $(a_{ij})_{i,j=1}^{l}$. Rewriting $\lambda = (\lambda_{1}, \ldots, \lambda_{l}) = (d_{1}^{-m_{1}}, \ldots, d_{r}^{-m_{r}})$ ($d_{1} > \cdots > d_{r}$) (this means that $\lambda$ contains $m$-many components equal to $d_{r}$), Aut $M$ can be expressed in matrix form as

$$
\begin{pmatrix}
GL_{m_{1}}(\mathfrak{o}/p^{d_{1}}) & \cdots & \text{Hom}((\mathfrak{o}/p^{d_{j}})^{\oplus m_{j}}, (\mathfrak{o}/p^{d_{k}})^{\oplus m_{k}}) \\
\vdots & \ddots & \vdots \\
\text{Hom}((\mathfrak{o}/p^{d_{1}})^{\oplus m_{1}}, (\mathfrak{o}/p^{d_{k}})^{\oplus m_{k}}) & \cdots & GL_{m_{k}}(\mathfrak{o}/p^{d_{k}})
\end{pmatrix},
$$

with respect to the ordered basis $e$. Here, the block matrix in the diagonal

$$A \in GL_{m_{i}}(\mathfrak{o}/p^{d_{i}})$$

is of size $m_{i} \times m_{i}$ and has elements of $\mathfrak{o}/p^{d_{i}}$ in its components, satisfying $\pi \equiv \det A$. Also, the block matrix at $(i, j)$-position ($i \neq j$)

$$A \in \text{Hom}((\mathfrak{o}/p^{d_{i}})^{\oplus m_{i}}, (\mathfrak{o}/p^{d_{j}})^{\oplus m_{j}})$$

is of size $m_{i} \times m_{j}$ and in its components has elements of $p^{d_{i} - \min(d_{i}, d_{j})}(\mathfrak{o}/p^{d_{i}})$, that is, for $i < j$ ($\Longrightarrow d_{i} > d_{j}$) elements of $p^{d_{i} - d_{j}}(\mathfrak{o}/p^{d_{i}})$, and for $i > j$ ($\Longrightarrow d_{i} < d_{j}$) elements of $\mathfrak{o}/p^{d_{i}}$.

4. MAIN RESULTS

For the case $\lambda_{1} > \lambda_{2} = \lambda_{3}$, we can state our main result as follows:

**Theorem 2.** Assume $\lambda_{2} = \lambda_{3}$. Then $R(\mathfrak{o})$ contains a normal subgroup $N$ such that

$$R(\mathfrak{o})/N \cong \text{Aut} \mathfrak{o}/p^{\lambda_{2}},$$

$$N \cong \begin{cases}
k^{\lambda_{1} - \lambda_{2}} \quad & \lambda_{2} = \lambda_{3} > 2, \\
(k \times)^{\lambda_{1} - \lambda_{2}} \quad & \lambda_{2} = \lambda_{3} = 1.
\end{cases}
$$

The case $\lambda_{1} \geq \lambda_{2} > \lambda_{3}$ turns out to be rather complicated. The rest of this section is dedicated to explain our main result for this case.

Let $i \geq 1$. For $a, b \in \mathfrak{o}$, we write

$$a \equiv b \, \text{mod} \, p^{i}$$

to mean $a - b \in p^{i}$. With abuse of notation, we write $p^{i}$ also to denote this equivalence relation. Then obviously we have $p^{i} > p^{j} > p^{j+1} > \cdots$. On the other hand, put $u_{i} = 1 + p^{i} \subset \mathfrak{o}$ ($i \geq 1$). For $a, b \in \mathfrak{o}$, write

$$a \sim b \, \text{mod} \, u_{i}$$

if $a \in u_{i}b$. Clearly this defines an equivalence relation on $\mathfrak{o}$. Again with abuse of notation, we just write $u_{i}$ to denote this relation. Then note that we have $u_{1} > u_{2} > u_{3} > \cdots$. Also note that $p^{i} > u_{i}$ holds for all $i \geq 1$.

**Lemma 3.** The union of relations $p^{i} \cup u_{j}$ is an equivalence relation for all $i, j \geq 1$.

Hence we have $p^{i} \cup u_{j} = p^{j} \cup u_{j}$, and it makes sense to denote the quotient set by $\mathfrak{o}/p^{i}/u_{j} = \mathfrak{o}/u_{j}/p^{i} = \mathfrak{o}/p^{i} \cup u_{j}$ for all $i, j \geq 1$.

Now we proceed to the following lemma.

**Lemma 4.** Let $\varphi \in R(\mathfrak{o})$ be given. There exist bijective maps $\tau : \mathfrak{o} \to \mathfrak{o}$ and $\sigma : \mathfrak{o} \to \mathfrak{o}$ such that $\varphi(\mathfrak{a}e_{1} + e_{2}) = \varphi(\mathfrak{a}(e_{1} + e_{2}) = \mathfrak{a}(e_{1} + \sigma(\mathfrak{a})e_{2})$ for all $\mathfrak{a} \in \mathfrak{o}$. $\tau$ and $\sigma$ induce bijections $\tau : \mathfrak{o}/p^{\lambda_{1}}/u_{x_{2}} \to \mathfrak{o}/p^{\lambda_{2}}/u_{x_{2}}$ and $\sigma : \mathfrak{o}/p^{\lambda_{2}} \to \mathfrak{o}/p^{\lambda_{2}}$, respectively, which are uniquely determined by $\varphi$.

Let $\varphi \in R(\mathfrak{o})$ be given and $\tau, \sigma$ as in the preceding lemma. We list in the following lemma some of the properties satisfied by $\tau$ and $\sigma$.

**Lemma 5.** We have

1. $\tau(1) \sim 1 \, \text{mod} \, p^{\lambda_{1}} \cup u_{x_{2}}$, $\sigma(1) \equiv 1 \, \text{mod} \, p^{\lambda_{2}}$,
2. $\tau(p) \subset p$, $\sigma(p) \subset p$,
3. $\tau(ab) \sim \tau(a)\tau(b) \, \text{mod} \, p^{\lambda_{1}} \cup u_{x_{2}}$ for all $a, b \in \mathfrak{o}$,
4. $\sigma(ab) \sim \sigma(a)\sigma(b) \, \text{mod} \, p^{\lambda_{2}} \cup u_{x_{2}}$ for all $a, b \in \mathfrak{o}$,
5. $\tau(a - b) \sim \tau(a) - \tau(b) \, \text{mod} \, p^{\lambda_{1}} \cup p^{\lambda_{2} + \mathfrak{p}(b)} \cup u_{x_{2}}$ for all $a, b \in \mathfrak{o}$,
(6): $\sigma(a - b) \sim \sigma(a) - \sigma(b) \mod p^{\lambda_1} \vee u_{\lambda_3}$ for all $a, b \in o$,  
(7): $\tau(a) \equiv \sigma(a^{-1})^{-1} \mod p^{\lambda_2}$ for all $a \in o^*$,  
(8): $\tau(a) \sim \sigma(a) \mod p^{\lambda_2} \vee u_{\lambda_3}$ for all $a \in o$.

Given three positive integers $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 1$, let

$$\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(o)$$

denote the set of bijections $\tau : o \rightarrow o$ that satisfy the following three conditions:

Valuation law: $\tau(p) \subset p$,  
Strict product law: $\tau(ab) \sim \tau(a)\tau(b) \mod p^{\lambda_1} \vee u_{\lambda_3}$ for all $a, b \in o$,  
Difference law: $\tau(a - b) \sim \tau(a) - \tau(b) \mod p^{\lambda_1} \vee p^{\lambda_2 + v(b)} \vee u_{\lambda_3}$ for all $a, b \in o$.

In this section we shall prove that this set forms a group and a $\tau \in \text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(o)$ induces a bijection $\tau : o/p^{\lambda_1}/u_{\lambda_2} \rightarrow o/p^{\lambda_1}/u_{\lambda_3}$. It turns out that $R(e)$ can be described by using this group.

**Lemma 6.** Let $\tau : o \rightarrow o$ be a bijective map that satisfies the valuation law and the strict product law. Then we have

$$\tau(p^i) = p^i$$

for all $i \in [0, \lambda_1]$; that is, we have

$$v(\tau(a)) = v(a)$$

for all $a \in o \setminus p^{\lambda_1}$.

**Lemma 7.** Let $i \leq \lambda_1$ and $j \leq \lambda_2$. Let $\tau : o \rightarrow o$ be a bijective map that satisfies the difference law and the condition $v(\tau(a)) = v(a)$ for all $a \in o \setminus p^{\lambda_1}$. For $a, b \in o$, we have $a \sim b \mod p^i \vee u_j$ if and only if $\tau(a) \sim \tau(b) \mod p^i \vee u_j$. That is, there exists a unique bijective map $\overline{\tau}$ that makes the diagram below commutative:

$$\begin{array}{ccc}
o/p^{\lambda_1}/u_{\lambda_2} & \xrightarrow{\tau} & o/p^{\lambda_1}/u_{\lambda_3} \\
\downarrow & & \downarrow \\
o/p^i/u_j & \xrightarrow{\overline{\tau}} & o/p^i/u_j
\end{array}$$

**Proposition 8.** $\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(o)$ forms a subgroup of $\text{Sym}(o)$.

Now denote by $\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(o)u_{\lambda_2}$ the stabilizer of $u_{\lambda_2}$ in $\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}$. That is,

$$\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(o)u_{\lambda_2} = \{ \tau \in \text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(o) \mid \tau(u_{\lambda_2}) = u_{\lambda_2} \}.$$  

Then define

$$\Delta_{\lambda_2}^{-1}(\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(o)u_{\lambda_2} \times \text{Aut}_{\lambda_2, \lambda_2, \lambda_3}(o)u_{\lambda_2})$$

to be the set of $(\tau, \sigma) \in \text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(o)u_{\lambda_2} \times \text{Aut}_{\lambda_2, \lambda_2, \lambda_3}(o)u_{\lambda_2}$ satisfying the conditions

$$\begin{cases}
\tau(a)^{-1} \equiv \sigma(a^{-1}) \mod p^{\lambda_2} \quad \forall a \in o^*, \\
\tau(a) \sim \sigma(a) \mod p^{\lambda_2} \vee u_{\lambda_3} \quad \forall a \in o.
\end{cases}$$

Note that since $\tau(a)\tau(a^{-1}) \sim 1 \mod p^{\lambda_1} \vee u_{\lambda_3}$ whence $\tau(a)^{-1} \sim (a^{-1}) \mod p^{\lambda_1} \vee u_{\lambda_3}$ for all $a \in o^*$, the first condition $\tau(a)^{-1} \equiv \sigma(a^{-1}) \mod p^{\lambda_2}$ implies the second condition $\tau(a) \sim \sigma(a) \mod p^{\lambda_1} \vee u_{\lambda_3}$.

**Lemma 9.** The set

$$\Delta_{\lambda_2}^{-1}(\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(o)u_{\lambda_2} \times \text{Aut}_{\lambda_2, \lambda_2, \lambda_3}(o)u_{\lambda_2})$$

forms a subgroup of $\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(o)u_{\lambda_2} \times \text{Aut}_{\lambda_2, \lambda_2, \lambda_3}(o)u_{\lambda_2}$.

We have observed that $\tau \in \text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(o)$ induces a bijective map $\tau : o/p^{\lambda_1}/u_{\lambda_2} \rightarrow o/p^{\lambda_1}/u_{\lambda_2}$. That is to say, there exists a natural group homomorphism $\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(o) \rightarrow \text{Sym}(o/p^{\lambda_1}/u_{\lambda_2})$. Let us define

$$\text{Aut}_{\lambda_3}(o/p^{\lambda_1}/u_{\lambda_2})$$

$$\text{Aut}_{\lambda_3}(o/p^{\lambda_2})$$
to be the images of \( \text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(0) \to \text{Sym}(\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_3}) \) and \( \text{Aut}_{\lambda_2, \lambda_2, \lambda_3}(0) \to \text{Sym}(\mathfrak{p}^{\lambda_2}) \), respectively. Furthermore, let
\[
\begin{align*}
\text{Aut}_{\lambda_1}(\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_3})_1 \\
\text{Aut}_{\lambda_2}(\mathfrak{p}^{\lambda_2})_1 
\end{align*}
\]
be the subgroups of the above two, corresponding to \( \text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(0) \) and \( \text{Aut}_{\lambda_2, \lambda_2, \lambda_3}(0) \), respectively. Lastly, we denote by
\[
\Delta_{\lambda_2}^{-1}(\text{Aut}_{\lambda_3}(\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_3})_1 \times \text{Aut}_{\lambda_2}(\mathfrak{p}^{\lambda_2})_1)
\]
the subgroup of \( \text{Aut}_{\lambda_3}(\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_3})_1 \times \text{Aut}_{\lambda_2}(\mathfrak{p}^{\lambda_2})_1 \) that corresponds to \( \Delta_{\lambda_2}^{-1}(\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(0) \times \text{Aut}_{\lambda_2, \lambda_2, \lambda_3}(0)) \). Now we can state our:

**Theorem 10 (Main Isomorphism Theorem).** We have
\[
R(e) \cong \Delta_{\lambda_2}^{-1}(\text{Aut}_{\lambda_2}(\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_3})_1 \times \text{Aut}_{\lambda_2}(\mathfrak{p}^{\lambda_2})_1)
\]
if \( \lambda_2 > \lambda_3 \).

Note that one way of isomorphism
\[
\Phi : R(e) \to \Delta_{\lambda_2}^{-1}(\text{Aut}_{\lambda_3}(\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_3})_1 \times \text{Aut}_{\lambda_2}(\mathfrak{p}^{\lambda_2})_1)
\]
is already given, by sending \( \Phi : \varphi \mapsto (\tau, \sigma) \). In order to compute \( R(e) \), this theorem allows us to compute \( \Delta_{\lambda_2}^{-1}(\text{Aut}_{\lambda_3}(\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_3})_1 \times \text{Aut}_{\lambda_2}(\mathfrak{p}^{\lambda_2})_1) \) instead. Let
\[
\Lambda : \Delta_{\lambda_2}^{-1}(\text{Aut}_{\lambda_3}(\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_3})_1 \times \text{Aut}_{\lambda_2}(\mathfrak{p}^{\lambda_2})_1) \to \text{Aut}_{\lambda_3}(\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_3})_1
\]
be the "projection" map to the first component; i.e., \( \Lambda : (\tau, \sigma) \mapsto \tau \). Then \( \text{Ker} \Lambda \) is the set of \((1, \sigma)\) satisfying
\[
\begin{cases}
\sigma(a) \equiv a \text{ mod } p^{\lambda_2} & a \in o^{\times}/p^{\lambda_2}, \\
\sigma(a) \sim a \text{ mod } p^{\lambda_2} \vee u_{\lambda_3} & a \in p^{\lambda_2}.
\end{cases}
\]

Let \( K \) be the kernel of the natural map \( \text{Aut}_{\lambda_3}(\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_3}) \to o^{\times}/p^{\lambda_2} \), that is,
\[
K = \{ \sigma \in \text{Aut}_{\lambda_3}(\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_3})_1 | \sigma(a) = a \text{ for all } a \in o^{\times}/p^{\lambda_2} \}.
\]

**Lemma 11.** We have
\[
\text{Ker} \Lambda \cong K.
\]

We shall show that the group in question, \( \Delta_{\lambda_2}^{-1}(\text{Aut}_{\lambda_3}(\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_3})_1 \times \text{Aut}_{\lambda_2}(\mathfrak{p}^{\lambda_2})_1) \), is isomorphic to a semidirect product of \( K \) and the first component \( \text{Aut}_{\lambda_3}(\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_3})_1 \):

**Proposition 12.** The sequence
\[
1 \to K \to \Delta_{\lambda_2}^{-1}(\text{Aut}_{\lambda_3}(\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_3})_1 \times \text{Aut}_{\lambda_2}(\mathfrak{p}^{\lambda_2})_1) \to \text{Aut}_{\lambda_3}(\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_3})_1 \to 1
\]
is exact and splitting. In other words, we have
\[
\Delta_{\lambda_2}^{-1}(\text{Aut}_{\lambda_3}(\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_3})_1 \times \text{Aut}_{\lambda_2}(\mathfrak{p}^{\lambda_2})_1) \cong \text{Aut}_{\lambda_3}(\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_3})_1 \rtimes K.
\]

This result divides our investigation into two parts: the analysis of the structure of \( K \) and that of \( \text{Aut}_{\lambda_3}(\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_3})_1 \). We begin with the former.

Recall that
\[
K = \{ \sigma \in \text{Aut}_{\lambda_3}(\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_3})_1 | \sigma(a) = a \forall a \in (o^{\times}/p^{\lambda_2}) \}
\]
\[
= \{ \sigma \in \text{Aut}_{\lambda_3}(\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_3})_1 | \sigma(a) = a \forall a \in (o^{\times}/p^{\lambda_2}) \text{ and } \sigma(a) \sim a \text{ mod } p^{\lambda_2} \vee u_{\lambda_3} \forall a \in p^{\lambda_2} \}.
\]

For the sake of convenience, we shall analyze groups slightly larger than \( K \); namely,
\[
\tilde{K} = \{ \sigma \in \text{Aut}_{\lambda_3}(\mathfrak{p}^{\lambda_1})_1 | \sigma(a) \sim a \text{ mod } p^{\lambda_2} \vee u_{\lambda_3} \forall a \in \mathfrak{p}^{\lambda_2} \},
\]
and
\[
\tilde{K}_1 = \{ \sigma \in \text{Aut}_{\lambda_3}(\mathfrak{p}^{\lambda_1})_1 | \sigma(a) \sim a \text{ mod } p^{\lambda_2} \vee u_{\lambda_3} \forall a \in \mathfrak{p}^{\lambda_2} \}.
\]

Of course we have \( \tilde{K}_1 = \{ \sigma \in \tilde{K} | \sigma(1) = 1 \} \).
Proposition 13. \( \tilde{K} \) decomposes into a direct product as
\[ \tilde{K} \cong Q_0 \times Q_1 \times \cdots \times Q_{\lambda_2-\lambda_3-1} \]
where each factor \( Q_i \) (defined for \( 0 \leq i \leq \lambda_2 - 1 \)) is given by
\[ Q_i = \left\{ \sigma \in \tilde{K} \mid \sigma(a) = a \text{ for all } a \in (\mathfrak{a} \setminus q_i)/\mathfrak{p}^{\lambda_3} \right\}. \]

Corollary 14. \( \tilde{K}_1 \) and \( K \) decompose into direct products as
\[ \tilde{K}_1 \cong \overline{Q}_0 \times Q_1 \times \cdots \times Q_{\lambda_2-\lambda_3-1} \text{ and} \]
\[ K \cong Q_1 \times Q_2 \times \cdots \times Q_{\lambda_2-\lambda_3-1}, \]
respectively, where
\[ \overline{Q}_0 = \left\{ \sigma \in \tilde{K}_1 \mid \sigma(a) = a \forall a \in \mathfrak{p}/\mathfrak{p}^{\lambda_3} \right\} = \{ \sigma \in Q_0 \mid \sigma(1) = 1 \}. \]

We now focus on the calculation of each \( Q_i \). First, we give a description of generators of \( Q_i \). We begin with the following lemma.

Lemma 15. Let \( \sigma \in \tilde{K} \) and \( 0 \leq j \leq \lambda_2 - \lambda_3 \). If \( a \equiv b \mod p^j \), then \( \sigma(a) - a \equiv \sigma(b) - b \mod p^{j+\lambda_3} \).

We apply this lemma particularly, to \( Q_i \) (\( 0 \leq i \leq \lambda_2 - \lambda_3 - 1 \)). For \( j \in [i+1, \lambda_2 - \lambda_3] \), let \( (p^{j+\lambda_3-1}/p^{\lambda_3})^{q_j/p^j} \) denote the set of maps \( z : q_i/p^j \to p^{j+\lambda_3-1}/p^{\lambda_3} \). Note that since \( p^{j+\lambda_3-1}/p^{\lambda_3} \) is an abelian group, so becomes \( (p^{j+\lambda_3-1}/p^{\lambda_3})^{q_j/p^j} \) naturally. Given \( j \in [i+1, \lambda_2 - \lambda_3] \)
and \( z : q_i/p^j \to p^{j+\lambda_3-1}/p^{\lambda_3} \), define a map \( g_{j,z} : o/p^{\lambda_3} \to o/p^{\lambda_3} \) by
\[ g_{j,z}(a) = \begin{cases} a + z(a \mod p^j) & a \in q_i, \\ a & \text{otherwise.} \end{cases} \]

We shall show that \( g_{j,z} \) is in \( Q_i \); more precisely,

Proposition 16. We have
\[ Q_i = \langle g_{j,z} \rangle_{j \in [i+1, \lambda_2 - \lambda_3]} = \langle g_{j,z} \rangle_{z \in (p^{j+\lambda_3-1}/p^{\lambda_3})^{q_j/p^j}} \]
where \( (S^{p^{j+\lambda_3-1}/p^{\lambda_3}})^{q_j/p^j} \) denotes the set of maps \( z : q_i/p^j \to S^{p^{j+\lambda_3-1}/p^{\lambda_3}} \).

Using these generators, we give two sorts of descriptions of \( Q_i \). The former turns out to be useful particularly for the case \( \lambda_3 = 1 \), whereas the latter being useful for the case \( \lambda_3 \geq \frac{1}{2} \lambda_2 \).

For \( j \in [i+1, \lambda_2 - \lambda_3 + 1] \), define
\[ L_i^{(j)} = \left\{ \sigma \in Q_i \mid \sigma(a) \equiv a \mod p^{j+\lambda_3-1} \forall a \in o/p^{\lambda_3} \right\}. \]
Then clearly we have \( Q_i \triangleright L_i^{(j)} \) for each \( j \) whence obtain a chain of normal subgroups \( L_i^{(j)} \).

Proposition 17. We have a normal series of \( Q_i \):
\[ Q_i = L_i^{(i+1)} \triangleright L_i^{(i+2)} \triangleright \cdots \triangleright L_i^{(\lambda_2-\lambda_3+1)} = 1, \]
where the factors of the series are given by
\[ L_i^{(j)}/L_i^{(j+1)} \cong (o/p)^{q_j/p^j}, \]
for all \( j \in [i+1, \lambda_2 - \lambda_3] \).

Proposition 18. We have
\[ L_i^{(j)} = \langle g_{n,z} \rangle_{n \in [j, \lambda_2 - \lambda_3]} = \langle g_{n,z} \rangle_{z \in (S^{p^{n+\lambda_3-1}/p^{\lambda_3}})^{q_j/p^j}} \]

Lemma 19. We have the exact sequence of groups
\[ 1 \to L_i^{(\lambda_2-\lambda_3)} \to Q_i \to Q_{i+1} \to 1. \]
This sequence splits if \( \lambda_3 = 1 \).
Proposition 20. If $\lambda_3 = 1$, then $Q_i$ decomposes into a wreath product as

$$Q_i \simeq k_1 k_2 \cdots k_{\lambda_2}$$

where 1 is to act on $k^x$ trivially and $k$ on by addition. To put it more concisely,

$$Q_i \simeq (k_1^{(\lambda_2-1)})^x.$$

We present another way of describing the structure of $Q_i$. In order to do this, let us define

$$U_i^{(j)} = \{ \sigma \in Q_i \mid \sigma(a) - a \equiv \sigma(b) - b \text{ mod } p^{\lambda_2} \text{ if } a \equiv b \text{ mod } p^j \},$$

where $i \leq j \leq \lambda_2 - \lambda_3$. This gives us a filtration of $Q_i$:

$$Q_i = U_i^{(\lambda_2-\lambda_3)} \supset U_i^{(\lambda_2-\lambda_3-1)} \supset \cdots \supset U_i^{(i+1)} \supset U_i^{(i)} = T_i.$$ 

Here, $T_i$ is the group of translations, i.e.,

$$T_i = \{ \sigma \in Q_i \mid \sigma(a) - a = \sigma(b) - b \text{ for all } a, b \in \mathfrak{p}^{\lambda_2} \}.$$

Lemma 21. We have

$$U_i^{(j)} = \langle g_{n,z} \rangle \quad n \in [i+1,j] \quad z \in (\mathfrak{p}^{j+\lambda_3-1}/\mathfrak{p}^{\lambda_2})^x.$$ 

Now for each $j \in [i+1, \lambda_2 - \lambda_3]$ define

$$H_i^{(j)} = \{ \sigma \in U_i^{(j)} \mid \sigma(a) = a \text{ if } a_{j-1} = 0 \in S \},$$

with $a \in \mathfrak{p}^{\lambda_2}$ being written as $a = \sum_{n=1}^{\lambda_2-1} a_n \pi^n$ with $a_n \in S$. Obviously the definition of $H_i^{(j)}$ depends on the choice of $S$.

Proposition 22. The subgroups $H_i^{(j)}$ are abelian; more precisely, we have

$$H_i^{(j)} \cong (\mathfrak{p}^{j+\lambda_3-1}/\mathfrak{p}^{\lambda_2})^{(\mathfrak{p}^x/x)} \cong (\mathfrak{p}^{\lambda_2-\lambda_3-j+1}/\mathfrak{p}^{\lambda_2})^{(\mathfrak{p}^x/x)}.$$ 

Proposition 23. $Q_i$ decomposes into a product of abelian subgroups $H_i^{(j)} \subset Q_i$ ($i+1 \leq j \leq \lambda_2 - \lambda_3$) as

$$Q_i = H_i^{(i+1)} H_i^{(i+2)} \cdots H_i^{(\lambda_2-\lambda_3)},$$

with the properties

$$\left( H_i^{(i+1)} H_i^{(i+2)} \cdots H_i^{(j)} \right) \cap H_i^{(j+1)} = 1,$$

$$\left( H_i^{(i+1)} H_i^{(i+2)} \cdots H_i^{(j)} \right) H_i^{(j+1)} = H_i^{(j+1)} \left( H_i^{(i+1)} H_i^{(i+2)} \cdots H_i^{(j)} \right).$$

Lemma 24. If $i + \lambda_3 \geq \lambda_2 - \lambda_3$, then we have

$$Q_i \cong \bigoplus_{j=i}^{\lambda_2-\lambda_3-1} (\mathfrak{p}^{j-1-\lambda_3}/\mathfrak{p}^{\lambda_2})^{\times} \times k^x.$$ 

Proposition 25. If $\lambda_3 \geq \frac{1}{2}(\lambda_2 - 1)$, then $K$ is abelian and

$$K \cong \bigoplus_{i=1}^{\lambda_2-\lambda_3-1} \bigoplus_{j=i}^{\lambda_2-\lambda_3-1} (\mathfrak{p}^{j-1-\lambda_3}/\mathfrak{p}^{\lambda_2})^{\times} \times k^x.$$ 

Also, if $\lambda_3 \geq \frac{1}{2}\lambda_2$, then $Q_0$ is abelian and

$$Q_0 \cong \bigoplus_{j=0}^{\lambda_2-\lambda_3-1} (\mathfrak{p}^{j-1}/\mathfrak{p}^{\lambda_2})^{\times} \times k^x.$$
Now we describe the structure of $Q_0$, which we shall need later in computing $\text{Aut}_{\lambda_3} (a/p^{\lambda_1}/u_{\lambda_2})_1$. So for each $j \in [1, \lambda_2 - \lambda_3]$, put

$$L_0^{(j)} = \{ \sigma \in Q_0 \mid \sigma(a) \equiv a \mod p^{j+\lambda_2-1} \forall a \in a/p^{\lambda_2} \}.$$  

Evidently we have $Q_0 \triangleright L_0^{(j)}$ for each $j$.

**Proposition 26.** We have a normal series of $Q_0$:

$$Q_0 = L_0^{(1)} \triangleright L_0^{(2)} \triangleright \cdots \triangleright L_0^{(\lambda_2-\lambda_3+1)} = 1,$$

where the factors of the series are given by

$$L_0^{(j)}/L_0^{(j+1)} \cong (a/p)^{x(1)/p^j}.$$

for each $j \in [1, \lambda_2 - \lambda_3]$.

**Proposition 27.** Assume $\lambda_3 = 1$. Then $Q_0$ decomposes into a semidirect product as

$$Q_0 \cong k^{(x_1)/p^{(x_2)}} \rtimes k^{(x_3)/p^{(x_4)}} \cdots \rtimes k^{(x_{\lambda_3})/p^{(x_{\lambda_2})}}.$$

**Proposition 28.** $Q_0$ decomposes into a product of abelian subgroups as

$$Q_0 \cong H_0^{(1)} H_0^{(2)} \cdots H_0^{(\lambda_2-\lambda_3)}$$

where

$$H_0^{(1)} = \{ \sigma \in H_0^{(1)} \mid \sigma(1) = 1 \} \cong (a/p^{\lambda_2-\lambda_3})^{x(1)}.$$

Now we shift our attention to calculating the structure of $\text{Aut}_{\lambda_3} (a/p^{\lambda_1}/u_{\lambda_2})_1$. Let $N$ and $\overline{N}$ be the kernels of the natural homomorphisms

$$\text{Aut}_{\lambda_3} (a/p^{\lambda_1}/u_{\lambda_2})_1 \to \text{Aut} a/p^{\lambda_3}$$

and

$$\text{Aut}_{\lambda_3} (a/p^{\lambda_1}/u_{\lambda_2})_1 \to \text{Aut}(a/p^{\lambda_1}/u_{\lambda_3}),$$

respectively. That is, $N = \{ \tau \in \text{Aut}_{\lambda_3} (a/p^{\lambda_1}/u_{\lambda_2})_1 \mid \tau(a) \equiv a \mod p^{\lambda_3} \forall a \in a/p^{\lambda_2} \}$, and $\overline{N} = \{ \tau \in \text{Aut}_{\lambda_3} (a/p^{\lambda_1}/u_{\lambda_2})_1 \mid \tau(a) \sim a \mod p^{\lambda_1} \forall a \in a/p^{\lambda_2} \}$.

Thus we have a normal series

$$\text{Aut}_{\lambda_3} (a/p^{\lambda_1}/u_{\lambda_2})_1 \triangleright N \triangleright \overline{N} \triangleright 1.$$

**Theorem 30.** The following holds.

1. We have

$$\text{Aut}_{\lambda_3} (a/p^{\lambda_1}/u_{\lambda_2})_1 / N \cong \text{Aut} a/p^{\lambda_3}.$$

2. We have

$$N/\overline{N} \cong \begin{cases}
  k & \lambda_3 \geq 2, \\
  k^x & \lambda_3 = 1.
\end{cases}$$

3. We have

$$\overline{N} \cong Q_0 \times Q_0^{\lambda_1-\lambda_2} \times K.$$

Hence in particular $\overline{N}$ is abelian if $\lambda_3 \geq \frac{1}{2}\lambda_2$.

We can show that $N$ is abelian for certain types of $\lambda$:

**Proposition 31.** If $\lambda_3 > \frac{1}{2}\lambda_1$, then $N$ is abelian.
The structure of $\overline{N}$ resembles that of $K$; We obtain a decomposition of $\overline{N}$ similar to that of $K$:

**Proposition 32.** $\overline{N}$ decomposes into a direct product as

$$\overline{N} \cong V_0 \times V_1 \times \cdots \times V_{\lambda_1 - \lambda_3 - 1},$$

where each factor $V_i$ (defined for $0 \leq i \leq \lambda_1 - 1$) is given by

$$V_i = \{ \tau \in N \mid \tau(a) = a \forall a \in (o \cap q_i)/p^{\lambda_1}/u_{\lambda_2} \}.$$

**Lemma 33.** We have the following.

1. $V_i \cong Q_i - \lambda_1 + \lambda_2$, for $i \in \{ \lambda_1 - \lambda_2 + 1, \lambda_1 - \lambda_3 - 1 \}$.
2. $V_0 \cong \mathbb{Z}_{p^{m}} \times \mathbb{Z}_{\mathbb{Z}'}''$.
3. $V_0 \cong \mathbb{Z}_{p'}$.

Lastly, we consider the situation in which the residue field $k$ is the finite field $\mathbb{F}_q$. Then $\text{Aut} \ L(M)$ is evidently finite, and by the structural theorem we can compute the order of the group. There is not much to do for case $\lambda_2 = \lambda_3$, so assume $\lambda_2 > \lambda_3$. We start with computing the order $|Q_i|$. We can use either the $L$-sequence of $Q_i$ or $H$-decomposition. Let us choose the former this time:

$$|Q_i| = q^{(q-1)^i q + (q-1)q^2 + \cdots + (q-1)q^{\lambda_2 - \lambda_3 - i}}.$$

In particular, we get $|Q_0| = q^{-1 + q^{\lambda_2 - \lambda_3}}$. Since $K = \prod_{i=1}^{\lambda_2 - \lambda_3 - 1} Q_i$, we see that

$$|K| = \prod_{i=1}^{\lambda_2 - \lambda_3 - 1} |Q_i| = q^{|\sum_{i=1}^{\lambda_2 - \lambda_3 - 1} (-1 + q^i)}.$$

Also, by $L$-sequence or $H$-decomposition of $\overline{Q}_0$, we see that $|\overline{Q}_0| q^{\lambda_2 - \lambda_3} = |Q_0|$. So we compute:

$$|\overline{N}| = |\overline{Q}_0| \cdot |Q_0|^{\lambda_1 - \lambda_2} \cdot |K| = q^{-\lambda_2 + \lambda_3} q^{-1 + q^{\lambda_2 - \lambda_3}} \left( q^{-1 + q^{\lambda_2 - \lambda_3}} \right)^{\lambda_1 - \lambda_2} \cdot q^{|\sum_{i=1}^{\lambda_2 - \lambda_3 - 1} (-1 + q^i)}.$$