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ON THE AUTOMORPHISM GROUP OF THE SUBGROUP LATTICE
OF A FINITE ABELIAN p-GROUP; SOME GENERALIZATIONS

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ABSTRACT. The automorphism group $\text{Aut } \mathcal{L}(M)$ of the submodule lattice $\mathcal{L}(M)$ of a finite-length module $M$ over complete discrete valuation ring $\mathfrak{o}$ is studied. Let $\lambda = (\lambda_1, \cdots, \lambda_l)$ be the type of $M$. We show that for those $M$ with $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 1$, $\text{Aut } \mathcal{L}(M)$ can be analyzed by computing a certain subgroup of the bijections on a quotient of the scalar ring $\mathfrak{o}$. In particular, when the residue field $k = \mathfrak{o}/p$ is a finite field $\mathbb{F}_q$, we compute the order of the group.

1. OBJECTIVE

Let $\mathfrak{o}$ be a discrete valuation ring with the maximal ideal $\mathfrak{p}$, a prime element $\pi$ ($\text{i.e.}, \mathfrak{o}\pi = \mathfrak{p}$) and the valuation function $v: \mathfrak{o} \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}$. Let $k \cong \mathfrak{o}/\mathfrak{p}$ denote the residue field. Let $M$ be an $\mathfrak{o}$-module of finite length. Then, since $\mathfrak{o}$ is a principal ideal domain, $M$ can be written as a sum of cyclic $\mathfrak{o}$-submodules:

$$M \cong \mathfrak{o}/\mathfrak{p}^{\lambda_1} \oplus \cdots \oplus \mathfrak{o}/\mathfrak{p}^{\lambda_l},$$

with $\lambda = (\lambda_1, \cdots, \lambda_l)$ being some partition of a non-negative integer (That is, we have $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$). $\lambda$ is called the type of $M$. Now since we have $\mathfrak{o}/\mathfrak{p}^{i} \cong \overline{\mathfrak{o}}/\overline{\mathfrak{p}}^{i}$ where $\overline{\mathfrak{o}}$ is the completion of $\mathfrak{o}$ and $\overline{\mathfrak{p}}$ its maximal ideal, without loss of generality we can assume $\mathfrak{o}$ to be complete. Let $\mathcal{L}(M)$ denote the set of $\mathfrak{o}$-submodules of $M$. $\mathcal{L}(M)$ inherits a lattice structure by inclusion relation. Our main objective is to compute $\text{Aut } \mathcal{L}(M)$, the automorphism group of the lattice $\mathcal{L}(M)$, for such $\lambda$ as $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 1$.

When $\mathfrak{o} = \mathbb{Z}_p$, the ring of $p$-adic integers, $M$ becomes nothing but a finite abelian $p$-group and $\mathcal{L}(M)$ the subgroup lattice of $M$. This can be generalized by considering the case $\mathfrak{o} = W[\mathbb{F}_q]$, the ring of Witt vectors over the finite field $\mathbb{F}_q$, for $W[\mathbb{F}_p] \cong \mathbb{Z}_p$. Another example of $\mathfrak{o}$ is the ring $k[[t]]$ of formal power series in one variable $t$.

We call $e = (e_1, \cdots, e_l) \in M^l$ an ordered basis for $M$ if $M = \bigoplus_{i=1}^{l} \mathfrak{o}e_i$ and $\mathfrak{o}e_i \cong \mathfrak{o}/\mathfrak{p}^{\lambda_i}$. Let $e$ be fixed. We denote by $R(e)$ the set of $\mathcal{L}(M)$ satisfying

$$\begin{align*}
\varphi(\mathfrak{o}e_i) &= \mathfrak{o}e_i & \forall i \in [1, l] \\
\varphi(\mathfrak{o}(e_1 + e_i)) &= \mathfrak{o}(e_1 + e_i) & \forall i \in [2, l]
\end{align*}$$

In most cases it boils down to computing $R(e)$ in order to analyze $\text{Aut } \mathcal{L}(M)$, in the sense we describe as follows.

Since an automorphism of $\mathfrak{o}$-module $M$ induces an automorphism of the lattice $\mathcal{L}(M)$, we have the natural group homomorphism

$$\Theta: \text{Aut } M \rightarrow \text{Aut } \mathcal{L}(M).$$

It can be directly checked that $\text{Ker } \Theta \cong (\mathfrak{o}/\mathfrak{p}^{\lambda_1})^* \times$ and that $\text{Aut } M$ can be expressed in matrix form, as described in the sequel. Naturally $\mathcal{L}(M)$ contains a subgroup isomorphic to $\text{Aut } M/\text{Ker } \Theta$, and we let $\text{PAut } M$ denote this subgroup.

It turns out that $\text{Aut } \mathcal{L}(M)$ is a product of these two subgroups $R(e)$ and $\text{PAut } M$. Namely, we have

Lemma 1.

$$R(e) \cdot \text{PAut } M = \text{Aut } \mathcal{L}(M)$$

$$R(e) \cap \text{PAut } M = 1.$$
2. Historical Background

Let us mention the relation with earlier results. The structure of Aut $\mathcal{L}(M)$ is well-known for the case $\lambda_1 = \lambda_2 = \lambda_3$, which is essentially the result of Baer [2]. In this case, we have Aut $\mathcal{L}(M) \cong R(e) \ltimes \text{PAut } M$, and

$$R(e) \cong \text{Aut } o / p^{\lambda_3},$$

where Aut $o / p^{\lambda_3}$ is the group of automorphisms of ring $o / p^{\lambda_3}$. In particular, when $\lambda_1 = \cdots = \lambda_l = 1 (l \geq 3)$, $M$ becomes a vector space over the residue field $o$, and Aut $\mathcal{L}(M)$ is isomorphic to $PGL(l, k)$, the group of projective semi-linear automorphisms. This result is a variation of so-called the Fundamental Theorem of Finite Projective Geometry.

We next consider the case when the residue field of $o$ is the finite field $F_p$. Let $M = o / p \oplus o / p \cong F_p \oplus F_p$. Then Aut $\mathcal{L}(M)$ is isomorphic to the symmetric group $S_{p+1}$ and PAut $M$ isomorphic to the projective general linear group $PGL(2, p)$ (Note that $|PGL(2, p)| = (p + 1)(p - 1)$). In this case, $R(e)$ is a subgroup that fixes three points and isomorphic to $S_{p-2}$. More generally, for $M = \mathbb{Z}/p^{\lambda_3} \mathbb{Z}_p \oplus \mathbb{Z}/p^{\lambda_2} \mathbb{Z}_p$ ($o = \mathbb{Z}_p$ is the ring of $p$-adic integers), Holmes’ result [5] states that Aut $\mathcal{L}(M)$ is isomorphic to $S_{p}^{(\lambda_3 - 1)} \ltimes S_{p+1}$, where $S_{p}^{\omega}$ means $S_p \ltimes \cdots \ltimes S_p$ ($\omega$ times) and $\lambda_1 > \lambda_2 > \lambda_3$.

Holmes [5] also obtains a result for the case $\lambda_1 > \lambda_2 > \lambda_3 = 0$: Aut $\mathcal{L}(M) \cong G^2 \rtimes H^{\lambda_1 - 1}$, where $G = S_{p}^{\lambda_3}$ and $H = S_{p}^{(\lambda_2 - 1)} \ltimes S_{p-1}$.

There have been works to bridge the gap between Baer’s result and Holmes’. Costantini-Holmes-Zacher [3] and Costantini-Zacher [4] treated the case of abelian $p$-groups in a rather general framework. Yasuda [11] studied the case of finite abelian $p$-groups for $\lambda_1 > \lambda_2 = \lambda_3$ with explicit computation of $R(e)$ and Aut $\mathcal{L}(M)$. In this work, we shall treat the case $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 1$, in the general setting of finite-length modules over (complete) discrete valuation ring.

3. Notations and Notions

Here we give some supplementary definitions and notations. Put $q_i = p_i^i \setminus p_i^{i+1}$ for $i \geq 1$; i.e., $q_i = \{a \in o \mid v(a) = i\}$. We define $q_0 = o \setminus p = o^\times$, the set of invertible elements. For $a, b \in o$ such that $v(a) \geq v(b)$ ($b \neq 0$), there exists an element $x \in o$ such that $a = xb$. As $o$ is a domain, $x$ must be unique. We use the notation $\frac{a}{b} = x$.

Given a set $X$, Map$(X)$ denotes the set of maps $f : X \to X$. Sym$(X)$ denotes the set of bijections $f : X \to X$. Map$(X)$ forms a monoid with respect to function composition, whereas Sym$(X)$ forms a group. Given two sets $X$ and $Y$, we define $X^Y$ to be the set of maps $f : X \to Y$.

Let $G$ be a group, and $H$ a group acting on a set $X$. Let $f, g : X \to G$ be two maps, and define a map $f \circ g : X \to G$ by $f \circ g(x) = f(x) \cdot g(x)$ where $\cdot$ is the product in $G$. Then $G^X$ becomes a group with respect to this $\circ$. Let $h \in H$ and $f \in G^X$. We define a semidirect product $G^X \rtimes H$ with respect to the group homomorphism $H \to \text{Aut } G^X$ ($h \mapsto (f \mapsto fh^{-1})$). We write $G \rtimes H$ to denote this semidirect product, and call it the wreath product of $G$ and $H$.

We now give the description of the automorphism group Aut $M$ of an $o$-module $M$ in matrix form, as promised. Let $e$ be fixed. The action of $f \in \text{Aut } M$ is then determined by its action on $e = (e_1, \cdots, e_l)$. Write

$$f(e_j) = \sum_{i=1}^{l} a_{ij} e_i$$

Also, we remark that if $e$ and $e'$ are ordered base for $M$, then it is easily checked that $\varphi R(e) \varphi^{-1} = R(e')$, where $\varphi \in \text{PAut } M$ is the lattice automorphism induced by the module automorphism of $M$ defined by $e_i \mapsto e'_i$ ($1 \leq i \leq l$). Hence the isomorphism type of $R(e)$ does not depend on the choice of $e$. We content ourselves with computing $R(e)$ instead of computing Aut $\mathcal{L}(M)$ for our purpose.
and express $f$ as the matrix $(a_{ij})_{i,j=1}^{l}$. Rewriting $\lambda = (\lambda_{1}, \ldots, \lambda_{l}) = (d_{1}^{\alpha_{1}}, \ldots, d_{r}^{\alpha_{r}})$ ($d_{1} > \cdots > d_{r}$) (this means that $\lambda$ contains $m_r$-many components equal to $d_{r}$), $\text{Aut} M$ can be expressed in matrix form as

$\begin{pmatrix} GL_{m_{1}}(\mathfrak{o}/\mathfrak{p}^{d_{1}}) & \ldots & \text{Hom}((\mathfrak{o}/\mathfrak{p}^{d_{1}})^{\oplus m_{1}}, (\mathfrak{o}/\mathfrak{p}^{d_{1}})^{\oplus m_{1}}) \\ \vdots & \ddots & \vdots \\ \text{Hom}((\mathfrak{o}/\mathfrak{p}^{d_{1}})^{\oplus m_{1}}, (\mathfrak{o}/\mathfrak{p}^{d_{1}})^{\oplus m_{r}}) & \ldots & GL_{m_{r}}(\mathfrak{o}/\mathfrak{p}^{d_{r}}) \end{pmatrix}$

with respect to the ordered basis $e$. Here, the block matrix in the diagonal

$A \in GL_{m_{1}}(\mathfrak{o}/\mathfrak{p}^{d_{1}})$

is of size $m_{1} \times m_{1}$ and has elements of $\mathfrak{o}/\mathfrak{p}^{d_{1}}$ in its components, satisfying $\pi \not\mid \det A$. Also, the block matrix at $(i, j)$-position ($i \neq j$)

$A \in \text{Hom}((\mathfrak{o}/\mathfrak{p}^{d_{j}})^{\oplus m_{j}}, (\mathfrak{o}/\mathfrak{p}^{d_{j}})^{\oplus m_{j}})$

is of size $m_{i} \times m_{j}$ and in its components has elements of $\mathfrak{p}^{d_{i}} \cdot \min(d_{i}, d_{j}) (\mathfrak{o}/\mathfrak{p}^{d_{j}})$, that is, for $i < j$ ($\Rightarrow d_{i} > d_{j}$) elements of $\mathfrak{p}^{d_{i} - d_{j}} (\mathfrak{o}/\mathfrak{p}^{d_{j}})$, and for $i > j$ ($\Rightarrow d_{i} < d_{j}$) elements of $\mathfrak{o}/\mathfrak{p}^{d_{i}}$.

4. MAIN RESULTS

For the case $\lambda_{1} > \lambda_{2} = \lambda_{3}$, we can state our main result as follows:

**Theorem 2.** Assume $\lambda_{2} = \lambda_{3}$. Then $R(e)$ contains a normal subgroup $N$ such that

$R(e)/N \cong \text{Aut} \mathfrak{o}/\mathfrak{p}^{\lambda_{3}}$,

$N \cong \begin{cases} k^{\lambda_{1}-\lambda_{2}} & \lambda_{2} = \lambda_{3} > 2, \\ (k^{\times})^{\lambda_{1}-\lambda_{2}} & \lambda_{2} = \lambda_{3} = 1. \end{cases}$

The case $\lambda_{1} \geq \lambda_{2} > \lambda_{3}$ turns out to be rather complicated. The rest of this section is dedicated to explain our main result for this case.

Let $i \geq 1$. For $a, b \in \mathfrak{o}$, we write

$a \equiv b \pmod{\mathfrak{p}}$

to mean $a - b \in \mathfrak{p}^{i}$. With abuse of notation, we write $\mathfrak{p}^{i}$ also to denote this equivalence relation. Then obviously we have $\mathfrak{p} \succ \mathfrak{p}^{2} \succ \mathfrak{p}^{3} \succ \cdots$. On the other hand, put $u_{i} = 1 + \mathfrak{p}^{i} \subset \mathfrak{o}$ ($i \geq 1$). For $a, b \in \mathfrak{o}$, write

$a \sim b \pmod{u_{i}}$

if $a \in u_{i}b$. Clearly this defines an equivalence relation on $\mathfrak{o}$. Again with abuse of notation, we just write $u_{i}$ to denote this relation. Then note that we have $u_{i} \succ u_{2} \succ u_{3} \succ \cdots$. Also note that $u_{i} \succ u_{i+1}$ holds for all $i \geq 1$.

**Lemma 3.** The union of relations $\mathfrak{p}^{i} \cup u_{j}$ is an equivalence relation for all $i, j \geq 1$.

Hence we have $\mathfrak{p}^{i} \cup u_{j} = \mathfrak{p}^{i} \cup u_{j}$, and it makes sense to denote the quotient set by $\mathfrak{o}/\mathfrak{p}^{i} \cup u_{j} = \mathfrak{o}/\mathfrak{p}^{i} \cup u_{j}$ for all $i, j \geq 1$.

Now we proceed to the following lemma:

**Lemma 4.** Let $\varphi \in R(e)$ be given. There exist bijective maps $\tau : \mathfrak{o} \rightarrow \mathfrak{o}$ and $\sigma : \mathfrak{o} \rightarrow \mathfrak{o}$ such that

$\varphi(\mathfrak{o}e_{1} + e_{2}) = \sigma(\mathfrak{o}e_{1} + e_{2})$ and $\varphi(\mathfrak{o}e_{1} + \mathfrak{o}e_{2}) = \sigma(\mathfrak{o}e_{1} + \mathfrak{o}e_{2})$ for all $a \in \mathfrak{o}$. $\tau$ and $\sigma$ induce bijections $\tau : \mathfrak{o}/\mathfrak{p}^{\lambda_{3}}/u_{j} \rightarrow \mathfrak{o}/\mathfrak{p}^{\lambda_{3}}/u_{j}$ and $\sigma : \mathfrak{o}/\mathfrak{p}^{\lambda_{2}} \rightarrow \mathfrak{o}/\mathfrak{p}^{\lambda_{2}}$, respectively, which are uniquely determined by $\varphi$.

Let $\varphi \in R(e)$ be given and $\tau, \sigma$ as in the preceding lemma. We list in the following lemma some of the properties satisfied by $\tau$ and $\sigma$.

**Lemma 5.** We have

1. $\tau(1) \sim 1 \pmod{\mathfrak{p}^{\lambda_{1}} \cup u_{\lambda_{2}}}$, $\sigma(1) \equiv 1 \pmod{\mathfrak{p}^{\lambda_{2}}}$,
2. $\tau(\mathfrak{p}) \subset \mathfrak{p}$, $\sigma(\mathfrak{p}) \subset \mathfrak{p}$,
3. $\tau(ab) \sim \tau(a)\tau(b) \pmod{\mathfrak{p}^{\lambda_{3}} \cup u_{\lambda_{2}}}$ for all $a, b \in \mathfrak{o}$,
4. $\sigma(ab) \sim \sigma(a)\sigma(b) \pmod{\mathfrak{p}^{\lambda_{2}} \cup u_{\lambda_{3}}}$ for all $a, b \in \mathfrak{o}$,
5. $\tau(a - b) \sim \tau(a) - \tau(b) \pmod{\mathfrak{p}^{\lambda_{1}} \cup \mathfrak{p}^{\lambda_{2} + v(b)} \cup u_{\lambda_{3}}}$ for all $a, b \in \mathfrak{o}$,
Given three positive integers \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 1 \), let
\[
\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{p})
\]
denote the set of bijections \( \tau : \mathfrak{o} \to \mathfrak{o} \) that satisfy the following three conditions:

Valuation law: \( \tau(p) \subseteq p \),

Strict product law: \( \tau(ab) = \tau(a)\tau(b) \mod p^{\lambda_1} \cup u_{\lambda_2} \) for all \( a, b \in \mathfrak{o} \),

Difference law: \( \tau(a - b) = \tau(a) - \tau(b) \mod p^{\lambda_1} \cup p^{\lambda_3 + \nu(b)} \cup u_{\lambda_3} \) for all \( a, b \in \mathfrak{o} \).

In this section we shall prove that this set forms a group and a \( \tau \in \text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o}) \) induces a bijection \( \tau : \mathfrak{o}/p^{\lambda_1}/u_{\lambda_2} \to \mathfrak{o}/p^{\lambda_1}/u_{\lambda_2} \). It turns out that \( R(e) \) can be described by using this group.

**Lemma 6.** Let \( \tau : \mathfrak{o} \to \mathfrak{o} \) be a bijective map that satisfies the valuation law and the strict product law. Then we have
\[
\tau(p^i) = p^i
\]
for all \( i \in [0, \lambda_1] \); that is, we have
\[
v(\tau(a)) = v(a)
\]
for all \( a \in \mathfrak{o} \setminus p^{\lambda_1} \).

**Lemma 7.** Let \( i \leq \lambda_1 \) and \( j \leq \lambda_2 \). Let \( \tau : \mathfrak{o} \to \mathfrak{o} \) be a bijective map that satisfies the difference law and the condition \( v(\tau(a)) = v(a) \) for all \( a \in \mathfrak{o} \setminus p^{\lambda_1} \). For \( a, b \in \mathfrak{o} \), we have \( a \sim b \mod p^i \cup u_j \) if and only if \( \tau(a) \sim \tau(b) \mod p^i \cup u_j \). That is, there exists a unique bijective map \( \tau \) that makes the diagram below commutative:

\[
\begin{array}{ccc}
\mathfrak{o}/p^{\lambda_1}/u_{\lambda_2} & \xrightarrow{\tau} & \mathfrak{o}/p^{\lambda_1}/u_{\lambda_2} \\
\downarrow & & \downarrow \\
\mathfrak{o}/p^i/u_j & \xrightarrow{\tau} & \mathfrak{o}/p^i/u_j \\
\end{array}
\]

**Proposition 8.** \( \text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o}) \) forms a subgroup of \( \text{Sym}(\mathfrak{o}) \).

Now denote by \( \text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o})u_{\lambda_3} \) the stabilizer of \( u_{\lambda_3} \) in \( \text{Aut}_{\lambda_1, \lambda_2, \lambda_3} \). That is, \( \text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o})u_{\lambda_3} = \{ \tau \in \text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o}) \mid \tau(u_{\lambda_3}) = u_{\lambda_3} \} \).

Then define
\[
\Delta_{\lambda_3}^{-1}(\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o})u_{\lambda_3} \times \text{Aut}_{\lambda_2, \lambda_2, \lambda_3}(\mathfrak{o})u_{\lambda_3})
\]
to be the set of \( (\tau, \sigma) \in \text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o})u_{\lambda_3} \times \text{Aut}_{\lambda_2, \lambda_2, \lambda_3}(\mathfrak{o})u_{\lambda_3} \) satisfying the conditions
\[
\begin{cases}
\tau(a)^{-1} \equiv \sigma(a^{-1}) \mod p^{\lambda_3} & \forall a \in \mathfrak{o}^x, \\
\tau(a) \sim \sigma(a) \mod p^{\lambda_3} \cup u_{\lambda_3} & \forall a \in \mathfrak{o}.
\end{cases}
\]

Note that since \( \tau(a)\tau(a^{-1}) = 1 \mod p^{\lambda_1} \cup u_{\lambda_3} \) whence \( \tau(a)^{-1} \sim \tau(a^{-1}) \mod p^{\lambda_1} \cup u_{\lambda_3} \) for all \( a \in \mathfrak{o}^x \), the first condition \( \tau(a)^{-1} \equiv \sigma(a^{-1}) \mod p^{\lambda_3} \) implies the second condition \( \tau(a) \sim \sigma(a) \mod p^{\lambda_3} \cup u_{\lambda_3} \).

**Lemma 9.** The set
\[
\Delta_{\lambda_3}^{-1}(\text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o})u_{\lambda_3} \times \text{Aut}_{\lambda_2, \lambda_2, \lambda_3}(\mathfrak{o})u_{\lambda_3})
\]
forms a subgroup of \( \text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o})u_{\lambda_3} \times \text{Aut}_{\lambda_2, \lambda_2, \lambda_3}(\mathfrak{o})u_{\lambda_3} \).

We have observed that \( \tau \in \text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o}) \) induces a bijective map \( \tau : \mathfrak{o}/p^{\lambda_1}/u_{\lambda_2} \to \mathfrak{o}/p^{\lambda_1}/u_{\lambda_2} \). That is to say, there exists a natural group homomorphism \( \text{Aut}_{\lambda_1, \lambda_2, \lambda_3}(\mathfrak{o}) \to \text{Sym}(\mathfrak{o}/p^{\lambda_1}/u_{\lambda_2}) \).

Let us define
\[
\text{Aut}_{\lambda_2}(\mathfrak{o}/p^{\lambda_1}/u_{\lambda_2})
\]
\[
\text{Aut}_{\lambda_3}(\mathfrak{o}/p^{\lambda_2})
\]
to be the images of $\text{Aut}_{\lambda_1,\lambda_2,\lambda_3}(\mathfrak{a}) \rightarrow \text{Sym}(\mathfrak{a}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})$ and $\text{Aut}_{\lambda_2,\lambda_3,\lambda_3}(\mathfrak{a}) \rightarrow \text{Sym}(\mathfrak{a}/\mathfrak{p}^{\lambda_2})$, respectively. Furthermore, let

$$\begin{align*}
\text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \\
\text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_2})_1
\end{align*}$$

be the subgroups of the above two, corresponding to $\text{Aut}_{\lambda_1,\lambda_2,\lambda_3}(\mathfrak{a})_{u_{\lambda_2}}$ and $\text{Aut}_{\lambda_2,\lambda_3,\lambda_3}(\mathfrak{a})_{u_{\lambda_2}}$, respectively. Lastly, we denote by

$$\Delta_{\lambda_3}^{-1}(\text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \times \text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_2})_1)$$

the subgroup of $\text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \times \text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_2})_1$ that corresponds to $\Delta_{\lambda_3}^{-1}(\text{Aut}_{\lambda_1,\lambda_2,\lambda_3}(\mathfrak{a})_{u_{\lambda_2}} \times \text{Aut}_{\lambda_2,\lambda_3,\lambda_3}(\mathfrak{a})_{u_{\lambda_2}})$. Now we can state our:

**Theorem 10 (Main Isomorphism Theorem).** We have

$$R(\varepsilon) \cong \Delta_{\lambda_3}^{-1}(\text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \times \text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_2})_1)$$

if $\lambda_2 > \lambda_3$.

Note that one way of isomorphism

$$\Phi : R(\varepsilon) \rightarrow \Delta_{\lambda_3}^{-1}(\text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \times \text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_2})_1)$$

is already given, by sending $\Phi : \varphi \mapsto (\tau, \sigma)$. In order to compute $R(\varepsilon)$, this theorem allows us to compute $\Delta_{\lambda_3}^{-1}(\text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \times \text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_2})_1)$ instead. Let

$$\Lambda : \Delta_{\lambda_3}^{-1}(\text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \times \text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_2})_1) \rightarrow \text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1$$

be the "projection" map to the first component; i.e., $\Lambda : (\tau, \sigma) \mapsto \tau$. Then $\text{Ker}\ \Lambda$ is the set of $(1, \sigma)$ satisfying

$$\left\{ \begin{array}{l}
\sigma(a) \equiv a \mod \mathfrak{p}^{\lambda_2}, \\
\sigma(a) \sim a \mod \mathfrak{p}^{\lambda_2} \lor \mathfrak{u}_{\lambda_3}, \\
\mathfrak{u}_{\lambda_3} \in \mathfrak{p}^{\lambda_2}.
\end{array} \right.$$ 

Let $K$ be the kernel of the natural map $\text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_2})_1 \rightarrow \mathfrak{a}^{\times}/\mathfrak{p}^{\lambda_2}$, that is,

$$K = \left\{ \sigma \in \text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_2})_1 \mid \sigma(a) = a \text{ for all } a \in \mathfrak{a}^{\times}/\mathfrak{p}^{\lambda_2} \right\}.$$

**Lemma 11.** We have

$$\text{Ker}\ \Lambda \cong K.$$

We shall show that the group in question, $\Delta_{\lambda_3}^{-1}(\text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \times \text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_2})_1)$, is isomorphic to a semidirect product of $K$ and the first component $\text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1$;

**Proposition 12.** The sequence

$$1 \rightarrow K \rightarrow \Delta_{\lambda_3}^{-1}(\text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \times \text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_2})_1) \rightarrow \text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \rightarrow 1$$

is exact and splitting. In other words, we have

$$\Delta_{\lambda_3}^{-1}(\text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \times \text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_2})_1) \cong \text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1 \ltimes K.$$

This result divides our investigation into two parts: the analysis of the structure of $K$ and that of $\text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_1}/\mathfrak{u}_{\lambda_2})_1$. We begin with the former.

Recall that

$$K = \left\{ \sigma \in \text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_2})_1 \mid \sigma(a) = a \equiv a \in (\mathfrak{a}^{\times}/\mathfrak{p}^{\lambda_2}) \right\}$$

$$= \left\{ \sigma \in \text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_2})_1 \mid \sigma(a) = a \equiv a \in (\mathfrak{a}^{\times}/\mathfrak{p}^{\lambda_2}) \text{ and } \sigma(a) \sim a \mod \mathfrak{p}^{\lambda_2} \lor \mathfrak{u}_{\lambda_3}, \mathfrak{u}_{\lambda_3} \in \mathfrak{p}^{\lambda_2} \right\}.$$

For the sake of convenience, we shall analyze groups slightly larger than $K$; namely,

$$\hat{K} = \left\{ \sigma \in \text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_2})_1 \mid \sigma(a) \sim a \mod \mathfrak{p}^{\lambda_2} \lor \mathfrak{u}_{\lambda_3}, \mathfrak{u}_{\lambda_3} \in \mathfrak{p}^{\lambda_2} \right\},$$

and

$$\hat{K}_1 = \left\{ \sigma \in \text{Aut}_{\lambda_3}(\mathfrak{a}/\mathfrak{p}^{\lambda_2})_1 \mid \sigma(a) \sim a \mod \mathfrak{p}^{\lambda_2} \lor \mathfrak{u}_{\lambda_3}, \mathfrak{u}_{\lambda_3} \in \mathfrak{p}^{\lambda_2} \right\}.$$

Of course we have $\hat{K}_1 = \{ \sigma \in \hat{K} \mid \sigma(1) = 1 \}$. 

{$\lambda_2 \succ \lambda_3$.}
Proposition 13. $\tilde{K}$ decomposes into a direct product as

$$\tilde{K} \cong Q_0 \times Q_1 \times \cdots \times Q_{\lambda_2 - \lambda_3 - 1}$$

where each factor $Q_i$ (defined for $0 \leq i \leq \lambda_2 - 1$) is given by

$$Q_i = \left\{ \sigma \in \tilde{K} \mid \sigma(a) = a \text{ for all } a \in (a \setminus q_i)/\mathfrak{p}^{\lambda_3} \right\}.$$  

Corollary 14. $\tilde{K}_1$ and $K$ decompose into direct products as

$$\tilde{K}_1 \cong \overline{Q}_0 \times Q_1 \times \cdots \times Q_{\lambda_2 - \lambda_3 - 1} \text{ and}$$

$$K \cong Q_1 \times Q_2 \times \cdots \times Q_{\lambda_2 - \lambda_3 - 1},$$

respectively, where

$$\overline{Q}_0 = \left\{ \sigma \in \tilde{K}_1 \mid \sigma(a) = a \forall a \in \mathfrak{p}/\mathfrak{p}^{\lambda_3} \right\} = \left\{ \sigma \in Q_0 \mid \sigma(1) = 1 \right\}.$$  

We now focus on the calculation of each $Q_i$. First, we give a description of generators of $Q_i$. We begin with the following lemma.

Lemma 15. Let $\sigma \in \tilde{K}$ and $0 \leq j \leq \lambda_2 - \lambda_3$. If $a \equiv b \mod \mathfrak{p}^j$, then $\sigma(a) - a \equiv \sigma(b) - b \mod \mathfrak{p}^{j+\lambda_3}$.

We apply this lemma particularly to $Q_i$ ($0 \leq i \leq \lambda_2 - \lambda_3 - 1$). For $j \in [i+1, \lambda_2 - \lambda_3]$, let $(\mathfrak{p}^{j+\lambda_3-1}/\mathfrak{p}^{\lambda_2})^{\sigma_i}/\mathfrak{p}^j$ denote the set of maps $z : q_i/\mathfrak{p}^j \to \mathfrak{p}^{j+\lambda_3-1}/\mathfrak{p}^{\lambda_2}$. Note that since $\mathfrak{p}^{j+\lambda_3-1}/\mathfrak{p}^{\lambda_2}$ is an abelian group, so becomes $(\mathfrak{p}^{j+\lambda_3-1}/\mathfrak{p}^{\lambda_2})^{\sigma_i}/\mathfrak{p}^j$ naturally. Given $j \in [i+1, \lambda_2 - \lambda_3]$, define a map $g_{j,z}(a) : a \to a + z(a \mod \mathfrak{p}^j)$ by

$$g_{j,z}(a) = \begin{cases} a + z(a \mod \mathfrak{p}^j) & a \in q_i, \\ a & \text{otherwise.} \end{cases}$$

We shall show that $g_{j,z}$ is in $Q_i$; more precisely,

Proposition 16. We have

$$Q_i = \langle g_{j,z} : j \in [i+1, \lambda_2 - \lambda_3], z \in (\mathfrak{p}^{j+\lambda_3-1}/\mathfrak{p}^{\lambda_2})^{\sigma_i}/\mathfrak{p}^j \rangle,$$

where $(\mathfrak{p}^{j+\lambda_3-1})^{\sigma_i}/\mathfrak{p}^j$ denotes the set of maps $z : q_i/\mathfrak{p}^j \to \mathfrak{p}^{j+\lambda_3-1}/\mathfrak{p}^{\lambda_2}$.

Using these generators, we give two sorts of descriptions of $Q_i$. The former turns out to be useful particularly for the case $\lambda_3 = 1$, whereas the latter being useful for the case $\lambda_3 \geq \frac{1}{2}\lambda_2$.

For $j \in [i+1, \lambda_2 - \lambda_3 + 1]$, define

$$L_i^{(j)} = \{ \sigma \in Q_i \mid \sigma(a) \equiv a \mod \mathfrak{p}^{j+\lambda_3-1} \forall a \in \mathfrak{p}/\mathfrak{p}^{\lambda_3} \}.$$  

Then clearly we have $Q_i \triangleright L_i^{(j)}$ for each $j$ whence obtain a chain of normal subgroups $L_i^{(j)}$.

Proposition 17. We have a normal series of $Q_i$:

$$Q_i = L_i^{(i+1)} \triangleright L_i^{(i+2)} \triangleright \cdots \triangleright L_i^{(\lambda_2 - \lambda_3 + 1)} = 1,$$

where the factors of the series are given by

$$L_i^{(j)}/L_i^{(j+1)} \cong (a/\mathfrak{p})^{\mathfrak{p}_i}/\mathfrak{p}^j,$$

for all $j \in [i+1, \lambda_2 - \lambda_3]$.

Proposition 18. We have

$$L_i^{(j)} = \langle g_{n,z} : n \in [j, \lambda_2 - \lambda_3], z \in (\mathfrak{p}^{n+\lambda_3-1}/\mathfrak{p}^{\lambda_2})^{\sigma_i}/\mathfrak{p}^{n} \rangle = \langle g_{n,z} : n \in [j, \lambda_2 - \lambda_3], z \in (\mathfrak{p}^{n+\lambda_3-1})^{\sigma_i}/\mathfrak{p}^{n} \rangle.$$  

Lemma 19. We have the exact sequence of groups

$$1 \to L_i^{(\lambda_2 - \lambda_3)} \to Q_i \to Q_{i+1} \to 1.$$  

This sequence splits if $\lambda_3 = 1$. 


**Proposition 20.** If $\lambda_3 = 1$, then $Q_i$ decomposes into a wreath product as
\[ Q_i \simeq k(i)k \cdots k(i) \]
where $1$ is to act on $k^\times$ trivially and $k$ on $k$ by addition. To put it more concisely,
\[ Q_i \simeq (k^{(\lambda_2 - 1)})^{k^\times}. \]

We present another way of describing the structure of $Q_i$. In order to do this, let us define
\[ U_i^{(j)} = \{ \sigma \in Q_i \mid \sigma(a) - a \equiv \sigma(b) - b \pmod{p^{\lambda_2}} \text{ if } a \equiv b \pmod{p^j} \}, \]
where $i \leq j \leq \lambda_2 - \lambda_3$. This gives us a filtration of $Q_i$:
\[ Q_i = U_i^{(\lambda_2 - \lambda_3)} \supset U_i^{(\lambda_2 - \lambda_3 - 1)} \supset \cdots \supset U_i^{(i+1)} \supset U_i^{(i)} = T_i. \]

Here, $T_i$ is the group of translations, i.e.,
\[ T_i = \{ \sigma \in Q_i \mid \sigma(a) - a = \sigma(b) - b \text{ for all } a, b \in \mathfrak{o}/p^{\lambda_2} \}. \]

**Lemma 21.** We have
\[ U_i^{(j)} = \langle g_{n,z} \rangle_{n \in [i+1,j], z \in \mathfrak{o}/p^{\lambda_2}}. \]

Now for each $j \in [i+1, \lambda_2 - \lambda_3]$ define
\[ H_i^{(j)} = \{ \sigma \in U_i^{(j)} \mid \sigma(a) = a \text{ if } a_{j-1} = 0 \in S \}, \]
with $a \in \mathfrak{o}/p^{\lambda_2}$ being written as $a = \sum_{n=1}^{\lambda_2-1} a_n \pi^n$ with $a_n \in S$. Obviously the definition of $H_i^{(j)}$ depends on the choice of $S$.

**Proposition 22.** The subgroups $H_i^{(j)}$ are abelian; more precisely, we have
\[ H_i^{(j)} \cong \left( \frac{\mathfrak{o}/p^{\lambda_2-j}}{\mathfrak{o}/p^{\lambda_2}} \right)^{k^\times} \cong \left( \frac{\mathfrak{o}/p^{\lambda_2-\lambda_3-j+1}}{\mathfrak{o}/p^{\lambda_2-\lambda_3-j}} \right)^{k^\times}. \]

**Proposition 23.** $Q_i$ decomposes into a product of abelian subgroups $H_i^{(j)} \subset Q_i (i+1 \leq j \leq \lambda_2 - \lambda_3)$ as
\[ Q_i = H_i^{(i+1)}H_i^{(i+2)} \cdots H_i^{(\lambda_2 - \lambda_3)}, \]
with the properties
\[ \left\{ \left( H_i^{(i+1)}H_i^{(i+2)} \cdots H_i^{(j)} \right) \cap H_i^{(j+1)} = 1, \right\} \]
\[ \left( H_i^{(i+1)}H_i^{(i+2)} \cdots H_i^{(j)} \right) H_i^{(j+1)} = H_i^{(j+1)} \left( H_i^{(i+1)}H_i^{(i+2)} \cdots H_i^{(j)} \right). \]

**Lemma 24.** If $i + \lambda_3 \geq \lambda_2 - \lambda_3$, then we have
\[ Q_i \cong \bigoplus_{j=i}^{\lambda_2-\lambda_3} \left( \frac{\mathfrak{o}/p^{\lambda_2-j}}{\mathfrak{o}/p^{\lambda_2}} \right)^{k^\times}. \]

**Proposition 25.** If $\lambda_3 \geq \frac{1}{2}(\lambda_2 - 1)$, then $K$ is abelian and
\[ K \cong \bigoplus_{i=1}^{\lambda_2-\lambda_3-1} \bigoplus_{j=1}^{\lambda_2-\lambda_3-1} \left( \frac{\mathfrak{o}/p^{\lambda_2-j}}{\mathfrak{o}/p^{\lambda_2-j}} \right)^{k^\times}. \]

Also, if $\lambda_3 \geq \frac{1}{2}\lambda_2$, then $Q_0$ is abelian and
\[ Q_0 \cong \bigoplus_{j=0}^{\lambda_2-\lambda_3-1} \left( \frac{\mathfrak{o}/p^{\lambda_2-j}}{\mathfrak{o}/p^{\lambda_2-j}} \right)^{k^\times}. \]
Now we describe the structure of $\overline{Q}_0$, which we shall need later in computing $\text{Aut}_{\lambda_3} (o/p^{\lambda_1}/u_{\lambda_2})_1$. So for each $j \in [1, \lambda_2 - \lambda_3]$, put

$$L_0^{(j)} = \{ \sigma \in Q_0 | \sigma(a) \equiv a \mod p^j a \in o/p^{\lambda_2} \}.$$  

Evidently we have $\overline{Q}_0 \triangleright L_0^{(j)}$ for each $j$.

**Proposition 26.** We have a normal series of $\overline{Q}_0$:

$$\overline{Q}_0 = T_0^{(1)} \triangleright T_0^{(2)} \triangleright \cdots \triangleright T_0^{(\lambda_2 - \lambda_3 + 1)} = 1,$$

where the factors of the series are given by

$$T_0^{(j)}/T_0^{(j+1)} \cong (o/p^{\lambda_3})^{(o^x \setminus \{1\})/p^j}.$$

**Proposition 27.** Assume $\lambda_3 = 1$. Then $\overline{Q}_0$ decomposes into a semidirect product as

$$\overline{Q}_0 \cong k^{(o^x \setminus \{1\})/p^{\lambda_2}} \times k^{(o^x \setminus \{1\})/p^{\lambda_2 - 1}} \times \cdots \times k^{(o^x \setminus \{1\})/p}.$$

**Proposition 28.** $\overline{Q}_0$ decomposes into a product of abelian subgroups as

$$\overline{Q}_0 \cong \overline{H}_0^{(1)} H_0^{(2)} \cdots H_0^{(\lambda_2 - \lambda_3)}$$

where

$$\overline{H}_0^{(1)} = \{ \sigma \in H_0^{(1)} | \sigma(1) = 1 \} \cong (o/p^{\lambda_2 - \lambda_3})^{k^x \setminus \{1\}}.$$

**Proposition 29.** If $\lambda_3 \geq \frac{1}{2} \lambda_2$, then

$$\overline{Q}_0 \cong (o/p^{\lambda_3})^{k^x \setminus \{1\}} \oplus (o/p^{\lambda_2 - \lambda_3}) \oplus \cdots \oplus (o/p^{\lambda_2 - \lambda_3 - j}) \oplus (o/p^{j})^x.$$  

Now we shift our attention to calculating the structure of $\text{Aut}_{\lambda_3} (o/p^{\lambda_1}/u_{\lambda_2})_1$. Let $N$ and $\overline{N}$ be the kernels of the natural homomorphisms

$$\text{Aut}_{\lambda_3} (o/p^{\lambda_1}/u_{\lambda_2})_1 \rightarrow \text{Aut} o/p^{\lambda_3}$$
and

$$\text{Aut}_{\lambda_3} (o/p^{\lambda_1}/u_{\lambda_2})_1 \rightarrow \text{Aut}(o/p^{\lambda_1}/u_{\lambda_2}),$$

respectively. That is,

$$N = \{ \tau \in \text{Aut}_{\lambda_3} (o/p^{\lambda_1}/u_{\lambda_2})_1 | \tau(a) \equiv a \mod p^\lambda a \in o/p^{\lambda_1} \},$$

and

$$\overline{N} = \{ \tau \in \text{Aut}_{\lambda_3} (o/p^{\lambda_1}/u_{\lambda_2})_1 | \tau(a) \sim a \mod p^\lambda \forall a \in p^{\lambda_1}/u_{\lambda_2} \}.$$  

Thus we have a normal series

$$\text{Aut}_{\lambda_3} (o/p^{\lambda_1}/u_{\lambda_2})_1 \triangleright N \triangleright \overline{N} \triangleright 1.$$

**Theorem 30.** The following holds.

(1): We have

$$\text{Aut}_{\lambda_3} (o/p^{\lambda_1}/u_{\lambda_2})_1 / N \cong \text{Aut} o/p^{\lambda_3}.$$  

(2): We have

$$N/\overline{N} \cong \begin{cases} k & \lambda_3 \geq 2, \\ k^x & \lambda_3 = 1. \end{cases}$$

(3): We have

$$\overline{N} \cong \overline{Q}_0 \times Q_0^{\lambda_1 - \lambda_2} \times K.$$

Hence in particular $\overline{N}$ is abelian if $\lambda_3 \geq \frac{1}{2} \lambda_2$.

We can show that $N$ is abelian for certain types of $\lambda$:

**Proposition 31.** If $\lambda_3 > \frac{1}{2} \lambda_1$, then $N$ is abelian.
The structure of $\overline{N}$ resembles that of $K$; We obtain a decomposition of $\overline{N}$ similar to that of $K$:

**Proposition 32.** $\overline{N}$ decomposes into a direct product as

$$\overline{N} \cong V_0 \times V_1 \times \ldots \times V_{\lambda_1-\lambda_2-1},$$

where each factor $V_i$ (defined for $0 \leq i \leq \lambda_1 - 1$) is given by

$$V_i = \{ \tau \in N \mid \tau(a) = a^q \in (o \cap q_i)/p^{\lambda_1}/u_{\lambda_2} \}.$$

**Lemma 33.** We have the following.

1. $V_i \cong V_{\lambda_2 + i}$ for $i \in [\lambda_2 - \lambda_1 + 1, \lambda_2 - \lambda_3 - 1]$,
2. $V_1 \cong V_{\lambda_2 - 1} \cong V_{\lambda_2 - \lambda_3 - 1}$, and
3. $V_{\lambda_2 - \lambda_3 - 1} \cong V_0$ where $\lambda_1 > \lambda_2$.

Lastly, we consider the situation in which the residue field $k$ is the finite field $\mathbb{F}_q$. Then Aut $L(M)$ is evidently finite, and by the structural theorem we can compute the order of the group. There is not much to do for case $\lambda_2 = \lambda_3$, so assume $\lambda_2 > \lambda_3$. We start with computing the order $|Q_0|$. We can use either the $L$-sequence of $Q_i$ or $H$-decomposition. Let us choose the former this time:

$$|Q_i| = q^{(q-1)+(q-1)q+(q-1)q^2+\ldots+(q-1)q^{k-2}-(q^{k_i})-1} = q^{-1+q^{\lambda_2}-q^i}.$$

In particular, we get $|Q_0| = q^{-1+q^{\lambda_2}-q^i}$. Since $K = \prod_{i=1}^{\lambda_1-\lambda_2-1} Q_i$, we see that

$$|K| = \prod_{i=1}^{\lambda_2-\lambda_3-1} |Q_i| = q^{\sum_{i=1}^{\lambda_2-\lambda_3-1} (-1+q^i)} = q^{-\lambda_2+\lambda_3+1+\sum_{i=1}^{\lambda_2-\lambda_3-1} q^i}.$$

Also, by $L$-sequence or $H$-decomposition of $\overline{Q}_0$, we see that $|\overline{Q}_0| = q^{\lambda_2-\lambda_3} = |Q_0|$. So we compute:

$$|N| = |\overline{Q}_0| \cdot |Q_0|^{\lambda_1-\lambda_2} \cdot |K|$$

$$= q^{-\lambda_2+\lambda_3} q^{-1+q^{\lambda_2}-\lambda_3} \left(q^{-1+q^{\lambda_2}-\lambda_3}\right)^{\lambda_1-\lambda_2} \cdot q^{\sum_{i=1}^{\lambda_2-\lambda_3-1} (-1+q^i)}$$

$$= q^{-\lambda_2+\lambda_3} q^{-1+q^{\lambda_2}-\lambda_3} q^{-\lambda_1+\lambda_3+(\lambda_1-\lambda_2)q^{\lambda_2}-\lambda_3} q^{-\lambda_2+\lambda_3+1+\sum_{i=1}^{\lambda_2-\lambda_3-1} q^i}$$

$$= q^{-\lambda_2+\lambda_1+2\lambda_3+(\lambda_1-\lambda_2)q^{\lambda_1-\lambda_2}+\sum_{i=1}^{\lambda_2-\lambda_3} q^i}.$$

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