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瞬間的動作を受けた後の密度成層完全流体の支配方程式系導出

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1. Introduction

Let us consider the stably and weakly stratified ideal fluid under the uniform gravitational acceleration. Various characteristic flows can be seen owing to the existence of restoring force caused by the difference of buoyancy and gravitational forces. In order to analyze these flows, many studies have been done. The impulsive-start problem is one of them. The examples of these flows are, an impulsive start of uniform flow over topography, an impulsive start of sink flow, an impulsive movement of obstacle in a stratified fluid at uniform velocity, and so on. The original set of governing equations and initial condition for these problems is the following:

Governing eqations : The Euler set of equations for a compressible fluid
Initial condition : A state of rest

However, many prior studies use the following set for analyzing these problems:

Governing eqations: The Euler set of equations for an incompressible fluid
Initial condition: Flow velocity is divergence free, momentum vector is rotation free, where the boundary condition just after the start of the impulsive motion is used, and the density profile is that of state at rest.

The latter set is that for an incompressible fluid and it is different from the former original set for a compressible fluid. Therefore, if we want to use the latter incompressible set, it must be derived in advance from the former original compressible set systematically under the suitable physical assumptions. When this derivation is completed, we can insist that the solution of the incompressible set represents the fluid motion we are now considering. In prior study, only its equations were derived (Spiegel & Veronis [1960]). However, no consideration is given to its initial condition which is important in the analysis of the general initial-value problem. Moreover, the systematic derivation on the basis of the series expansion has not been done. We can say, therefore, that the derivation of the incompressible set of equations and initial condition has not yet been done systematically on the basis of the former original set of compressible type.

In this paper, this problem is resolved. That is, through a systematic procedure from the original set for a compressible fluid, we derive the sets of governing equations for an incompressible fluid and the corresponding initial condition not only at the lowest order but also for the higher orders. The physical assumptions are the following five: (i) the fluid is ideal; (ii) the geometry of the boundary is smooth; (iii) the fluid is at rest, weakly stratified, and satisfies the hydrostatic condition before the impulsive motion starts; (iv) the impulsive motion is defined as the continuous change of the flow velocity on the boundary from zero to a given stationary value during the time for the sound wave to proceed over the distance of the fluid depth; (v) the flow region is open so that the sound wave is propagated away. In what follows, for the sake of specific explanation, we consider a flow initiated by the discharge of the fluid, or the sink flow. If the above five assumptions are satisfied, other impulsive-start problems like the flow over topography and the flow past a body can be treated in a similar way. That is, the flow speed of the discharge imposed on the boundary (or sink) should be replaced by that of the flow over topography or the normal velocity component to the boundary of a moving body.

In the meantime for the sake of concise explanation, the fluid is assumed to be a gas. So the sound speed is of the order of the square root of the ratio of the pressure to the density and its dependency on the pressure and the density is moderate, that is, its derivative with respect to the pressure or the density is of the order of the ratio of the sound speed to the pressure or the density. Moreover we assume that the stratification is caused thermodynamically and not by the concentration of solvent or species. When a fluid is liquid, or the above condition for the sound speed is not satisfied, we can use the result of the gas with a straightforward manipulation. This point is discussed in Section 4. The case where the fluid is subject to stratification by the concentration of solvent or species without diffusion is also discussed in Section 4.

2. Problem

Let us consider an ideal fluid between two parallel infinite plates at rest located at $X_3 = 0$ and d, where X_i is the rectangular space coordinate system (see figure 1). The fluid is subject to a uniform gravitational force g in the negative X_3 direction, i.e., in the direction normal to the plates. The fluid is at rest and satisfies the following hydrostatic condition:

$$\frac{\partial p}{\partial X_i} = -\rho g \delta_{i3}, \tag{1}$$

where p is the pressure of the fluid, ρ is the density. This condition determines the pressure gradient if the density profile is given. The density variation results not only from pressure variation but also from temperature variation. Here the stratified fluid with this density variation is assumed to be stable [or $\partial\rho/\partial X_3 < -\rho g/(\text{sound speed})^2$], and the magnitude of this density gradient is smaller than ρ_0/d, where the subscript 0 indicates the value on the lower plate $X_3 = 0$:

$$\frac{\max|\partial\rho/\partial X_3|}{\rho_0/d} = O(\varepsilon), \tag{2}$$

where ε is a given small parameter characterizing the degree of stratification. In contrast, the maximum pressure difference in the X_3 direction is of the order of $\rho_0 g d$ from (1) and the density change due to this pressure change is of the order of $\rho_0 g d / a_0^2$ where a_0 is the sound speed of the fluid on the lower plate. Here this density change is assumed to be of the order of ε^M relative to ρ_0, i.e.,

$$\frac{gd}{a_0^2} = \varepsilon^M G, \tag{3}$$

where G is a constant of the order of unity and M is a natural number ($M \geq 1$). The left-hand side of (3) is identical with the inverse of the Froude number.

Now consider intakes of smooth geometrical shape set up in the fluid. We are interested in the flow after starting the discharge with suction velocity profile $V(X_{wi},\tau)$ normal to the intake boundary, where X_{wi} is the space coordinate of the intake and τ is time. $V(X_{wi},\tau)$ is zero for $\tau<0$. It then continuously varies for $0<\tau\leq O(d/a_0)$, or during the time for the sound wave to proceed over the distance d. After this variation it keeps a stationary value. Its length scale of variation is assumed to be the distance of the fluid depth d, and its magnitude is of the order of ε^k relative to the sound speed:

$$\frac{\max|V|}{a_0}=O(\varepsilon^k), \tag{4}$$

where k is a real positive constant.

The problem is to systematically derive simplified sets of equations that describe the asymptotic behavior for small ε of the fluid motion after the start of discharge of the weakly stratified fluid. For that purpose, first consider a system that deviates only slightly from an initial state at rest such that the linearized Euler set of equations is valid for the analysis. It will be made clear that this case corresponds to the case where the parameter range for k and M is given by

(I) $2k>>M+1$ (or $2k>>M$).

It will be useful here to note the physical meaning of this parameter range. The magnitude of ε^{2k} represents the effect of convection, since it is of the order of the square of the suction speed V. The magnitude of ε^{M+1} represents the effect of buoyancy, since it is of the order of the product of the gravitational force and the maximum density difference at the initial state. Therefore, the physical meaning of (I) is that the characteristic convective force is much smaller than that of buoyancy.

Next we consider the case

(II) $2k=M+1$.

The physical meaning of this parameter range is that the characteristic convective force is comparable with that of buoyancy. Therefore, the nonlinear convective terms will be the same order quantities as the buoyancy term and should be retained in the analysis, that is, the analysis has to be carried out on the basis of the original (nonlinear) Euler set of equations. In the above two parameter ranges (I) and (II), the buoyancy force is dominant or is comparable with that of convection. So the set of Boussinesq equations will be derived as their leading-order set.

Lastly, we consider the case where the buoyancy effect is not important. The parameter range for k and M is given by

(III) $2k<<M+1$ (or $2k<<M$).

In this case, the set of equations for a homogeneous incompressible fluid will be derived as the leading-order set, and the resulting flow will be irrotational at the leading order. In the following analysis, because of the limitation for the number of pages, only the case (I) is presented (Section 3).

3. Linear theory (Case I : $2k>>M$)

3.1 Basic equation

The following variables are introduced: the time $\tau=(d/a_0)t$, the rectangular space coordinate $X_i=dx_i$, the space coordinate of the intake $X_{wi}=dx_{wi}$, the flow velocity a_0u_i, the suction speed $V = a_0 \upsilon$, the pressure $\rho_0 a_0^2(\bar{p}+p')$, and the density $\rho_0(\bar{\rho}+\rho')$, where the variables with overbar indicate those evaluated at the initial state at rest, i.e., $\bar{p}$ and $\bar{\rho}$ are the pressure and the density at the initial state. They are quantities of the order of unity and given functions of x_3 whose length scale of variation is the distance of the fluid depth d, or in dimesionless form, the distance x_3 of the order of unity. From (1)-(3), they satisfy the following relations:

$$\frac{d\bar{p}}{dx_3} = -\varepsilon^M G\bar{\rho}, \quad \text{(Hydrostatic condition)} \tag{5}$$

$$\bar{\rho} = 1 + O(\varepsilon), \tag{6}$$

In contrast, the variables with prime represent deviations from the corresponding values at the initial state, i.e., p' and ρ' are the deviations of pressure and density, respectively. Here we consider the behavior of the fluid that slightly deviates from the initial state at rest, that is, the case $|u_i| << 1$, $p' << 1$, $\rho' << 1$. Then, the mathematical problem is the following initial-boundary-value problem of the linearized Euler set of equations for a compressible fluid: The nondimensional form of the basic equations is

$$\frac{\partial \rho'}{\partial t} + u_3 \frac{d\bar{\rho}}{dx_3} + \bar{\rho}\frac{\partial u_i}{\partial x_i} = 0, \tag{7a}$$

$$\bar{\rho}\frac{\partial u_i}{\partial t} = -\frac{\partial p'}{\partial x_i} - \varepsilon^M G\delta_{i3}\rho', \tag{7b}$$

$$\frac{\partial \rho'}{\partial t} + u_3 \frac{d\bar{\rho}}{dx_3} = \frac{1}{\bar{a}^2}\left(\frac{\partial p'}{\partial t} + u_3 \frac{d\bar{p}}{dx_3}\right), \tag{7c}$$

where $\bar{a}^2$, representing the square of sound speed nondimensionalized by a_0^2, is a moderately varying function of $\bar{p}$ and $\bar{\rho}$ whose magnitude is of the order of unity [$\bar{a}^2 = O(1)$, $\partial\bar{a}^2/\partial\bar{p} = O(1)$, $\partial\bar{a}^2/\partial\bar{\rho} = O(1)$]:

$$\bar{a}^2 = \bar{a}^2(\bar{p}, \bar{\rho}). \tag{8}$$

The initial condition is

$$u_i = 0, \tag{9a}$$

$$p' = 0, \tag{9b}$$

$$\rho' = 0. \tag{9c}$$

The boundary condition is

$$u_i n_i = \upsilon \text{ at } x_i = x_{wi}, \tag{10a}$$

$$u_i n_i = 0 \text{ at } x_3 = 0, 1. \tag{10b}$$

where $\upsilon = V/a_0 << 1$ is a moderately varying function of x_{wi} and t that varies from zero to a given stationary value during the nondimensional time $0 < t \le O(1)$. n_i is the unit normal vector to the boundary.

3.2 Fluid motion at the start of discharge

Variation of the suction speed takes place continuously during the dimensionless time of the order of unity, or in dimensional form, during the time for the sound wave to proceed over the distance of the fluid depth. The fluid motion at the start of discharge is, therefore, described by the set of acoustic equations at the leading order. Thus, the magnitude of the unknowns u_i,

p' and ρ' is expected to be the same order as that of the nondimensional suction speed υ. Bearing this in mind, we consider the fluid motion whose appreciable variation occurs in the distance of the fluid depth d and the time for the sound wave to proceed over the distance of the fluid depth, i.e., in the dimensionless distance and the dimensionless time of the order of unity [$\partial\phi/\partial x_i = O(\phi)$, $\partial\phi/\partial t = O(\phi)$ with $\phi = u_i, p', \rho'$]. The asymptotic behavior for small ε describing this situation can be sought in power series of ε. However, in the analysis of the next subsection, where the subsequent time development is considered, we encounter series expansion in the square root of ε. Therefore, we look for the moderately varying solution of Eqs. (7a-c) with (8), (9a-c) and (10a,b), in power series of $\sqrt{\varepsilon}$. That is, introducing

$$\delta = \sqrt{\varepsilon}, \tag{11}$$

we seek the solution in power series of δ:

$$u_{iS} = u_{iS0} + \delta u_{iS1} + \delta^2 u_{iS2} + \cdots, \tag{12a}$$

$$p'_S = p'_{S0} + \delta p'_{S1} + \delta^2 p'_{S2} + \cdots, \tag{12b}$$

$$\rho'_S = \rho'_{S0} + \delta \rho'_{S1} + \delta^2 \rho'_{S2} + \cdots, \tag{12c}$$

which correspond to the expansion of the dimensionless suction speed υ:

$$\upsilon = \upsilon_0 + \delta\upsilon_1 + \delta^2\upsilon_2 + \cdots, \tag{13}$$

where $\upsilon_n (n = 0,1,2,\cdots)$ is a given function of x_{wi} and t that varies its value from zero to a stationary function of x_{wi} during $0 < t \le O(1)$. In (12a-c) the subscript S is attached to discriminate the solution of sound wave region.

Corresponding to this expansion, the given functions $\bar{p}$ and $\bar{\rho}$ are also expanded in δ:

$$\bar{p} = \bar{p}_0 + \delta^{2M}\bar{p}_{2M} + \delta^{2M+1}\bar{p}_{2M+1} + \cdots, \tag{14a}$$

$$\bar{\rho} = 1 + \delta^2\bar{\rho}_2 + \delta^3\bar{\rho}_3 + \cdots, \tag{14b}$$

Here $\bar{p}_0$ is a given constant. The component functions $\bar{p}_n (n = 2M, 2M+1, 2M+2, \cdots)$ and $\bar{\rho}_n$ $(n = 2,3,4,\cdots)$ are given functions of x_3. In (14a-c), the series of the variable parts start from $O(\delta^{2M})$ and $O(\delta^2)$, respectively following Eqs.(5) and (6). These component functions are related by the hydrostatic condition (5) as

$$\frac{d\bar{p}_n}{dx_3} = \begin{cases} -G & (n = 2M) \\ 0 & (n = 2M+1) \\ -G\bar{\rho}_{n-2M} & (n \ge 2M+2) \end{cases} \tag{15}$$

Substituting the series (12a-c) and (14a-c) into the linearized Euler set of equations (7a-c), we obtain a series of equations for u_{iSn}, p'_{Sn} and $\rho'_{Sn} (n = 0,1,2,\cdots)$ as follows:

At $n = 0,1$, the acoustic equations:

$$\frac{\partial p'_{Sn}}{\partial t} + \frac{\partial u_{iSn}}{\partial x_i} = 0, \tag{16a}$$

$$\frac{\partial u_{iSn}}{\partial t} + \frac{\partial p'_{Sn}}{\partial x_i} = 0, \quad (n = 0,1) \tag{16b}$$

with

$$\frac{\partial \rho'_{Sn}}{\partial t} - \frac{\partial p'_{Sn}}{\partial t} = 0. \tag{16c}$$

For $n \geq 2$, the acoustic equations with source terms:

$$\frac{\partial p'_{Sn}}{\partial t} + \frac{\partial u_{iSn}}{\partial x_i} = A_n, \tag{17a}$$

$$\frac{\partial u_{iSn}}{\partial t} + \frac{\partial p'_{Sn}}{\partial x_i} = B_{in}, \tag{17b}$$

with

$$\frac{\partial \rho'_{Sn}}{\partial t} - \frac{\partial p'_{Sn}}{\partial t} = C_n, \tag{17c}$$

where A_n, B_{in} and $C_n (n \geq 2)$ are source terms composed of the lower order solutions. Their components are classified into the following two kinds: the terms representing inhomogeneity of fluids AH_n, BH_{in}, CH_n, and those representing effects of gravity AG_n, BG_{in}, CG_n, i.e.,

$$A_n = AH_n + AG_n, \tag{18a}$$

$$B_{in} = BH_{in} + BG_{in}, \quad (n \geq 2) \tag{18b}$$

$$C_n = CH_n + CG_n. \tag{18c}$$

Specifically, AH_n, BH_{in} and CH_n are given by

$$AH_n = -\left\langle (\bar{\rho}\bar{a}^2 - 1)\frac{\partial u_{iS}}{\partial x_i} \right\rangle_n, \tag{19a}$$

$$BH_{in} = -\left\langle (\bar{\rho} - 1)\frac{\partial u_{iS}}{\partial t} \right\rangle_n, \quad (n \geq 2) \tag{19b}$$

$$CH_n = \left\langle \left(\frac{1}{\bar{a}^2} - 1\right)\frac{\partial p'_S}{\partial t} \right\rangle_n - \left\langle u_{3S}\frac{d\bar{\rho}}{dx_3} \right\rangle_n, \tag{19c}$$

where the notation $\langle \cdots \rangle_n$ indicates the n-th order component function of the expansion with respect to δ, for example $\langle (\bar{\rho}-1)\partial u_{iS}/\partial t \rangle_3 = \bar{\rho}_2 \partial u_{iS1}/\partial t + \bar{\rho}_3 \partial u_{iS0}/\partial t$ and

$$\left\langle \frac{1}{\bar{a}^2} \right\rangle_n = \begin{cases} 1 \quad (n=0), \\ 0 \quad (n=1), \\ -\left\langle \dfrac{\partial \bar{a}^2}{\partial \bar{p}} \right\rangle_0 \bar{p}_2 - \left\langle \dfrac{\partial \bar{a}^2}{\partial \bar{\rho}} \right\rangle_0 \bar{\rho}_2 \quad (n=2), \\ \cdots\cdots, \end{cases}$$

AG_n, BG_{in} and CG_n are zero for $n \leq 2M - 1$, and for $n \geq 2M$,

$$AG_n = G\langle \bar{\rho} u_{3S} \rangle_{n-2M}, \tag{20a}$$

$$BG_{in} = -G\delta_{i3}\rho'_{Sn-2M}, \quad (n \geq 2M) \tag{20b}$$

$$CG_n = -G\left\langle \frac{\bar{\rho} u_{3S}}{\bar{a}^2} \right\rangle_{n-2M}. \tag{20c}$$

The initial condition is obtained by substituting the series (12a-c) into (9a-c) and arranging the same order quantities in δ:

$$u_{iSn} = 0, \tag{21a}$$

$$\rho'_{Sn} = 0, \quad (n \geq 0) \tag{21b}$$

$$p'_{Sn} = 0. \tag{21c}$$

The boundary condition is obtained from (10a,b) by the same procedure as

$$u_{iSn} n_i = \upsilon_n \text{ at } x_i = x_{wi}, \quad (n \geq 0) \tag{22a}$$

$$u_{iSn} n_i = 0 \text{ at } x_3 = 0,1. \tag{22b}$$

We analyze the above initial-boundary-value problem (16)-(22). Note that the purpose of the present study is to derive the set of governing equations after the acoustic region. For that purpose, we need the solution as $t \to \infty$ that can be the initial condition of the subsequent time development. First consider the case of $n = 0,1$. The governing equations for u_{iSn} and p'_{Sn} are the acoustic equations (16a,b), so that the sound wave is generated and propagates inside the fluid when the discharge commences. Since the flow region is open and the suction speed υ_n is steady after a given time of the order of unity, any sound wave is propagated away if the time elapses sufficiently, i.e., the flow becomes steady as $t \to \infty$ at this order. u_{iSn} and p'_{Sn} at $n = 0,1$ are, therefore, independent of t as $t \to \infty$ and we obtain the following equations from Eqs.(16a,b),:

$$\left. \frac{\partial u_{iSn}}{\partial x_i} \right|_{t \to \infty} = 0, \tag{23a}$$

$$\left. \frac{\partial p'_{Sn}}{\partial x_i} \right|_{t \to \infty} = 0, \tag{23b}$$

where the variable with $\cdots|_{t \to \infty}$ is evaluated as $t \to \infty$. Another equation for $u_{iSn}|_{t \to \infty}$ can be derived by taking the rotation of (16b) and integrating with respect to t under the initial condition (21a) as

$$\mathbf{curl}\, \boldsymbol{u}_{Sn}|_{t \to \infty} = 0, \quad (n = 0,1) \tag{23c}$$

where $\boldsymbol{u}_{Sn}$ is the vector notation of u_{iSn} and **curl** is that of rotation. Thus, $u_{iSn}|_{t \to \infty}$ can be obtained by solving (23a) and (23c) under the boundary condition (22a,b) as $t \to \infty$. The remaining variable $\rho'_{Sn}|_{t \to \infty}$ is deducible from $p'_{Sn}|_{t \to \infty}$ by (16c) as

$$\rho'_{Sn}|_{t \to \infty} = p'_{Sn}|_{t \to \infty}, \quad (n = 0,1) \tag{23d}$$

Next consider the case of $n = 2$. The governing equations for u_{iS2} and p'_{S2} are (17a,b), or the acoustic equations with source terms. Using (23a,c), however, we find that these source terms disappear as $t \to \infty$ and governing equations approach the usual acoustic equations as $t \to \infty$. So u_{iS2} and p'_{S2} are independent of t as $t \to \infty$. We then obtain the equations for them from (17a,b). If we continue the analysis for the higher order successively, the situation is the same up to $n = 2M - 1$. This is because the source terms AG_n and BG_{in} representing the effect of gravity never appear for $n \leq 2M - 1$. Thus, we obtain

$$\left. \frac{\partial u_{iSn}}{\partial x_i} \right|_{t \to \infty} = 0, \tag{24a}$$

$$\left.\frac{\partial p'_{Sn}}{\partial x_i}\right|_{t\to\infty} = 0, \quad (2 \le n \le 2M-1;\ \text{with } M \ne 1) \qquad (24b)$$

$$\left.\mathbf{curl}\left(\boldsymbol{u}_{Sn} + \sum_{m=2}^{n} \bar{\rho}_m \boldsymbol{u}_{Sn-m}\right)\right|_{t\to\infty} = 0. \qquad (24c)$$

Eq.(24c) indicates that the momentum vector is rotation free. The remaining variable $\rho'_{Sn}|_{t\to\infty}$ is deducible from $p'_{Sn-m}|_{t\to\infty}$ $(0 \le m \le n)$ and u_{iSn-m} $(2 \le m \le n)$ by (17c) as

$$\rho'_{Sn}|_{t\to\infty} = \sum_{m=0}^{n} p'_{Sn-m}|_{t\to\infty}\left\langle\frac{1}{\bar{a}^2}\right\rangle_m - \sum_{m=2}^{n}\int_0^{\infty}\left(u_{3Sn-m} - u_{3Sn-m}|_{t\to\infty}\right)dt\frac{d\bar{\rho}_m}{dx_3} - \sum_{m=2}^{n} u_{3Sn-m}|_{t\to\infty}\frac{d\bar{\rho}_m}{dx_3}t.$$

$$(2 \le n \le 2M-1) \qquad (24d)$$

The last term on the right-hand side represents the dependency of $\rho'_{Sn}|_{t\to\infty}$ on t.

When $n = 2M$ and $2M+1$, the source terms of (17a,b) are not zero due to contribution of AG_n and BG_{in}. They are found to approach constant values as $t \to \infty$ because u_{3Sn-2M} and ρ'_{Sn-2M} included in AG_n and BG_{3n} take stationary values other than zero as $t \to \infty$ [see Eqs. (23a,c,d)]. So we get, from (17a,b),

$$\left.\frac{\partial u_{iSn}}{\partial x_i}\right|_{t\to\infty} = G u_{3Sn-2M}|_{t\to\infty}, \quad (n = 2M, 2M+1) \qquad (25a)$$

$$\left.\frac{\partial p'_{Sn}}{\partial x_i}\right|_{t\to\infty} = -G\delta_{i3}\rho'_{Sn-2M}|_{t\to\infty}. \qquad (25b)$$

Taking the rotation of (17b) and integrating with respect to t under the initial condition (21a), we have

$$\left.\mathbf{curl}\left(\boldsymbol{u}_{Sn} + \sum_{m=2}^{n} \bar{\rho}_m \boldsymbol{u}_{Sn-m}\right)\right|_{t\to\infty} = -G\,\mathbf{curl}\int_0^{\infty} \rho'_{Sn-2M}\boldsymbol{\delta}_3 dt. \quad (n = 2M, 2M+1) \qquad (25c)$$

where $\boldsymbol{\delta}_3$ is the unit vector in the positive x_3 direction. The remaining variable $\rho'_{Sn}|_{t\to\infty}$ at $n = 2M$ and $2M+1$ is given by (24d) with additional term

$$-G\int_0^{\infty}\left(u_{3Sn-2M} - u_{3Sn-2M}|_{t\to\infty}\right)dt - G u_{3Sn-2M}|_{t\to\infty} t, \qquad (25d)$$

on its right-hand side.

We can proceed with the analysis in a similar way. $\partial u_{iSn}/\partial x_i|_{t\to\infty}$ and $\partial p'_{Sn}/\partial x_i|_{t\to\infty}$ are obtained from (17a,b). The rotation of the momentum vector as $t \to \infty$ is given by taking the rotation of (17b) and integrating with respect to t under the initial condition (21a). $\rho'_{Sn}|_{t\to\infty}$ is deducible from $p'_{Sn-m}|_{t\to\infty}$ $(0 \le m \le n)$ and u_{iSn-m} $(2 \le m \le n)$ by (17c). For $2(M+1)J \le n \le 2(M+1)(J+1)-1$ (J is a natural number), $\partial u_{iSn}/\partial x_i|_{t\to\infty}$, $\partial p'_{Sn}/\partial x_i|_{t\to\infty}$ and the rotation of the momentum vector as $t \to \infty$ are given by series of t up to the order of $2J-2$, $2J-1$ and $2J$, respectively. The time dependency of $\rho'_{Sn}|_{t\to\infty}$ is staggered and it is a series of t up to the order of $2J+1$ for $2(M+1)J+2 \le n < 2(M+1)(J+1)+1$. From these results, it is clear that the n-th component functions increase their dependency on t by the order of two as n is increased by $2(M+1)$.

3.3 Fluid motion after the acoustic region

We consider the subsequent fluid motion after the acoustic region. The basic equations are (7a-c) with (8). The initial condition is the steady part of the solution as $t \to \infty$ of the acoustic region obtained in the preceding subsection. It is written as

$$u_i = u_{i0} + \delta u_{i1} + \delta^2 u_{i2} + \cdots, \tag{26a}$$

$$p' = p'_0 + \delta p'_1 + \delta^2 p'_2 + \cdots, \tag{26b}$$

$$\rho' = \rho'_0 + \delta \rho'_1 + \delta^2 \rho'_2 + \cdots, \tag{26c}$$

where the component functions u_{in}, p'_n and ρ'_n ($n = 0,1,2,\cdots$) on the right-hand side are the steady part of the corresponding component functions as $t \to \infty$ of the acoustic region. These are independent of t and satisfy (23a-d), (24a-c), (25a-c) and the steady part of (24d). The boundary condition is the steady part of (10a,b) and the steady-state solution of the acoustic region prescribed at far field $\sqrt{x_1^2 + x_2^2} \to \infty$ [This boundary condition is numbered (10'a-c)].

We analyze the above initial-boundary value problem. Let us estimate the time scale of this solution. Assuming that the time scale is $O(\delta^{-\alpha})$ where α is an undetermined positive constant, we introduce a shortened coordinate with respect to t:

$$T = \delta^{\alpha} t. \tag{30}$$

The value α is determined by considering a balance among time-derivative terms of (7a-c) and the leading-order terms that can contribute to the time development following the acoustic region. We then find

$$\alpha = M + 1. \tag{33}$$

(33) is consistent with the fact that the n-th component functions of the solution as $t \to \infty$ in the acoustic region increase their dependency on t by the order of two as n is increased by $2(M+1)$. The order of each variable u_i, p' and ρ' under this time scale is given by substituting (33) into (31b-d) and then into (26a-c) as

$$u_i = O\left(u_i|_{T\to 0}\right), \tag{34a}$$

$$p' = O\left(\delta^{-M+1} u_3|_{T\to 0}\right), \tag{34b}$$

$$\rho' = O\left(\delta^{-M+1} u_3|_{T\to 0}\right). \tag{34c}$$

Based on these estimates, we look for the solution of Eqs. (7a-c) with (8) whose appreciable variation occurs in the dimensionless distance x_i and the dimensionless time T of the order of unity [$\partial\phi/\partial x_i = O(\phi)$, $\partial\phi/\partial T = O(\phi)$ with $\phi = u_i, p', \rho'$], in power series of δ:

$$u_{iB} = u_{iB0} + \delta u_{iB1} + \delta^2 u_{iB2} + \cdots, \tag{35a}$$

$$p'_B = \delta^{-M+1} p'_{B-M+1} + \delta^{-M+2} p'_{B-M+2} + \delta^{-M+3} p'_{B-M+3} + \cdots, \tag{35b}$$

$$\rho'_B = \delta^{-M+1} \rho'_{B-M+1} + \delta^{-M+2} \rho'_{B-M+2} + \delta^{-M+3} \rho'_{B-M+3} + \cdots, \tag{35c}$$

where the subscript B is attached to discriminate the type of solution. In the following analysis, derivation of sets of governing equations are given in Section 3.3.1 and their corresponding initial conditions and boundary conditions are arranged in Section 3.3.2.

3.3.1 Derivation of sets of governing equations

Substituting (35a-c) and (14a-c) into the original governing equations (7a-c), and arranging the terms by the order in δ, we obtain a series of equations. First $2M$ sets are those for $p'_{Bm}(-M+1 \le n \le M)$. They are derived from the momentum equation (7b) at the order from $-M+1$ to M:

$$\frac{\partial p'_{B-M+1}}{\partial x_i}=\frac{\partial p'_{B-M+2}}{\partial x_i}=\cdots=\frac{\partial p'_{BM}}{\partial x_i}=0\,. \tag{36}$$

Using the boundary condition (10'c) at far field $\sqrt{x_1^2+x_2^2}\to\infty$ that prescribes the steady pressure distribution, we obtain constant values for $p'_{Bm}\,(-M+1\le n\le M)$:

$$p'_{B-M+1}=\text{Const}\,,\ \ p'_{B-M+2}=\text{Const},\ \ \cdots\cdots,\ \ p'_{BM}=\text{Const}\,. \tag{37}$$

In the subsequent orders, sets of equations for u_{iBn}, p'_{Bn+M+1} and ρ'_{Bn-M+1} $(n=0,1,2,\cdots)$ are successively derived. It is useful here to note that the order of each variable in the same set is staggered, that is, p'_B is of higher order by $M+1$ and ρ'_B is of lower order by $M-1$ than u_{iB}. p'_B being higher order is a necessary condition for equations to be those for an incompressible fluid. Reflecting this situation, the initial sets up to $n=2M-3$ are classified as those for an incompressible fluid, i.e., for $0\le n\le 2M-3$ (with $M\ne 1$),

$$\frac{\partial u_{iBn}}{\partial x_i}=0\,, \tag{38a}$$

$$\frac{\partial u_{iBn}}{\partial T}+\sum_{m=2}^{n}\bar{\rho}_m\frac{\partial u_{iBn-m}}{\partial T}=-\frac{\partial p'_{Bn+M+1}}{\partial x_i}-G\delta_{i3}\rho'_{Bn-M+1}\,, \tag{38b}$$

$$\frac{\partial \rho'_{Bn-M+1}}{\partial T}+\sum_{m=2}^{n+2}u_{3Bn+2-m}\frac{d\bar{\rho}_m}{dx_3}=0\,, \tag{38c}$$

where Eq.(37) is used to derive the equation of incompressibility (38c). At $n=0$ or 1, the second term on the left-hand side of momentum equation (38b) disappears and Eqs.(38a-c) are nothing but the set of linearized Boussinesq equations.

The next derived sets ($n=2M-2,2M-1$) are classified as those for a compressible fluid with a special feature, that is, they can be reduced to the form for an incompressible fluid by easy manipulation. They are the combination of (38a,b) with $n=2M-2$, $2M-1$ and the following:

$$\frac{\partial \rho'_{Bn-M+1}}{\partial T}+\sum_{m=2}^{n+2}u_{3Bn+2-m}\left[\frac{d\bar{\rho}_m}{dx_3}+G\left\langle\frac{\bar{\rho}}{\bar{a}^2}\right\rangle_{m-2M}\right]=0\,, \tag{39c}$$

where the hydrostatic equation (15) and Eq.(37) are used to derive (39c) and $\langle\bar{\rho}/\bar{a}^2\rangle_{m-2M}$ is zero for $m\le 2M-1$. Although $\langle\bar{\rho}/\bar{a}^2\rangle_{m-2M}$ in the third term on the left-hand side of (39c) is not zero at $m=2M$, the sets of equations (38a,b) and (39c) can be reduced to the incompressible Euler sets by regarding $\bar{\rho}_{2M}+G\int_0^{x_3}\langle\bar{\rho}/\bar{a}^2\rangle_0 dx_3=\bar{\rho}_{2M}+Gx_3$ as the corresponding component function of the initial density because $\bar{\rho}_{2M}$ does not appear at the other place. When $M=1$, they are the lowest order sets ($n=0,1$) and correspond with the sets of Boussinesq equations. When $M\ge 2$ (or $n\ge 2$), they include the non-Boussinesq terms on the left-hand side of the momentum equation and have a role of being corrections to the preceding lower-order sets (38a-c).

At the next order, or $n=2M$, $2M+1$, we obtain the linearized Euler sets of equations for a compressible fluid:

$$\frac{\partial u_{iBn}}{\partial x_i}=Gu_{3Bn-2M}\,, \tag{40a}$$

(38b) with $n=2M$, $2M+1$ (40b)

$$\frac{\partial}{\partial T}(\rho'_{Bn-M+1}-p'_{Bn-M+1})+\sum_{m=2}^{n+2}u_{3Bn+2-m}\left[\frac{d\bar{\rho}_m}{dx_3}+G\left\langle\frac{\bar{\rho}}{\bar{a}^2}\right\rangle_m\right]=0, \tag{40c}$$

where Eq.(39c) is used to derive (40a). The differences from those of the previous sets are the right-hand side of (40a) and the $\partial p'_{Bn-M+1}/\partial T$ term on the left-hand side of (40c). Owing to these terms and the fact that $\bar{\rho}_m$ $(m=2M,2M+1)$ appear on the left-hand side of (40b) so that $\bar{\rho}_m+G\int_0^{x_3}\langle\bar{\rho}/\bar{a}^2\rangle_{m-2M}dx_3$ $(m=2M,2M+1)$ cannot be regarded as the corresponding component function of the initial density, this set cannot be reduced to the incompressible type.

Finally we note that the analysis can be continued in a similar way and the higher-order sets of equations for a compressible fluid are successively derived.

3.3.2 Initial conditions and boundary conditions

Initial conditions for the incompressible Euler sets of equations (38a-c) and (39a-c) (They are the Boussinesq sets at $n=0,1$, and the non-Boussinesq sets for $n\geq 2$) are arranged in the following form from Eqs.(27a-d) and (28a-d):

For $0\leq n\leq M$,

$$\frac{\partial u_{iBn}}{\partial x_i}=0, \tag{41a}$$

$$\mathbf{curl}\left(\boldsymbol{u}_{Bn}+\sum_{m=2}^{n}\bar{\rho}_m\boldsymbol{u}_{Bn-m}\right)=0, \tag{41b}$$

(irrotational at $n=0,1$; momentum vector being rotation free for $n\geq 2$)

$$\rho'_{Bn-M+1}=\begin{cases}0 & (0\leq n\leq M-2),\\ p'_{Bn-M+1} & (n=M-1,M).\end{cases} \tag{41c}$$

For $M+1\leq n\leq 2M-1$ $(M\neq 1)$,

$$\frac{\partial u_{iBn}}{\partial x_i}=0, \tag{42a}$$

$$\mathbf{curl}\left(\boldsymbol{u}_{Bn}+\sum_{m=2}^{n}\bar{\rho}_m\boldsymbol{u}_{Bn-m}\right)=0, \tag{42b}$$

$$\rho'_{Bn-M+1}=\sum_{m=M-1}^{n}p'_{Bn-m}\left\langle\frac{1}{\bar{a}^2}\right\rangle_{m-M+1}-\sum_{m=M+1}^{n}\int_0^{\infty}(u_{3Sn-m}-u_{3Bn-m})dt\frac{d\bar{\rho}_{m-M+1}}{dx_3}. \tag{42c}$$

Here the condition for p'_{Bn+M+1} is not presented in (41) and (42). It is because it can be obtained as a solution of the equation that is derived by taking the divergence of the momentum equation (38b) or (39b) and applying to it the divergence free condition for flow velocity from the equation of mass conservation (38a) or (39a). The derived equation is the second-order partial differential equation for p'_{Bn+M+1} with respect to x_i. Its boundary condition is a normal component, with respect to boundary, of the momentum equation whose time derivative term of the velocity component normal to the boundary is zero.

An important aspect of the above initial conditions (41a-c) and (42a-c) is that a deviation of the density from its initial value at rest is not zero for $n\geq M-1$. Recalling that the terms representing compressibility of fluid appears in the governing equations at the higher order $n=2M$, the correct evaluation of initial conditions may be more important than taking the compressibility of fluids into account in decreasing the error for using a set of equations for an incompressible fluid.

Lastly, the initial condition for the compressible Euler set of equations (40a-c) ($n=2M$, $2M+1$) is given. From Eqs.(28d) and (29a-d),

$$\frac{\partial u_{iBn}}{\partial x_i}=Gu_{3Bn-2M}, \tag{43a}$$

$$\mathbf{curl}\left(\boldsymbol{u}_{Bn}+\sum_{m=2}^{n}\bar{\rho}_m\boldsymbol{u}_{Bn-m}\right)=-G\,\mathbf{curl}\int_0^\infty \rho'_{Sn-2M}\delta_3 dt, \tag{43b}$$

(42c) with $n=2M$ or $2M+1$. (43c)

The condition for p'_{Bn+M+1} is not presented because it can be obtained by a similar procedure as that for an incompressible type. In this case, the equation for p'_{Bn+M+1} is derived by taking the divergence of the momentum equation (40b) and subtracting G times the x_3 component of the momentum equation (40b) at $n=0$ from it before using the mass conservation equation (40a). In (43a), the flow velocity is not divergence free. This is consistent with the equation of mass conservation (40a) for a compressible fluid. Moreover the momentum vector is not rotation free from this order.

In the linearized theory discussed above, we neglected the nonlinear terms of the deviation from the initial state like $u_j\partial u_i/\partial x_j$ which is of the order of δ^{4k} [recall that u_i is $O(\delta^{2k})$] but retained the quantities like $\bar{\rho}\partial u_i/\partial t$ and its higher order which are of the order of $\delta^{2k+M+1+n}$ [see (30) and (33)] where n is zero or any positive integer. This means that $2k>>M+1+n$, or $2k>>M+1$ (or $2k>>M$). The sets of linearized equations derived in this section 3 reflect this situation.

4. Discussion

Let us extend the results to the case of a liquid, as commented at the last paragraph of Section 1. We assume that the derivatives of the square of the sound speed with respect to the pressure and the density are of the orders of $O(\varepsilon^{b_1}/\rho_0)$ and $O(\varepsilon^{b_2}a_0^2/\rho_0)$, respectively, where b_1 and b_2 are given integers. Then the results for this case are obtained only by the reevaluation of the orders of $\partial\bar{a}^2/\partial\bar{p}$ and $\partial\bar{a}^2/\partial\bar{\rho}$ that appear in the derived sets of equations and initial conditions presented in the previous sections as

$$\frac{\partial\bar{a}^2}{\partial\bar{p}}=O(\varepsilon^{b_1}),\quad \frac{\partial\bar{a}^2}{\partial\bar{\rho}}=O(\varepsilon^{b_2}). \tag{44a,b}$$

We note that the magnitude of the square of the sound speed itself does not affect the results.

Next, consider the case where the fluid is also subject to the stratification due to the concentration of solvent or species without diffusion. Then we have to add another unknown variable representing the concentration $\bar{c}+c'$ where $\bar{c}(x_3)$ represents initial concentration profile and c' that of deviation. This variable appears as another independent variable of a^2. That is, $\bar{a}^2=\bar{a}^2(\bar{p},\bar{\rho},\bar{c})$ in (8) for the linear theory, where $\bar{c}$ is subject to the equations:

In the linear theory,

$$\frac{\partial c'}{\partial t} + u_3 \frac{d\bar{c}}{dx_3} = 0 \tag{45}$$

The initial condition is

$$c' = 0 \,. \tag{46}$$

In the linear theory no modification is needed to the derived sets of equations and initial conditions in Section 3.3. They remain the same form, since effects of concentration are already included in the density profile.

Now, based on the results obtained in the preceding sections, we estimate the error for using an approximate set of equations. The error is defined as

$$\frac{\left| \begin{pmatrix} \text{Solution of the compressible} \\ \text{Euler set} \end{pmatrix} - \left(\text{Solution of an approximate set} \right) \right|}{\left| \begin{matrix} \text{Solution of the compressible} \\ \text{Euler set} \end{matrix} \right|} . \tag{48}$$

We conduct an error estimate for the following three kinds of approximate sets:

(i) Use of the set of equations for an incompressible fluid, where the condition of incompressibility is employed as an equation of state.

(ii) Use of the set of Boussinesq equations, where the constant value is substituted into the density of inertial terms in the momentum equation of the incompressible Euler set.

(iii) Use of the conventional initial condition where flow velocity is divergence free, the momentum vector is rotation free, and the density deviation is zero.

The results of the error estimates are arranged in Table 1. From their results, we see that the error due to the initial condition (iii) is equal to or larger than that due to the use of incompressible Euler set. This indicates that the correct evaluation of initial conditions may be more important than incorporating the terms representing compressibility of fluids into the governing equations to decrease an error for using an approximate set of equations. This tendency is amplified as M increases, or the effect of gravity decreases.

Table 1

$(M \geq 1)$	(i) The error of incompressible Euler set	(ii) The error of the Boussinesq set	(iii) The error of the initial condition
(I) Linear theory $2k >> M+1$	ε^{M}	ε	$\varepsilon^{\frac{M+1}{2}}$
(II) Weakly nonlinear theory $2k = M+1$	$\varepsilon^{2k-1}\left(= \varepsilon^{M}\right)$		$\varepsilon^{k}\left(= \varepsilon^{\frac{M+1}{2}}\right)$
(III) Flow with no buoyancy effect $2k << M+1$	ε^{2k}		ε^{k}

References

Spiegel, E.A. and Veronis, G. (1960), On the Boussinesq approximation for a compressible fluid, *Astrophys. J.* **131**, 442-447.