<table>
<thead>
<tr>
<th>Title</th>
<th>Mass Normalization of Collapses in the Theory of Self-Interacting Particles (Mathematical Aspects and Applications of Nonlinear Wave Phenomena)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>Suzuki, Takashi</td>
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</tr>
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</table>

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Mass Normalization of Collapses in the Theory of Self-Interacting Particles

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1 Introduction

This paper is concerned with the elliptic-parabolic system of cross-diffusion,

\[
\begin{align*}
\begin{cases}
  u_t &= \nabla \cdot (\nabla u - u \nabla v) \\
  0 &= \Delta v - av + u
\end{cases}
\end{align*}
\]

in \( \Omega \times (0,T) \)

\[
\begin{align*}
\frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega \times (0,T) \\
u|_{t=0} &= u_0(x) \quad \text{in} \quad \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^2 \) is a bounded domain with smooth boundary \( \partial \Omega \), \( a > 0 \) is a constant, and \( \nu \) is the outer unit normal vector on \( \partial \Omega \). It is proposed by Nagai [8] as a simplified form of the ones given by Jäger and Luckhaus [6], Nanjundiah [12], Keller and Segel [7], and Patlak [14] to describe the chemotactic feature of cellular slime molds. It is also a description of the non-equilibrium mean field of self-attractive particles subject to the second law of thermodynamics. Actually, this physical principle is realized by introducing the friction and fluctuations of particles. See Bavaud [1] and Wolansky [23], [24]. On the other hand, the mathematical study has a long history, and we refer to [21] for the background, known results, and standard arguments.

Actually, it follows from Yagi [25] and Biler [2] that the unique classical solution exists locally in time if the initial value is smooth, and that the solution becomes positive if the initial value is non-negative and not identically

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Letting $T_{\text{max}} > 0$ to be the supremum of the existence time of the solution, we say that the solution blows-up in finite time if $T_{\text{max}} < +\infty$. Then, it is proven in Senba and Suzuki [16] that in the case of $T_{\text{max}} < +\infty$ there exists a finite set $S \subset \overline{\Omega}$ and a non-negative function $f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus S)$ such that

$$u(x, t)dx \rightarrow \sum_{x_0 \in S} m(x_0) \delta_{x_0}(dx) + f(x)dx \quad \text{in} \quad \mathcal{M}(\overline{\Omega}) \quad (2)$$

as $t \uparrow T_{\text{max}}$ with

$$m(x_0) \geq m_*(x_0) \quad (x_0 \in S), \quad (3)$$

where $\mathcal{M}(\overline{\Omega})$ denotes the set of measures on $\overline{\Omega}$, $\rightarrow$ the $*$-weak convergence there, and

$$m_*(x_0) \equiv \begin{cases} 8\pi & (x_0 \in \Omega) \\ 4\pi & (x_0 \in \partial\Omega). \end{cases}$$

It follows from $T_{\text{max}} < +\infty$ that

$$\lim_{t \rightarrow T_{\text{max}}} \|u(t)\|_\infty = +\infty$$

and $S$ is actually the blowup set of $u$. That is, $x_0 \in \overline{\Omega}$ belongs to $S$ if and only if there exist $x_k \rightarrow x_0$ and $t_k \uparrow T_{\text{max}}$ such that $u(x_k, t_k) \rightarrow +\infty$. Because

$$\|u(t)\|_1 = \|u_0\|_1 \quad (4)$$

holds for $t \in [0, T_{\text{max}})$, inequality (2) with (3) implies that

$$2 \cdot \#(\Omega \cap S) + \#(\partial\Omega \cap S) \leq \|u_0\|_1 / (4\pi). \quad (5)$$

We have, furthermore, that $S \neq \emptyset$ if $T_{\text{max}} < +\infty$, and therefore, $\|u_0\|_1 < 4\pi$ implies $T_{\text{max}} = +\infty$. This fact on the existence of the solution globally in time was proven independently by Nagai, Senba, and Yoshida [11], Biler [2], and Gajewski and Zacharias [4], while relation (2) was conjectured by Nanjundiah [12]. It is referred to as the formation of chemotactic collapses, and each collapse

$$m(x_0) \delta_{x_0}(dx)$$

is regarded as a spore created by the cellular slime molds.

In 1996, Herrero and Velázquez [5] constructed a family of radially symmetric blowup solutions by the method of matched asymptotic expansion,
where it holds that $m(x_0) = m_*(x_0)$ with $x_0 = 0 \in \Omega \cap S$. Also, Nagai [9] and Senba and Suzuki [17] showed that if
\[ ||u_0||_1 > 4\pi \quad \text{and} \quad \int_{\Omega} |x - x_0|^2 u_0(x) dx \ll 1 \]
hold for $x_0 \in \partial \Omega$, then it follows that $T_{\max} < +\infty$. This means that the mass of collapses made by those solutions can be close to $4\pi$ as we like. However, it may be always $4\pi$, and under those considerations it was suspected that $m(x_0) = m_*(x_0)$ for any $x_0 \in S$.

This problem, referred to as the mass normalization in the present paper, is related to the blowup rate, and we say that $x_0 \in S$ is of type (I) if
\[ \limsup_{t \to T} \sup_{|x-x_0| \leq Cr(t)} r(t)^2 u(x, t) < +\infty \]
holds for any $C > 0$, and that it is of type (II) for the other case that
\[ \limsup_{t \to T} \sup_{|x-x_0| \leq Cr(t)} r(t)^2 u(x, t) = +\infty \]
holds with some $C > 0$, where $T = T_{\max} < +\infty$ and $r(t) = (T - t)^{1/2}$. It is expected that type (I) blowup point never arises. Here, we shall show the following.

**Theorem 1** If $x_0 \in S$ is of type (II), then the mass normalization $m(x_0) = m_*(x_0)$ occurs.

## 2 Preliminaries

We suppose that $T = T_{\max} < +\infty$, and take the standard backward self-similar transformation
\[ z(y, s) = (T - t)u(x, t) \]
for $y = (x - x_0)/(T - t)^{1/2}$ and $s = -\log(T - t)$, where $x_0 \in S$ denotes the blowup point in consideration. The zero extension of $z(y, s)$ is always taken to the region where it is not defined.

The following fact is proven similarly to [20] concerning Jäger - Luckhaus model, where
\[ \{m_*(y_0)\delta_{y_0}(dy) \mid y_0 \in B\} \]
and $F(y)dy = \mu_{a.c.}(dy)$ are called the sub-collapses and the residual term, respectively. It is referred to as the formation of sub-collapses, and the proof is quite similar to the one given in [19] concerning the blowup in infinite time for the pre-scaled system. Here and henceforth, $\mu_s(dy)$ and $\mu_{a.c.}(dy)$ denote the singular and the absolutely continuous parts of $\mu(dy) \in \mathcal{M}(\mathbb{R}^2)$ relative to the Lebesgue measure $dy$, respectively.

**Lemma 2** Any $s_n \to +\infty$ admits $\{s'_n\} \subset \{s_n\}$ such that

$$z(y, s'_n)dy \to \mu_0(dy)$$

as $n \to \infty$ in $\mathcal{M}(\mathbb{R}^2)$, where $\text{supp} \mu_0(dy) \subset \overline{L}$ and

$$\mu_0(dy) = \sum_{y_0 \in B} m_*(y_0)\delta_{y_0}(dy) + F(y)dy$$

(6)

with

$$m_*(y_0) = \begin{cases} 8\pi & (y_0 \in L) \\ 4\pi & (y_0 \in \partial L), \end{cases}$$

$$0 \leq F \in L^1(L) \cap C(\overline{L} \setminus B),$$

and

$$L = \begin{cases} \mathbb{R}^2 & (x_0 \in \Omega) \\ H & (x_0 \in \partial \Omega). \end{cases}$$

Here, $H$ denotes the half space in $\mathbb{R}^2$ with $\partial H$ containig the origin and parallel to the tangent line of $\partial \Omega$ at $x_0$, and the case $B = \emptyset$ is admitted.

On the other hand, the following fact is referred to as the existence of the **parabolic envelope**.

**Lemma 3** We have

$$m(x_0) = \mu_0(\overline{L}) = \sum_{y_0 \in B} m_*(y_0) + \int_L F(y)dy.$$  

(7)

**Proof:** First, we take

$$\varphi = \varphi_{x_0, R', R}$$
for $x_0 \in S$ and $0 < R' < R$ satisfying $0 \leq \varphi \leq 1$, supp $\varphi \subset B(x_0, R)$, $\varphi = 1$ on $B(x_0, R')$, and $\frac{\partial \varphi}{\partial \nu} = 0$ on $\partial \Omega$. Then, we set

$$M_R(t) = \int_{\Omega} \psi(x)u(x,t)dx$$

for $\psi = \varphi_{x_0,R,2R}^4$. Relation (2) implies that

$$\lim_{R \downarrow 0} \lim_{t \to T} M_R(t) = m(x_0).$$

On the other hand, in [16] it is proven that

$$\left| \frac{d}{dt} M_R(t) \right| \leq C \left( \lambda^2 R^{-2} + \lambda R^{-1} \right)$$

with a constant $C > 0$ determined by $\Omega$, and hence we obtain

$$|M_R(T) - M_R(t)| \leq C \left( \lambda^2 R^{-2} + \lambda R^{-1} \right) (T - t).$$

Putting

$$R = br(t) = b(T - t)^{1/2}$$

to this inequality with a constant $b > 0$, we get that

$$\left| M_{br(t)}(T) - M_{br(t)}(t) \right| \leq C \left( \lambda^2 b^{-2} + \lambda b^{-1} (T - t)^{1/2} \right),$$

and therefore, for

$$\overline{m}_b(x_0) = \lim_{t \to T} \sup M_{br(t)}(t) \quad \text{and} \quad \underline{m}_b(x_0) = \lim_{t \to T} \inf M_{br(t)}(t)$$

it holds that

$$m(x_0) - C\lambda^2 b^{-2} \leq \overline{m}_b(x_0) \leq \overline{m}_b(x_0) \leq m(x_0) + C\lambda^2 b^{-2}$$

by $m(x_0) = \lim_{t \to T} M_{br(t)}(T)$. We note that this inequality is indicated as

$$\overline{m}_b(x_0) - C\lambda^2 b^{-2} \leq m(x_0) \leq \overline{m}_b(x_0) + C\lambda^2 b^{-2}. \quad (8)$$

Here, we have

$$\int_{B(x_0,R) \cap \Omega} u(x,t)dx \leq M_R(t) \leq \int_{B(x_0,2R) \cap \Omega} u(x,t)dx$$
and hence it follows that
\[ \int_{B(0,b)} z(y, s) dy \leq M_{br(t)}(t) \leq \int_{B(0,2b)} z(y, s) dy. \]

Thus we obtain
\[ \mu_0(B(0, b - 1)) \leq m_b(x_0) \leq \overline{m}_b(x_0) \leq \mu_0(B(0, 2b + 1)), \]
and hence it follows that
\[ \lim_{b \to +\infty} m_b(x_0) = \lim_{b \to +\infty} \overline{m}_b(x_0) = \mu_0(\mathbb{R}^2) = \mu_0(\overline{L}). \]

Then, (7) is obtained by (8).

### 3 Movement of Sub-collapses

Similarly to the pre-scaled system treated in [18], Lemma 2 is refined in the following way. Namely, any \( s_n \to +\infty \) admits \( \{s'_n\} \subset \{s_n\} \) such that
\[ z(y, s + s'_n) dy \to \mu(dy, s) \]
in \( C_* ((-\infty, +\infty), \mathcal{M}(\mathbb{R}^2)) \), where \( \text{supp} \mu(dy, s) \subset \overline{L}, m(x_0) = \mu(\overline{L}, s) \), and
\[ \mu_s(dy, s) = \sum_{y_0 \in B_s} m_s(y_0) \delta_{y_0}(dy) \]
with
\[ 8\pi \cdot \#(L \cap B_s) + 4\pi \cdot \#(\partial L \cap B_s) + \mu_{a.c.}(L, s) = m(x_0). \]

This \( \mu(dy, s) \) becomes a weak solution to
\[
\begin{align*}
\frac{\partial z}{\partial s} &= \nabla \cdot (\nabla z - z \nabla p) \quad \text{in} \quad L \times (-\infty, \infty) \\
\frac{\partial z}{\partial s} &= 0 \quad \text{on} \quad \partial L \times (-\infty, \infty),
\end{align*}
\]
where \( p = w + \frac{|y|^2}{4} \) and
\[ \nabla_y w(y, s) = \int_{L} \nabla_y G_0(y, y') z(y', s) dy \]
for the reflection $y'^*$ of $y'$ with respect to $\partial H$. The proof is similar to the one for the pre-scaled case ([18]), and the precise notion of weak solution is not necessary for later arguments. However, let us note that the zero extension of $\mu(dy, s)$ to $\mathbb{R}^2 \setminus L$ is always taken in the case of $x_0 \in \partial \Omega$, following the agreement for $z(y, s)$, and furthermore, that if $\eta \in C_0(\overline{L}) \cap C^2(\overline{L})$ satisfies $\frac{\partial \eta}{\partial \nu}|_{\partial L} = 0$, then the mapping

$$s \in [0, \infty) \mapsto \int_{\overline{L}} \eta(y) \mu(dy, s)$$

is locally absolutely continuous, where $C_0(\overline{L})$ is the set of continuous functions on $\overline{L}$ taking the value zero at infinity.

If $F(y, s)dy = \mu_{a.c.}(dy, s)$, then $F(y, s) \geq 0$ is smooth in

$$D = \bigcup_{s \in \mathbb{R}} (\overline{L} \setminus B_s) \times \{s\}.$$  

Actually, this is a consequence of the parabolic and elliptic regularity, and $F(y, s)$ satisfies there that

$$F_s = \nabla \cdot (\nabla F - F \nabla p)$$

with smooth $p$. As a consequence, if $G \subset \overline{L}$ is relatively open, if $\eta \in C^2(G) \cap C(\overline{G})$ satisfies $\eta|_{\partial G} = 0$ and $\frac{\partial \eta}{\partial \nu}|_{\partial L} = 0$, and if $\text{supp} \mu_s(dy, s) \cap \partial G = \emptyset$ holds for $s \in J$ with the time interval $J \neq \emptyset$, then

$$s \in J \mapsto \int_{\overline{L}} \eta(y) \mu(dy, s)$$

is locally absolutely continuous.

First, we study a special case of Theorem 1, making use of

$$\left[ \int_{\mathbb{R}^2} (R^2 - |y|^2)_+ \mu(dy, s') \right]_{s' = s}^{s' = s + \Delta s} \geq \int_s^{s + \Delta s} ds' \cdot \left\{ \int_{B_R} (-4 - |y|^2) \mu(dy, s') + \frac{4}{m_*(y_0)} \mu(B_R, s')^2 \right\},$$

(11)

where $R > 0$, $B_R = B(0, R)$, and $0 \leq s < s + \Delta s$. 

\*\*\*\*\*
In fact, in use of the standard backward self-similar transformation given in the previous section,
\[ z(y, s) = (T - t)u(x, t) \quad \text{and} \quad w(y, s) = v(x, t) \]
with \( y = x / (T - t)^{1/2} \) and \( s = -\log(T - t) \), it follows that
\[
\begin{align*}
  z_s &= \nabla \cdot (\nabla z - z \nabla w - y z / 2) \\
  0 &= \Delta w + z - ae^{-s}w \\
  \frac{\partial z}{\partial \nu} &= \frac{\partial w}{\partial \nu} = 0 \\
  z|_{s=-\log T} &= z_0
\end{align*}
\]
for \( z_0(y) = Tu_0(x) \),
\[
\mathcal{O} = \bigcup_{s > -\log T} e^{s/2}(\Omega \setminus \{x_0\}) \times \{s\},
\]
and
\[
\Gamma = \bigcup_{s > -\log T} e^{s/2}(\partial \Omega \setminus \{x_0\}) \times \{s\}.
\]
Here, we have
\[
w(y, s) = v(x, t) = \int_{\Omega} G(x, x')u(x', t)dx' = \int_{\mathcal{O}(s)} G(e^{-s/2}y + x_0, e^{-s/2}y' + x_0) z(y', s)dy',
\]
and therefore, system (12) is reduced to
\[
\begin{align*}
  z_s &= \nabla \cdot (\nabla z - z \nabla p) \quad \text{in} \quad \mathcal{O} \\
  \frac{\partial z}{\partial \nu} &= 0 \quad \text{on} \quad \Gamma
\end{align*}
\]
with \( p = w + \frac{|w|^2}{4} \), where \( G = G(y, y') \) denotes the Green's function for \(-\Delta + a\)
in \( \Omega \) with \( \frac{\partial}{\partial \nu} \big|_{\partial \Omega} = 0 \).

Letting \( \varphi = (R^2 - |y|^2)^+ \),
we have
\[
\varphi|_{\partial B_R} = 0, \quad \frac{\partial \varphi}{\partial \nu}|_{\partial B_R} < 0, \quad \text{and} \quad \frac{\partial \varphi}{\partial \nu}|_{\partial H} = 0.
\]
with the last case valid only for $x_0 \in \partial \Omega$. Let us note that

$$B_R = B(0, R) = \{ y \in \mathbb{R}^2 \mid \varphi(y) > 0 \}.$$  

Then, from (12) we can deduce that

$$\frac{d}{ds} \int_{\mathbb{R}^2} \varphi(y)z(y, s)dy \geq \int_{B_R} (\Delta \varphi + \frac{y}{2} \cdot \nabla \varphi)z(y, s)dy + \frac{1}{2} \int \int_{B_R \times B_R} \rho^s_{\varphi}(y, y')z(y, s)z(y', s)dydy'$$  

with

$$\rho^s_{\varphi}(y, y') = \nabla \varphi(y) \cdot \nabla_y G^s(y, y') + \nabla \varphi(y') \cdot \nabla_{y'} G^s(y, y')$$

and $G^s(y, y') = G(e^{-s/2}y + x_0, e^{-s/2}y' + x_0)$.

Here, we have

$$\Delta \varphi + \frac{y}{2} \cdot \nabla \varphi = -4 - |y|^2$$

in $B_R$. Also we have for $\theta \in (0, 1)$ that

$$G(y, y') = G_0(y, y') + K_1(y, y')$$

with $K_1 \in C^{1+\theta}_{loc}(\Omega \times \overline{\Omega}) \cap C^{1+\theta}_{loc}(\overline{\Omega} \times \Omega)$. In the case of $x_0 \in \Omega$, those relations imply the continuity of $\rho^s_{\varphi}$ as well as the uniform convergence $\rho^s_{\varphi} \to \rho^0$ as $s \to +\infty$ on $\overline{B_R \times B_R}$, where

$$\rho^0(y, y') = \frac{2}{\pi} \cdot \frac{1}{\pi}.$$  

(14)

In the case of $x_0 \in \partial \Omega$, on the other hand, we can make use of

$$G(y, y') = G_0(X(y), X(y')) + G_0(X(y), X(y')^*) + K_2(y, y')$$

with $K_2 \in C^{\theta,1+\theta}(\Omega \cup \gamma \times \overline{\Omega}) \cap C^{1+\theta,\theta}(\overline{\Omega} \times \Omega \cup \gamma)$, where $X : \overline{\Omega} \to \mathbb{R}^2_+$ is the conformal mapping satisfying $X(x_0) = 0$, $\gamma$ is the connected component of $\partial \Omega$ containing $x_0$, and $\hat{\Omega}$ is the domain defined by $\partial \hat{\Omega} = \gamma$. Then, the above conclusion follows similarly, with (14) replaced by

$$\rho^0(y, y') = \frac{2}{\pi}.$$
Now, inequality (11) follows from (13) with $z(y, s)$ replaced by $z(y, s + s'_n)$ and sending $n \to \infty$. Here, we refer to [16], [22] for those facts on the Green's function.

In terms of $\nu(dy, s) = \mu(dy, s) - m_*(y_0)\delta_0(dy)$, inequality (11) reads;

$$
\left[ \int_{\mathbb{R}^2} \left( R^2 - |y|^2 \right)_+ \nu(dy, s') \right]_{s'=s+\Delta s}^{s'=s} 
\geq \int_s^{s+\Delta s} ds' \left\{ \int_{\mathbb{R}^2} \left( R^2 - |y|^2 \right)_+ \nu(dy, s') + I_R(s') \right\}
$$

(15)

with

$$I_R(s) = m_*(y_0)R^2 - (R^2 + 4)\mu(B_R, s) + \frac{4}{m_*(y_0)}\mu(B_R, s)^2.$$

Here, $0 < R \leq 2$ and

$$\mu(B_R, s) > m_*(y_0)$$

(16)

imply $I_R(s) > 0$. On the other hand, (16) follows from

$$\int_{\mathbb{R}^2} \left( R^2 - |y|^2 \right)_+ \nu(dy, s) > 0.$$

We now show that

$$0 < R \leq 2 \quad \text{with} \quad \int_{\mathbb{R}^2} \left( R^2 - |y|^2 \right)_+ \nu(dy, 0) > 0$$

(17)

gives a contradiction. In fact, applying (15) with $s = 0$, we see that

$$\left\{ s \in [0, \infty) \mid \int_{\mathbb{R}^2} \left( R^2 - |y|^2 \right)_+ \nu(dy, s') > 0 \text{ on } s' \in [0, s] \right\}$$

is right-closed from the above consideration. Its right-openess follows from $\mu(dy, s) \in C_*((-\infty, \infty), \mathcal{M}(\mathbb{R}^2))$, so that (17) induces

$$\int_{\mathbb{R}^2} \left( R^2 - |y|^2 \right)_+ \nu(dy, s) > 0$$

for any $s \in [0, \infty)$. Simultaneously, it also holds that $I_R(s) > 0$ for $s \in [0, \infty)$, and again (15) assures the monotone increasing of the mapping

$$s \in [0, \infty) \mapsto \int_{\mathbb{R}^2} \left( R^2 - |y|^2 \right)_+ \nu(dy, s).$$
Therefore, for $n = 1, 2, \cdots$ we have

\[
\int_{\mathbb{R}^2} (R^2 - |y|^2)^+ \nu(dy, n + 1) \geq \int_{\mathbb{R}^2} (R^2 - |y|^2)^+ \nu(dy, n) + \int_n^{n+1} ds' \cdot \int_{\mathbb{R}^2} (R^2 - |y|^2)^+ \nu(dy, s') \\
\geq 2 \int_{\mathbb{R}^2} (R^2 - |y|^2)^+ \nu(dy, n),
\]

which implies that

\[
\int_{\mathbb{R}^2} (R^2 - |y|^2)^+ \nu(dy, n) \geq 2^n \int_{\mathbb{R}^2} (R^2 - |y|^2)^+ \nu(dy, 0).
\]

However, this is impossible by $\mu(\mathbb{R}^2, s) = m(x_0) < +\infty$.

We have shown that (17) does not occur. If $0 \in \text{supp} \, \mu_s(dy, 0)$, then $\nu(dy, 0) \geq 0$ holds and this means that

\[
\nu(dy, 0) = 0 \quad \text{on} \quad B(0, 2),
\]

or equivalently, $\text{supp} \, \mu_s(dy, 0) \cap B(0, 2) = \{0\}$ and

\[
F(y, 0) = 0 \quad \text{for a.e.} \quad y \in B(0, 2).
\]

Recall the notation that $F(y, s)dy = \mu_{a.c.}(dy, s)$. Because $F(y, s) \geq 0$ satisfies the parabolic equation (10) with smooth coefficient $p$ in $\mathcal{D} = \bigcup_{s \in \mathbb{R}} (\overline{L} \setminus B_s) \times \{s\}$, the strong maximum principle guarantees $F(y, s) = 0$ there. Hence $\mu_{0.a.c.}(dy) = 0$ follows.

To treat the general case, we note that if $s \in [0, \infty) \mapsto y_0(s) \in \mathbb{R}^2$ is locally absolutely continuous, then inequality (11) is replaced by

\[
\left[ \int_{\mathbb{R}^2} (R^2 - |y - y_0(s')|^2)^+ \mu(dy, s') \right]_{s' = s + \Delta s}^{s' = s + \Delta s} \geq \int_s^{s + \Delta s} ds' \\
\{ \int_{B(y(s), R)} (2y'_0(s') \cdot (y - y_0(s')) - 4 - y \cdot (y - y_0(s'))) \mu(dy, s') \}
\frac{4}{m_*(y_0)} \mu(B(y_0(s'), R), s')^2.
\]

In terms of $\mu'(dy, s)$ defined by $\mu'(A, s) = \mu(A + \{y_0(s)\}, s)$, it is represented as

\[
\left[ \int_{\mathbb{R}^2} (R^2 - |y|^2)^+ \mu'(dy, s') \right]_{s' = s + \Delta s}^{s' = s + \Delta s} \geq \int_s^{s + \Delta s} ds'.
\]
\[ \{ \int_{B_{R}} (-4 - |y|^{2} + (2y_{0}'(s) - y_{0}(s)) \cdot y) \mu'(dy, s') \\
+ \frac{4}{m_{*}(y_{0})} \mu'(B_{R}, s')^{2} \} \]

If we take
\[ y_{0}(s) = y_{0}e^{s/2} , \]
then it is reduced to (11):
\[
\left[ \int_{\mathbb{R}^2} \left( R^{2} - |y|^{2} \right)_{+} \mu'(dy, s') \right]_{s'=s}^{s'=s+\Delta s} \geq \int_{s}^{s+\Delta s} ds' .
\]
We see that \( 0 \in \supp \mu_{s}'(dy, 0) \), or equivalently \( y_{0} \in \supp \mu_{s}(dy, 0) \), implies \( \mu_{a.c.}(dy, 0) = 0 \) and

\[ \supp \mu_{s}(dy, 0) \cap B(y_{0},2) = \{ y_{0} \} . \]

If \( x_{0} \in S \) is of type (II), then there is \( s_{n} \to +\infty \) such that \( z(y, s_{n})dy \to \mu_{0}(dy) \) in \( \mathcal{M}(\mathbb{R}^2) \) with \( \supp \mu_{0s}(dy) \neq \emptyset \). We now take \( s'_{n} \subset \{ s_{n} \} \) such that \( z(y, s + s'_{n})dy \to \mu(dy, s) \) in \( C_{*}((-\infty, \infty), \mathcal{M}(\mathbb{R}^2)) \) with \( \mu(dy, s) \) being the weak solution to (9). Because of \( \mu_{s}(dy, 0) = \mu_{0s}(dy) \neq 0 \), it follows from the above argument that \( \mu_{a.c.}(dy, s) \equiv 0 \). We also have \( \mu(dy, s) \in C_{*}((-\infty, \infty), \mathcal{M}(\overline{L})) \) and \( \mu(\{ y_{0} \}, s) = m_{*}(y_{0}) \) for any \( y_{0} \in \supp \mu_{s}(dy, s) \), and therefore, it holds that

\[ \mu(dy, s) = \sum_{i=1}^{n} m_{*}^{i} \delta_{y_{i}(s)}(dy) , \]

with \( s \in (-\infty, \infty) \leftrightarrow y_{i}(s) \in \overline{L} \) being continuous, \( y_{i}(s) \in \mathbb{L} \) or \( y_{i}(s) \in \partial \mathbb{L} \) exclusively in \( s \in \mathbb{R} \), and

\[ m_{*}^{i} = \begin{cases} 8\pi & (y_{i}(s) \in \mathbb{L}) \\
4\pi & (y_{i}(s) \in \partial \mathbb{L}) . \end{cases} \]

Then, again the above argument guarantees that

\[ |y_{i}(s) - y_{j}(s)| \geq 2 \quad (i \neq j, \ s \in \mathbb{R}) . \quad (18) \]
We also have
\[ m(x_0) = \sum_{i=1}^{n} m_*^i. \]

Now, we take \( i = 1, \cdots, n \), \( R \in (0, 2) \), and the interval
\[ J_i = \{ s \in [0, \infty) \mid \text{supp } \mu_*(dy, s') \cap \overline{B(y_i(0)e^{s'/2}, R)} = \{ y_i(s') \} \} \]
for any \( s' \in [0, s] \),
which is a right neighbourhood of 0. Then, we repeat the same argument for \( \nu(dy, s) = \mu'(dy, s) - m_*^i \delta_0(dy) \) with \( \mu'(A, s) = \mu\left( A + \{ y_i(0)e^{s/2} \} , s \right) \). This time, we have \( I_R'(s) = 0 \) for \( s \in J_i \), where
\[ I_R'(s) = m_*^i R^2 - (R^2 + 4) \mu'(B_R, s) + \frac{4}{m_*^i} \mu'(B_R, s)^2. \]
Furthermore,
\[ s \in J_i \Rightarrow \int_{\mathbb{R}^2} \left( R^2 - |y|^2 \right)_+ \nu(dy, s) \]
is locally absolutely continuous, and it holds by (15) that
\[ \frac{d}{ds} \int_{\mathbb{R}^2} \left( R^2 - |y|^2 \right)_+ \nu(dy, s) \geq \int_{\mathbb{R}^2} \left( R^2 - |y|^2 \right)_+ \nu(dy, s) \]
for a.e. \( s \in J_i \). Therefore, because of
\[ \int_{\mathbb{R}^2} \left( R^2 - |y|^2 \right)_+ \nu(dy, 0) = 0 \]
we obtain
\[ \int_{\mathbb{R}^2} \left( R^2 - |y|^2 \right)_+ \nu(dy, s) \geq 0, \]
or equivalently
\[ R^2 - \left| y_i(s) - y_i(0)e^{s/2} \right|^2 \geq R^2, \]
and hence \( y_i(s) = y_i(0)e^{s/2} \) follows for \( s \in J_i \).
This relation holds for each \( i = 1, \cdots, n \), so that
\[ d_i(s) = \min_{j \neq i} |y_i(s) - y_j(s)| \]
is increasing in \( s \). We have \( J_i = [0, \infty) \) and the relation \( y_i(s) = y_i(0)e^{s/2} \)
continues to hold for every \( s \in [0, \infty) \). Now, we translate the time variable
as $s \mapsto s - s_0$, repeat the same argument, and see that $y_i(s - s_0) = y_i(-s_0)e^{s/2}$ holds for any $s_0 \geq 0$. This implies $y_i(-s)e^{s} = y_i(0)$ for $s \geq 0$, so that

$$y_i(s) = y_i(0)e^{s/2} \quad (s \in \mathbb{R})$$

holds. Consequently,

$$\lim_{s \to -\infty} y_i(s) = 0$$

follows for $i = 1, \cdots, n$. However, this contradicts to (18) in the case of $n \geq 2$. We get $n = 1$, $m(x_0) = m_*(x_0)$, and

$$\mu(dy, s) = m_*(x_0)\delta_{y_0e^{s/2}}(dy) \quad (s \in \mathbb{R}),$$

and the proof is complete.

References


[22] Suzuki, T., Free energy and Self-Interacting Particles, to be published from Birkhauser, Boston.

