Numerical Radius Norm for Module Maps

モジュール写像の数域半径ノルム

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In the present note we will explain a couple of results related to numerical radius norm for module maps on $C^*$-algebras.

(1) One is to extend the Ando-Okubo's theorem concerning Schur multipliers in the infinite dimensional setting.

(2) The other is to characterize a completely bounded module map.

This is joint with Masaru Nagisa (Chiba University).

1. Schur products and Schur multipliers

Let $M_n(\mathbb{C})$ be the $n \times n$ matrix algebra over $\mathbb{C}$. For $a = [a_{ij}], b = [b_{ij}] \in M_n(\mathbb{C})$, the Schur product $\circ$ is defined by

$$a \circ b = [a_{ij}b_{ij}].$$

The Schur multiplier $S_a : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ for $a \in M_n(\mathbb{C})$ is defined by $S_a(x) = a \circ x$. The Schur norm for $a \in M_n(\mathbb{C})$ is defined by

$$\|a\|_s = \|S_a\| = \sup\{\|a \circ x\| : \|x\| = 1\}.$$

The following is due to Haagerup in which the Schur multiplier appears naturally in operator algebras.

Example [Haagerup, 5]  Let $G$ be a locally compact group, $C(G)$ the continuous functions and $R(G)$ the group von-Neumann algebra. For $\varphi \in C(G)$, if

$$M_\varphi : R(G) \ni \lambda_g \mapsto \varphi(g)\lambda_g \in R(G)$$
is normal (σ-weak - σ-weak continuous), then
\[ \|M_\varphi\|_{cb} = \sup\{\|\varphi(g_j^{-1}g_i)\|_s | g_i \in G, i \leq n, n \in \mathbb{N}\}. \]

The next result is unpublished.

**Theorem A** [Haagerup, 5] Let \( a = [a_{ij}] \in M_n(\mathbb{C}) \). Then the following are equivalent:

1) \( \|S_a\| \leq 1. \)
2) There are \( 0 \leq r_1, r_2 \in M_n(\mathbb{C}) \) such that
\[
\begin{bmatrix} r_1 & a \\ a^* & r_2 \end{bmatrix} \geq 0, \quad r_1 \circ I \leq I \text{ and } r_2 \circ I \leq I.
\]
3) \( a \) has a factorization \( a = b^*c \) such that \( b^*b \circ I \leq I, \ c^*c \circ I \leq I. \)
4) There are vectors \( \{\xi_i\}, \{\eta_i\} \subset \ell^2_n, (i = 1, \cdots, n) \) such that \( \|\xi_i\|, \|\eta_i\| \leq 1 \) and \( a_{ij} = (\xi_j|\eta_i) \).

We consider the numerical radius norm \( w(\cdot) \) on \( \mathcal{B}(\mathcal{H}) \):
\[
w(a) = \sup_{\xi \neq 0} \frac{|(a\xi|\xi)|}{\|\xi\|^2}.
\]
It is easy to see that \( w(a) \leq \|a\| \leq 2w(a) \).

We also consider the induced norm for \( S_a \) with respect to the numerical radius norm that will be denoted by \( \|S_a\|_w \):
\[
\|S_a\|_w \equiv \sup_{x \neq 0} \frac{w(a \circ x)}{w(x)}.
\]

The following is due to Ando and Okubo which looks similar to Theorem A but each condition is finer than the above.

**Theorem B** [Ando-Okubo, 2] Let \( a = [a_{ij}] \in M_n(\mathbb{C}) \). Then the following are equivalent.
1) \( \|S_a\|_w \leq 1. \)
2) There is a $0 \leq r \in M_n(\mathbb{C})$ such that
\[
\begin{bmatrix}
  r & a \\
  a^* & r
\end{bmatrix} \geq 0, \quad \text{and} \quad r \circ I \leq I.
\]

3) $a$ has a factorization $a = b^*d b$ such that $b^*b \circ I \leq I$, $d^*d \leq I$.

4) There are vectors $\{\xi_i\} \subset \ell^2_n, (i = 1, \cdots, n)$ and a contraction $d \in M_n(\mathbb{C})$ such that $||\xi_i|| \leq 1$ and $a_{ij} = (d \xi_j | \xi_i)$.

**Remark** The Haagerup’s theorem is derived from the Ando-Okubo’s theorem, because
\[
||S_a|| = ||S \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}||_w.
\]

To show 1) $\Rightarrow$ 2), $||S_a|| \leq 1$ implies
\[
||S \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix}||_w \leq 1.
\]

By the implication 1) $\Rightarrow$ 2), there exists
\[
0 \leq r = \begin{bmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \end{bmatrix} \in M_n(\mathbb{C}) \quad \text{such that}
\]
\[
\begin{bmatrix}
  r_{11} & r_{21} & 0 & a \\
  r_{12} & r_{22} & 0 & 0 \\
  0 & 0 & r_{11} & r_{21} \\
  a^* & 0 & r_{12} & r_{22}
\end{bmatrix} \geq 0.
\]

This implies that
\[
\begin{bmatrix}
  r_{11} & a \\
  a^* & r_{22}
\end{bmatrix} \geq 0.
\]

2. **Module maps on operator systems**

Let $\varphi : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$. Then the followings are equivalent:

1) There exists $a \in M_n(\mathbb{C})$ such that $\varphi = S_a$. 

2) \( \varphi(\lambda x \mu) = \lambda \varphi(\mu) \) \( \lambda \in M_n(\mathbb{C}), \)

\[
\lambda = \begin{bmatrix}
\lambda_1 \\
\vdots \\
0
\end{bmatrix} \quad \text{and} \quad \mu = \begin{bmatrix}
\mu_1 \\
\vdots \\
0
\end{bmatrix} \in M_n(\mathbb{C})
\]

(i.e. \( \ell_n^\infty \)-module map)

Let \( A \) be a \( C^* \)-algebra and \( V \) an operator system in \( \mathcal{B}(\mathcal{H}) \), i.e., \( V \) is a self-adjoint subspace in \( \mathcal{B}(\mathcal{H}) \) with the identity. Let \( T \) be a bounded linear map from \( (V, || \cdot ||) \) to \( (\mathcal{B}(\mathcal{H}), || \cdot ||) \). We denote by \( T \otimes \text{id}_n \) the linear map

\[
\mathcal{M}_n(V) \ni [x_{ij}] \mapsto [T(x_{ij})] \in \mathcal{M}_n(\mathcal{B}(\mathcal{H})).
\]

If \( \sup_n ||T \otimes \text{id}_n|| \) is bounded, then we say \( T \) is completely bounded and denote the supremum by \( ||T||_{cb} \). If \( T \otimes \text{id}_n \) is positive for all \( n \), then we say \( T \) is completely positive.

We denote by \( ||T||_w \) the operator norm of \( T \) viewed as a bounded linear map from \( (V, w(\cdot)) \) to \( (\mathcal{B}(\mathcal{H}), w(\cdot)) \), i.e.,

\[
||T||_w = \sup\{w(T(x)) | w(x) \leq 1, x \in V\}.
\]

Every completely bounded map from an operator space to \( \mathcal{B}(\mathcal{H}) \) is also completely bounded with respect to numerical radius norm. We use the following notation:

\[
||T||_{wcb} = \sup_{n \in \mathbb{N}} ||T \otimes \text{id}_n||_w.
\]

We call that an action of \( A \) on \( \mathcal{H} \) is locally cyclic if, for any \( n \) and \( \xi_1, \xi_2, \ldots, \xi_n \in \mathcal{H} \), there exists a vector \( \eta \in \mathcal{H} \) such that

\[
\xi_i \in \text{the norm closure of } \{a\eta | a \in A\}.
\]

We remark that, for \( x = (x_{ij}) \in \mathcal{M}_n(V) \subset \mathcal{B}(\mathcal{H}^n) \) and \( a = (a_{kl}) \in M_{nm}(A) \), we can see

\[
a^* \cdot x \cdot a = (\sum_{k,l}a_{ki}^* \cdot x_{kl} \cdot a_{lj}) \in \mathcal{M}_m(V) \subset \mathcal{B}(\mathcal{H}^m)
\]

and we get

\[
w(a^* \cdot x \cdot a) \leq ||a||^2 w(x).
\]

The condition (locally cyclic) implies that \( || \cdot ||_w = || \cdot ||_{wcb} \).
Proposition 1 Let $A$ be a unital $C^*$-algebra, $V$ an $A$-bimodule operator system and $T$ a bounded $A$-bimodule map from $V$ to $\mathcal{B}(\mathcal{H})$. If the action of $A$ on $\mathcal{H}$ is locally cyclic, then we have

$$||T||_w = ||T||_{wcb}.$$ 

Let $A$ be a unital $C^*$-algebra and $V$ an $A$-bimodule operator system in $\mathcal{B}(\mathcal{H})$. Set

$$N = \{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} | x, w \in \mathcal{B}(\mathcal{H}), y, z \in V \}.$$ 

Then $N$ is an $A$-bimodule operator system by the action

$$a \cdot \begin{pmatrix} x & y \\ z & w \end{pmatrix} \cdot b = \begin{pmatrix} axb & ayb \\ azb & awb \end{pmatrix} \text{ for } a, b \in A.$$ 

Theorem 2 Let $A$ be a unital $C^*$-algebra, $V$ an $A$-bimodule operator system in $\mathcal{B}(\mathcal{H})$ and $T$ a completely bounded $A$-bimodule map from $V$ to $\mathcal{B}(\mathcal{H})$. Then we have

$$||T||_{wcb} = \inf \{ ||S|| | \begin{pmatrix} S & T \\ T^* & S \end{pmatrix} : N \rightarrow M_2(\mathcal{B}(\mathcal{H})) \}$$

$$A\text{-bimodule completely positive} \}$$

$$= \inf \{ ||S|| | \begin{pmatrix} S & T \\ T^* & S \end{pmatrix} : N \rightarrow M_2(\mathcal{B}(\mathcal{H})) \}$$

$$\text{completely positive} \}$$

where

$$\begin{pmatrix} S & T \\ T^* & S \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} S(x) & T(y) \\ T(z^*)^* & S(w) \end{pmatrix},$$

for $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in N = \{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} | x, w \in \mathcal{B}(\mathcal{H}), y, z \in V \}$. 
To show this, we need the Wittstock's Hahn Banach type theorem for completely bounded maps. The key operator space $N_0$ is that

$$N_0 = \left\{ \begin{bmatrix} a+x & y \\ z & a-x \end{bmatrix} \mid a \in A, x \in \mathcal{B}(\mathcal{H}), y, z \in V \right\}$$

and we consider

$$\varphi_0 \left( \begin{bmatrix} a+x & y \\ z & a-x \end{bmatrix} \right) = a + \frac{1}{2}(T(y) + T(z^*)^*).$$

We can see that $\varphi_0$ is a completely positive $A$-bimodule map. Using the Wittstock's Hahn-Banach theorem [13], we get the unital completely positive $A$-bimodule map $\varphi$ from $N$ to $\mathcal{B}(\mathcal{H})$ which is an extension of $\varphi_0$. Set

$$S(x) = 2\varphi \left( \begin{bmatrix} x & 0 \\ 0 & 0 \end{bmatrix} \right)$$

Then we have a desired map.

**Remark** V. I. Paulsen and C. Y. Suen [9] introduced the norm \( |||T||| \) for a completely bounded map $T$ from a C*-algebra $A$ to $\mathcal{B}(\mathcal{H})$ as follows:

$$|||T||| = \inf \{ \|S\| \mid \begin{bmatrix} S & T \\ T^* & S \end{bmatrix} : \mathcal{M}_2(A) \rightarrow \mathcal{M}_2(\mathcal{B}(\mathcal{H})) \text{ completely positive} \}.$$  

By the injectivity of $\mathcal{B}(\mathcal{H})$, we can get

$$|||T||| = ||T|| _{wcb}.$$  

We set $\mathcal{A}(^*) = \{ x^* \in \mathcal{B}(\mathcal{H}) \mid x \in \mathcal{A} \}$.  


Theorem 3 Let $A$ be a norm closed unital algebra on $\mathcal{H}$ and $T$ a completely bounded left $A^{(*)}$-right $A$-module map from $\mathbb{K}(\mathcal{H})$ to $\mathcal{B}(\mathcal{H})$. Then there exist $t = (t_{ij}) \in \mathcal{B}(\ell^2(I))$ and $\{v_i \mid i \in I\} \subset A'$ such that

$$\|t\| = \|T\|_{wcb}, \quad \sum_{i \in I} v_i^*v_i \leq 1$$

$$T(x) = \sum_{i,j \in I} v_i^* t_{ij} x v_j \quad (x \in \mathbb{K}(\mathcal{H})).$$

To see this, we may regard $T$ as a normal completely bounded $A^{(*)}$--$A$-module map on $\mathcal{B}(\mathcal{H})$ Then there exist a $*$-representation $\pi$ of $\mathbb{K}(\mathcal{H})$ on a Hilbert space $\mathcal{K}$, an isometry $w : \mathcal{H} \to \mathcal{K}$ and an operator $s \in \pi(\mathbb{K}(\mathcal{H}))'$ such that

$$\|T\|_{wcb} = \|s\|, \quad T(\cdot) = w^* s \pi(\cdot) w.$$

Since all irreducible representations of $\mathbb{K}(\mathcal{H})$ are unitarily equivalent to the identity representation, we may assume that

$$\mathcal{K} = \mathcal{H} \otimes \ell^2(I), \quad \pi(x) = x \otimes 1, \quad s = (s_{ij}1_{\mathcal{B}(\mathcal{H})})_{i,j \in I}$$

$$(s_{ij} \in \mathbb{C}), \quad w = (w_i)_{i \in I} \in \mathcal{B}(\mathcal{H}, \mathcal{H} \otimes \ell^2(I)),$$

$$T(x) = w^* s (x \otimes 1) w = \sum_{i,j \in I} w_i^* s_{i,j} x w_j \text{ for } x \in \mathbb{K}(\mathcal{H}).$$

We can replace $\{w_i\}$ by $\{v_i\} \subset A'$.

Corollary 4 [Smith, 11] Let $A$ and $B$ be norm closed unital algebras on $\mathcal{H}$ and $T$ a completely bounded left $A$-right $B$-module map from $\mathbb{K}(\mathcal{H})$ to $\mathcal{B}(\mathcal{H})$. Then there exist $\{a_i \mid i \in I\} \subset A'$ and $\{b_i \mid i \in I\} \subset B'$ such that

$$T(x) = \sum_{i \in I} a_i x b_i, \quad \|\sum_{i \in I} a_i a_i^*\| \|\sum_{i \in I} b_i^* b_i\| = \|T\|_{cb}^2$$

for $x \in \mathbb{K}(\mathcal{H})$.

We define the left action of $A \oplus B^{(*)}$ and the right action of $A^{(*)} \oplus B$ on $\mathcal{M}_\kappa(\mathbb{K}(\mathcal{H}))$ and the left $A \oplus B^{(*)}$-right $A^{(*)}$-module completely bounded.
map $\tilde{T}$ from $\mathbb{M}_2(\mathbb{K}(\mathcal{H}))$ to $\mathbb{M}_2(\mathbb{B}(\mathcal{H}))$ as follows:

$$(a_1 \oplus b_1^*)(x \ y \ z \ w) (a_2^* \oplus b_2) = (a_1 x a_2^* \ a_1 y b_2) (b_1^* z a_2^* \ b_1^* w b_2)$$

$$\tilde{T}(x \ y \ z \ w) = \begin{pmatrix} 0 & T(y) \\ 0 & 0 \end{pmatrix}$$

where $x, y, z, w \in \mathbb{K}(\mathcal{H})$ and $a_1, a_2 \in A, \ b_1, b_2 \in B$. Then we show that $||\tilde{T}||_{wcb} = ||T||_{cb}$. Apply the previous theorem for $\tilde{T}$. Then we have the desired form for $T$.

For a Hilbert space $\mathcal{H}$, we choose a completely orthonormal system $\{e_i | i \in I\}$. We denote by $\ell^\infty$ the maximal abelian subalgebra of $\mathbb{B}(\mathcal{H})$ generated by $\{e_i \otimes e_i | i \in I\}$, where $(e_i \otimes e_j)(\xi) = (\xi|e_j)e_i$ for $\xi \in \mathcal{H}$. Let $T$ be a bounded $\ell^\infty$-bimodule map from $\mathbb{K}(\mathcal{H})$ to $\mathbb{B}(\mathcal{H})$. By the module property of $T$, we have the $I \times I$-matrix $a = (a_{ij})$ over $\mathbb{C}$ such that

$$T(e_i \otimes e_j) = a_{ij}(e_i \otimes e_j).$$

Since the set $\{a_{ij}\}_{i,j \in I}$ is bounded, we can define the bounded linear operator $a_T$ from $\ell^1$ to $\ell^\infty$ given by

$$a_T((\lambda_j)_{j \in I}) = (\sum_{j \in I} a_{ij} \lambda_j)_{i \in I} \text{ for } (\lambda_j)_{j \in I} \in \ell^1.$$ 

We will extend the Ando-Okubo's theorem.

**Theorem 4** Let $T$ be an $\ell^\infty$-bimodule bounded linear map from $\mathbb{K}(\mathcal{H})$ to $\mathbb{B}(\mathcal{H})$. Then the following are equivalent:

1. $||T||_w \leq 1$.
2. $||T||_{wcb} \leq 1$.
3. There exists a completely positive contraction $S$ from $\mathbb{K}(\mathcal{H})$ to $\mathbb{B}(\mathcal{H})$ such that $\begin{pmatrix} S & T \\ T^* & S \end{pmatrix} : \mathbb{M}_2(\mathbb{K}(\mathcal{H})) \to \mathbb{M}_2(\mathbb{B}(\mathcal{H}))$ is $\ell^\infty$-bimodule completely positive.
4. There exist a bounded linear operator $v$ from $\ell^1$ to $\ell^2$ and $b \in \mathbb{B}(\ell^2)$ such that $a_T = v^* b v$ and $||v||^2 ||b|| \leq 1$.
5. There exist $\{\xi_i | i \in I\} \subset \mathcal{H}$ and $b \in \mathbb{B}(\mathcal{H})$ such that $||\xi_i|| \leq 1, ||b|| \leq 1, a_{ij} = (b \xi_j | \xi_i)$. 

Note added in proof.

References


