<table>
<thead>
<tr>
<th>Title</th>
<th>DIFFERENCE IN PROJECTIONS (Structure of operators and related current topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Fujii, Masatoshi; Nakamoto, Ritsuo</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2003), 1312: 27-30</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2003-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/42935">http://hdl.handle.net/2433/42935</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
<tr>
<td>Publisher</td>
<td>Kyoto University</td>
</tr>
</tbody>
</table>
DIFFERENCE IN PROJECTIONS

ABSTRACT. Let $P$ and $Q$ be orthogonal projections. Then it is well known that

$$||P - Q|| = \max\{||PQ^\perp||, ||P^\perp Q||\}.$$  

For this formula, we give more precise estimations by elementary methods. Among others, an operator inequality

$$-||P^\perp Q|| \leq P - Q \leq ||PQ^\perp||$$  

is shown, in which the constants on both sides are optimal except the trivial cases. As a corollary, it is proved that $||R + S|| = 1 + ||RS||$ for orthogonal projections $R$ and $S$.

1. INTRODUCTION

Throughout this note, an operator means a bounded linear operator acting on a Hilbert space $\mathcal{H}$ and $\sigma(T)$ denotes the spectrum of an operator $T$.

The following result on the opening of two subspaces is well known:

(1) $$||P - Q|| = \max\{||P^\perp Q||, ||Q^\perp P||\}$$

for two (orthogonal) projections $P$ and $Q$ (see [1]), where $R^\perp = 1 - R$.

Izumino and Watatani [6] pointed out that (1) is assured by the following two facts:

(i) If $A$ and $B$ are positive operators with $AB = 0$, then

$$||A + B|| = \max\{||A||, ||B||\}.$$  

(ii) If $P$ and $Q$ are projections, then

$$(P - Q)^2 = Q^\perp PQ^\perp + PQ^\perp Q.$$  

By the way, the formula (1) says that $||P - Q|| \leq 1$. If $||P - Q|| < 1$ in particular, then $P$ and $Q$ are interchanged by a symmetry $U$ (i.e., $U = U^*$ and $U^2 = 1$), which is given as follows (see [5]):

$$U = Q\{1 - (P - Q)^2\}^{-1/2}P - Q^\perp\{1 - (P - Q)^2\}^{-1/2}P^\perp.$$  

Furthermore, Izumino and Watatani [6] proved that if $P$ and $Q$ are projections interchanged by a symmetry, then

(2) $$||P - Q|| = ||PQ^\perp|| = ||P^\perp Q||.$$  

In this note, we shall give more precise descriptions for the formula (1). Among others, we present an operator inequality

$$-||P^\perp Q|| \leq P - Q \leq ||PQ^\perp||.$$  

1991 Mathematics Subject Classification. 46A30, 47A66.

Key words and phrases. projection, norm, inequality.
in which the constants on both sides are optimal except the trivial cases. As an application, we have a result due to Duncan and Taylor [3] that \( \|P + Q\| = 1 + \|PQ\| \) for projections \( P \) and \( Q \). In addition, we pose an elementary proof of the formula (2) under the assumption \( \|P - Q\| < 1 \).

2. Results

First of all, we prove that the following norm equalities hold for two projections.

**Lemma 1.** Let \( P \) and \( Q \) be orthogonal projections on \( \mathcal{H} \). Then the following statements hold:

(i) If \( \|P - Q\| \in \sigma(P - Q) \), then \( \|P - Q\| = \|PQ^\perp\|\).

(ii) If \( -\|P - Q\| \in \sigma(P - Q) \), then \( \|P - Q\| = \|P^\perp Q\|\).

**Proof.** (i) If \( a = \|P - Q\| \in \sigma(P - Q) \), then using the Berberian representation ([2]) if necessary, we may assume that

\[
(P - Q)x = (Q^\perp - P^\perp)x = ax
\]

for some non-zero \( x \in \mathcal{H} \). Hence we have \( Q^\perp P x = aQ^\perp x \) and \( PQ^\perp x = aPx \), and so

\[
\|Q^\perp P x\| = a\|Q^\perp x\|, \quad (Q^\perp P x, x) = a\|Q^\perp x\|^2.
\]

Also, it is easy to see \( Px \neq 0 \) because \( a > 0 \).

Moreover, since

\[
(Q^\perp P x, x) = (x, PQ^\perp x) = a\|Px\|^2,
\]

we have \( \|Q^\perp x\| = \|Px\| \), and so \( \|Q^\perp P x\| = a\|Q^\perp x\| = a\|Px\| \). That is, \( a \leq \|Q^\perp P\| = \|P - Q\| = a \).

(ii) If \( -\|P - Q\| \in \sigma(P - Q) \), then \( \|Q - P\| \in \sigma(Q - P) \), so that (ii) is proved by (i).

\( \square \)

**Remark.** The following result is due to Izumino and Watatani [6]: If \( \|P - Q\| < 1 \) for orthogonal projections \( P \) and \( Q \) on \( \mathcal{H} \), then

\[
\|P - Q\| = \|PQ^\perp\| = \|P^\perp Q\|.
\]

As an application of Lemma 1, we give it an elementary proof with no use of the existence of a symmetry. As a matter of fact, it is sufficient to consider the case \( \|P - Q\| \in \sigma(P - Q) \). Then, Lemma 1 implies that \( \|P - Q\| = \|PQ^\perp\| \) holds. On the other hand, it follows from (3) that \( -QP^\perp x = \|P - Q\|Qx \), so that

\[
\|QP^\perp x\|\|Qx\| \geq \|(QP^\perp x, x)\| = \|P - Q\|(Qx, x) = \|P - Q\|\|Qx\|^2.
\]

Noting \( Qx \neq 0 \) as \( \|P - Q\| < 1 \), we have \( \|P - Q\| = \|QP^\perp\| \).

Based on our consideration in Lemma 1, we have the following estimation of \( P - Q \) itself.

**Theorem 2.** Let \( P \) and \( Q \) be orthogonal projections on \( \mathcal{H} \). Then

\[
-\|PQ^\perp\| \leq P - Q \leq \|PQ^\perp\|.
\]

Moreover the constants \( -\|PQ^\perp\| \) and \( \|PQ^\perp\| \) are optimal except the trivial cases (a) \( P = 1 \) and \( Q = 0 \), and (b) \( P = 0 \) and \( Q = 1 \), i.e., if \( -c \leq P - Q \leq d \) for \( c, d \geq 0 \), then \( c \geq \|PQ^\perp\| \) and \( \|PQ^\perp\| \leq d \).
Proof. If $P \geq Q$ or $P \leq Q$ holds, then the conclusion is clear.

Put $b = \sup \{ \lambda ; \lambda \in \sigma (P - Q) \} > 0$. Since we may assume that $b$ is a positive eigenvalue of $P - Q$, we have $b \leq \| PQ^\perp \|$ as in the proof of Theorem 1 because (3) can be assumed for $b$. On the other hand, since $Q^\perp P Q^\perp = Q^\perp (P - Q) Q^\perp \leq bQ^\perp$, we have $\| PQ^\perp \|^2 \leq b$.

Therefore it suffices to show that $b \geq \| PQ^\perp \|$ under the case $\| PQ^\perp \| < 1$. First of all, if $\| P - Q \| < 1$, then the existence of a symmetry $U$ with $Q = UP\U$ implies the symmetry of $\sigma (P - Q)$ with respect to the origin because

$$P - Q = P - UP\U = U(UP\U - P)U = U(Q - P)U = -U(P - Q)U.$$ 

Hence $\pm \| P - Q \| \in \sigma (P - Q)$ and so $b = \| P - Q \| \geq \| PQ^\perp \|$.

Next we suppose that $\| P - Q \| = 1$, and put $M = \{ x \in H ; Px = 0, Qx = x \}$. Then $M$ is the eigenspace of $-1$ for $P - Q$, i.e., $M = \{ x \in H ; (P - Q)x = -x \}$. As a matter of fact, if $x \in M$, then $(P - Q)x = -x$ easily. Conversely, if $(P - Q)x = -x \neq 0$, then $Px + Q^\perp x = 0, Q^\perp x = -Px$. Hence it follows that

$$\| Q^\perp x \| = \| P^\perp Px \| \leq \| Px \| = \| PQ^\perp x \| \leq \| Q^\perp x \|,$$

and so $\| PQ^\perp x \| = \| Q^\perp x \|$. If $Q^\perp x \neq 0$, then $\| PQ^\perp \| \geq 1$, which contradicts to the assumption $\| PQ^\perp \| < 1$. Namely we have $Q^\perp x = 0 = Px$.

So $M$ is a (nontrivial) reducing subspace of both $P$ and $Q$. We here put $P_1 = P|_{M^\perp}$ and $Q_1 = Q|_{M^\perp}$. Noting that $PQ^\perp|_{M} = 0$, we have $\| PQ^\perp \| = \| P_1 Q_1^\perp \|( < 1)$, and $b_1 = \sup \{ (P_1 - Q_1)x, x \in M^\perp, \| x \| = 1 \} = b$. Since $\| P_1 Q_1^\perp \| < 1$, we have $P_1 \wedge Q_1^\perp = 0$, where $\wedge$ means the infimum in the case of projections. Moreover we may assume that $P \wedge Q = P^\perp \wedge Q^\perp = 0$. Namely, $P_1$ and $Q_1$ are in generic position in the sense of Halmos [4]. So the structure theorem [4; Theorem 2] says that there exist commuting positive contractions $S$ and $C$ on some Hilbert space such that $S^2 + C^2 = 1$, ker $S = \ker C = 0$ and that $P_1$ and $Q_1$ are unitarily equivalent to

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix}.$$ 

Since it is easily checked that

$$EF^\perp E = \begin{pmatrix} S^2 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad (E - F)^2 = \begin{pmatrix} S^2 & 0 \\ 0 & S^2 \end{pmatrix},$$

we have $\| E - F \| = \| S \| = \| EF^\perp \| < 1$. Applying $P_1$, $Q_1$ and $b_1$ to the first case in this proof, we have

$$b = \| P_1 - Q_1 \| = \| P_1 Q_1^\perp \| = \| PQ^\perp \|.$$

Finally the other part $\| P^\perp Q \| \leq P - Q$ is equivalent to the inequality $Q - P \leq \| PQ^\perp \|$. So there is nothing to do.

As a direct consequence, we have the following result appeared in [3]:

**Corollary 3.** If $R$ and $S$ are orthogonal projections, then

$$\| R + S \| = 1 + \| RS \|.$$ 

**Proof.** We have

$$0 \leq 1 - \| R^\perp S \parallel \leq 1 - R - S \parallel \leq 1 + \| RS \|$$

by Theorem 2. As $R + S = 1 + R - S \parallel$ and the last inequality is optimal by Theorem 2 again, the conclusion $\| R + S \| = 1 + \| RS \|$ is obtained. \qed
Acknowledgement. The authors would like to express their thanks to Professor Araki and Professor Kosaki for their kind suggestion.

REFERENCES


*') Department of Mathematics, Osaka Kyoiku University, Kashiwara, Osaka 582-8582, Japan.
E-mail address: mfuji@cc.osaka-kyoiku.ac.jp

**) Faculty of Engineering, Ibaraki University, Nakanarusawa, Hitachi, Ibaraki 316-0033, Japan.
E-mail address: nakamoto@base.ibaraki.ac.jp