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Kyoto University
On classes of operators generalizing class A
and paranormality and related results

東京理科大学 伊藤公智 (Masatoshi Ito)
(Department of Mathematical Information Science, Tokyo University of Science)

This report is based on the following papers:


Abstract

Recently, we introduced class A defined by an operator inequality, and also the definition of class A is similar to that of paranormality defined by a norm inequality. As generalizations of class A and paranormality, Fujii-Nakamoto introduced class $F(p,r,q)$ and $(p,r,q)$-paranormality respectively. These classes are related to $p$-hyponormality and log-hyponormality.

In this report, we shall remove the assumption of invertibility from some results on invertible class $F(p,r,q)$ operators, and also we shall show that the families of class $F(p,r,\frac{r+q}{2})$ and $(p,r,\frac{r}{1+r})$-paranormality are proper on $p$. Moreover, we shall obtain the relations between Furuta-type inequalities as a generalization of the key theorem in the proofs of our main results.

1 Introduction

In this paper, a capital letter means a bounded linear operator on a complex Hilbert space $H$. An operator $T$ is said to be positive (denoted by $T \geq 0$) if $(Tx,x) \geq 0$ for all $x \in H$, and also an operator $T$ is said to be strictly positive (denoted by $T > 0$) if $T$ is positive and invertible.

As extensions of hyponormal operators, i.e., $T^*T \geq TT^*$, it is well known that $p$-hyponormal operators for $p > 0$ are defined by $(T^*T)^p \geq (TT^*)^p$ and invertible log-hyponormal operators are defined by $\log T^*T \geq \log TT^*$ for an invertible operator $T$, and also an operator $T$ is said to be $p$-quasihyponormal for $p > 0$ if $T^*((T^*T)^p - (TT^*)^p)T \geq 0$. We remark that we treat only invertible log-hyponormal operators in this paper (see also [26]). It is easily obtained that every $p$-hyponormal operator is $q$-hyponormal for
p > q > 0 by Löwner-Heinz theorem "A > B > 0 ensures A^α ≥ B^α for any α ∈ [0, 1]," and every invertible p-hyponormal operator for p > 0 is log-hyponormal since log t is an operator monotone function. We remark that log-hyponormality is sometimes regarded as 0-hyponormality since \( \frac{xp-1}{p} \to \log X \) as \( p \to +0 \) for \( X > 0 \). An operator \( T \) is paranormal if \( \|T^2x\| ≥ \|Tx\|^2 \) for every unit vector \( x \in H \). Ando [2] showed that every p-hyponormal operator for \( p > 0 \) and (invertible) log-hyponormal operator is paranormal.

Recently, in [15], we introduced class A defined by \( |T^2| ≥ |T|^2 \) where \( T = (T^*T)^{1/2} \), and we showed that every invertible log-hyponormal operator belongs to class A and every class A operator is paranormal. We remark that class A is defined by an operator inequality and paranormality is defined by a norm inequality, and their definitions appear to be similar forms. And also Fujii-Jung-S.H.Lee-M.Y.Lee-Nakamoto [9] introduced class A\( (p, r) \) and Yamazaki-Yanagida [28] introduced absolute-(p, r)-paranormality as follows: An operator \( T \) belongs to class A\( (p, r) \) for \( p > 0 \) and \( r > 0 \) if \( (|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} ≥ |T^*|^{2r} \), and also an operator \( T \) is absolute-(p, r)-paranormal if \( \|T^p|T^*|^r x\|^r ≥ \||T^*|^r x\|^{p+r} \) for every unit vector \( x \in H \). We remark that class A\( (1, 1) \) equals class A and also absolute-(1, 1)-paranormality equals paranormality. These classes are generalizations of class A\( (k) \) and absolute-k-paranormality introduced as two families of classes based on class A and paranormality in [15], and also absolute-(p, r)-paranormality is a generalization of p-paranormality in [7]. We should remark that the families of class A\( (p, r) \) determined by operator inequalities and absolute-(p, r)-paranormality determined by norm inequalities constitute two increasing lines on \( p > 0 \) and \( r > 0 \) whose origin is (invertible) log-hyponormality.

Moreover, as a continuation of the discussion in [9], Fujii-Nakamoto [10] introduced the following classes of operators.

**Definition ([10]).** For each \( p > 0 \), \( r ≥ 0 \) and \( q > 0 \),

(i) An operator \( T \) belongs to class \( F(p, r, q) \) if

\[
(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{1}{q}} ≥ |T^*|^{\frac{2(p+r)}{q}}. \tag{1.1}
\]

(ii) An operator \( T \) is \((p, r, q)-paranormal \) if

\[
\|T^pU|T^r x\|^{\frac{1}{q}} ≥ \|T^r|^{\frac{2(p+r)}{q}} x\| \tag{1.2}
\]

for every unit vector \( x \in H \), where \( T = U|T| \) is the polar decomposition of \( T \). In particular, if \( r > 0 \) and \( q ≥ 1 \), then (1.2) is equivalent to

\[
\|T^p|T^*|^r x\|^{\frac{1}{q}} ≥ \|T^r|^{\frac{2(p+r)}{q}} x\| \tag{1.3}
\]

for every unit vector \( x \in H \) ([18]).
We remark that class $F(p, r, \frac{p+r}{r})$ equals class $A(p, r)$ and also $(p, r, \frac{p+r}{r})$-paranormality equals absolute-$(p, r)$-paranormality. In [18], we obtained the parallel result to that of class $A(p, r)$ and absolute-$(p, r)$-paranormality that invertible class $F(p, r, q)$ and $(p, r, q)$-paranormality constitute two increasing lines on $p \geq \delta > 0$ and $r \geq r_0 > 0$ whose origin is $\delta$-quasihyponormality. And also we showed the result on powers of invertible class $F(p, r, q)$ operators. Thus many researchers have been discussed parallel families of classes of operators which are generalizations of class $A$ and paranormality.

In this report, we shall remove the assumption of invertibility from some results on invertible class $F(p, r, q)$ operators in [18], and also we shall show that the families of class $F(p, r, \frac{p+r}{r})$ and $(p, r, \frac{p+r}{r})$-paranormality are proper on $p$. Moreover, we shall obtain the relations between Furuta-type inequalities as a generalization of the result shown in [19] which is the key theorem in the proofs of our main results.

2 Preliminaries

Fujii-Nakamoto [10] observed that class $F(p, r, q)$ derives from the following Theorem 2.A shown in [11] and $(p, r, q)$-paranormality corresponds to class $F(p, r, q)$.

We remark that alternative proofs of Theorem 2.A were given in [5] and [21] and also an elementary one page proof in [12]. Tanahashi [23] showed that the domain drawn for $p, q$ and $r$ in the Figure 1 is the best possible one for Theorem 2.A.

**Theorem 2.A (Furuta inequality [11]).**

If $A \succeq B \succeq 0$, then for each $r \geq 0$,

(i) \[ (B^\frac{r}{2}A^pB^\frac{r}{2})^{\frac{1}{q}} \succeq (B^\frac{r}{2}B^pB^\frac{r}{2})^{\frac{1}{q}} \]

and

(ii) \[ (A^\frac{r}{2}A^pA^\frac{r}{2})^{\frac{1}{q}} \succeq (A^\frac{r}{2}B^pA^\frac{r}{2})^{\frac{1}{q}} \]

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

Fujii-Nakamoto [10] and the author [18] obtained the results on inclusion relations among the families of class $F(p, r, q)$ and $(p, r, q)$-paranormality.

**Theorem 2.B ([10]).**

(i) For a fixed $k > 0$, $T$ is $k$-hyponormal if and only if $T$ belongs to class $F(2kp, 2kr, q)$ for all $p > 0$, $r \geq 0$ and $q \geq 1$ with $(1+2r)q \geq 2(p+r)$, i.e., $T$ belongs to class $F(p, r, q)$ for all $p > 0$, $r \geq 0$ and $q \geq 1$ with $(k+r)q \geq p+r$. 

![Figure 1](image-url)
(ii) If $T$ belongs to class $F(p_0, r_0, q_0)$ for $p_0 > 0$, $r_0 \geq 0$ and $q_0 \geq 1$, then $T$ belongs to class $F(p_0, r_0, q)$ for any $q \geq q_0$.

(iii) If $T$ is $(p_0, r_0, q_0)$-paranormal for $p_0 > 0$, $r_0 \geq 0$ and $q_0 > 0$, then $T$ is $(p_0, r_0, q)$-paranormal for any $q \geq q_0$.

(iv) If $T$ belongs to class $F(p, r, q)$ for $p > 0$, $r \geq 0$ and $q \geq 1$, then $T$ is $(p, r, q)$-paranormal.

**Theorem 2.C ([18]).**

(i) For each $p > 0$ and $r > 0$,

(i-1) $T$ is $p$-quasihyponormal if and only if $T$ belongs to class $F(p, r, 1)$ if and only if $T$ is $(p, r, 1)$-paranormal.

(i-2) $T$ is $p$-quasihyponormal if and only if $T$ is $(p, 0, 1)$-paranormal.

(ii) Let $T$ be a class $F(p_0, r_0, \frac{p_0 + r_0}{\delta + r_0})$ operator for $p_0 > 0$, $r_0 \geq 0$ and $\delta > -r_0$.

(ii-1) If $T$ is invertible and $0 \leq \delta \leq p_0$, then $T$ belongs to class $F(p, r, \frac{p + r}{\delta + r})$ for any $p \geq p_0$ and $r \geq r_0$.

(ii-2) If $-r_0 < \delta \leq p_0$, then $T$ belongs to class $F(p_0, r, \frac{p_0 + r}{\delta + r})$ for any $r \geq r_0$.

(iii) Let $T$ be a $(p_0, r_0, \frac{p_0 + r_0}{\delta + r_0})$-paranormal operator for $p_0 > 0$, $r_0 \geq 0$ and $\delta > -r_0$.

(iii-1) If $0 \leq \delta \leq p_0$, then $T$ is $(p, r, \frac{p + r}{\delta + r})$-paranormal for any $p \geq p_0$ and $r \geq r_0$.

(iii-2) If $-r_0 < \delta \leq p_0$, then $T$ is $(p_0, r, \frac{p_0 + r}{\delta + r})$-paranormal for any $r \geq r_0$.

(iii-3) If $0 \leq \delta$, then $T$ is $(p, r_0, \frac{p + r}{\delta + r_0})$-paranormal for any $p \geq p_0$.

We remark that only (ii-1) of Theorem 2.C requires invertibility of $T$, and also we obtained in [19] that every class $A(p_0, r_0)$ operator for $p_0 > 0$ and $r_0 > 0$ belongs to class $A(p, r)$ for any $p \geq p_0$ and $r \geq r_0$ (without assumption of invertibility).

Figure 2 on the following page represents the inclusion relations among the families of class $F(p, r, q)$ and $(p, r, q)$-paranormality.

On the other hand, we obtained the results on powers of $p$-hyponormal, class $A(p, r)$ and invertible class $F(p, r, q)$ operators.
Theorem 2.D.

(i) Let $T$ be a $p$-hyponormal operator for $0 < p \leq 1$. Then $T^n$ is $\frac{p}{n}$-hyponormal for all positive integer $n$ ([1]).

(ii) Let $T$ be a class $A(p, r)$ operator for $0 < p \leq 1$ and $0 < r \leq 1$. Then $T^n$ belongs to class $A(\frac{p}{n}, \frac{r}{n})$ for all positive integer $n$ ([19]).

(iii) Let $T$ be an invertible class $F(p, r, q)$ operator for $0 < p \leq 1$, $0 \leq r \leq 1$ and $q \geq 1$ with $rq \leq p+r$. Then $T^n$ belongs to class $F(\frac{p}{n}, \frac{r}{n}, q)$ for all positive integer $n$ ([18]).

We remark that (iii) interpolates (i) and (ii) if $T$ is invertible in Theorem 2.D. In fact, (iii) yields (i) by putting $q = 1$ and $r = 0$, and also (iii) yields (ii) by putting $q = \frac{p+r}{r}$.

Moreover we have another result on powers of class A operators by combining [29, Theorem 1] and [19, Theorem 3].

**Theorem 2.1.** If $T$ is a class A operator, then

$$|T|^2 \leq |T^2| \leq \cdots \leq |T^n|^\frac{2}{n} \quad \text{and} \quad |T^*|^2 \geq |T^{2^*}| \geq \cdots \geq |T^{n^*}|^\frac{2}{n}$$

hold for all positive integer $n$.

We remark that (ii) of Theorem 2.D and Theorem 2.1 in case of invertible operators were shown in [27] and [17], respectively.
3 Main results

In this section, we shall show the results which remove the assumption of invertibility from (ii-1) of Theorem 2.C and (iii) of Theorem 2.D.

**Theorem 3.1.** Let $T$ be a class $F(p_0, r_0, \frac{p_0}{\delta+r_0})$ operator for $p_0 > 0$, $r_0 \geq 0$ and $0 \leq \delta \leq p_0$. Then $T$ belongs to class $F(p, r, \frac{p+r}{\delta+r})$ for any $p \geq p_0$ and $r \geq r_0$.

**Theorem 3.2.** Let $T$ be a class $F(p, r, q)$ operator for $0 < p \leq 1$, $0 \leq r \leq 1$ and $q \geq 1$ with $rq \leq p + r$. Then $T^n$ belongs to class $F(\frac{p}{n}, \frac{r}{n}, q)$ for all positive integer $n$.

We need the following two results in order to prove Theorem 3.1.

**Theorem 3.A ([19, Theorem 1]).** Let $A$ and $B$ be positive operators. Then for each $p \geq 0$ and $r \geq 0$,

(i) If $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$, then $A^p \geq (A^{\frac{r}{2}}B^rA^{\frac{r}{2}})^{\frac{r}{p+r}}$.

(ii) If $A^p \geq (A^{\frac{r}{2}}B^rA^{\frac{r}{2}})^{\frac{r}{p+r}}$ and $N(A) \subseteq N(B)$, then $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r$.

**Theorem 3.B ([29]).** If $A^{\alpha} \geq (A^{\frac{\alpha_0}{2}}B^{\beta_0}A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha_0+\beta_0}}$ holds for positive operators $A$ and $B$ and fixed $\alpha_0 > 0$ and $\beta_0 > 0$, then

$A^{\alpha} \geq (A^{\frac{\alpha_0}{2}}B^{\beta_0}A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha_0+\beta_0}}$

holds for any $\alpha \geq \alpha_0$. Moreover, for each fixed $\gamma \geq -\beta_0$,

$g_{\beta_0, \delta}(\alpha) = (B^{\frac{\beta_0}{2}}A^\alpha B^{\frac{\beta_0}{2}})^{\frac{\delta+\beta_0}{\alpha+\beta_0}}$

is an increasing function for $\alpha \geq \max\{\alpha_0, \delta\}$. Hence $(B^{\frac{\beta_0}{2}}A^{\alpha_1}B^{\frac{\beta_0}{2}})^{\frac{\alpha_1+\beta_0}{\alpha_1+\beta_0}} \geq B^{\frac{\beta_0}{2}}A^{\alpha_1}B^{\frac{\beta_0}{2}}$

holds for any $\alpha_1$ and $\alpha_2$ such that $\alpha_2 \geq \alpha_1 \geq \alpha_0$.

**Proof of Theorem 3.1.** In case $r_0 = 0$, it is already shown in (i) of Theorem 2.B since class $F(p_0, 0, \frac{p_0}{\delta})$ for $0 < \delta \leq p_0$ equals $\delta$-hyponormality. So we may assume $r_0 > 0$. Suppose that $T$ belongs to class $F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0})$ for $p_0 > 0$, $r_0 > 0$ and $0 \leq \delta \leq p_0$, i.e.,

$(|T^*|^r |T|^{2p_0} |T^*|^r)^{\frac{1}{p_0+r_0}} \geq |T^*|^{2(\delta+r_0)}$. (3.1)
Applying Löwner-Heinz theorem to (3.1), we have

\[ (|T^*|^r |T|^{2p} |T^*|^r) \frac{r_0}{p_0 + r_0} \geq |T^*|^{2r_0}, \]

and also we have

\[ |T|^{2p_0} \geq (|T|^{p_0} |T^*|^{2r_0} |T|^{p_0}) \frac{p_0}{p_0 + r_0} \quad (3.2) \]

by (i) of Theorem 3.A. By applying Theorem 3.B to (3.2), we obtain that

\[ g_{r_0, \delta}(p) = (|T^*|^r |T|^{2p} |T^*|^r) \frac{r_0}{p_0 + r_0} \]

is an increasing function for \( p \geq \max\{p_0, \delta\} = p_0 \).

Therefore we have

\[ \frac{(|T^*|^r |T|^{2p} |T^*|^r)^{s + r_0}}{p_0} \geq g_{r_0, \delta}(p) \quad (3.3) \]

for any \( p \geq p_0 \), i.e., \( T \) belongs to class \( F(p, r_0, \frac{r_0}{r_0}) \) for any \( p \geq p_0 \). Hence \( T \) belongs to class \( F(p, r, \frac{p + r}{p + r}) \) for any \( p \geq p_0 \) and \( r \geq r_0 \) by (ii-2) of Theorem 2.C.

To prove Theorem 3.2, we prepare the following result which is a slight modification of [29, Lemma 5].

**Lemma 3.3.** Let \( A, B \) and \( C \) be positive operators, \( p > 0, 0 < r \leq 1 \) and \( q \geq 1 \) with \( rq \leq p + r \leq (1 + r)q \). If \((B^\frac{r}{2} A^p B^\frac{r}{2})^{\frac{1}{q}} \geq B^{\frac{p + r}{q}}\) and \( B \geq C \), then \((C^\frac{r}{2} A^p C^\frac{r}{2})^{\frac{1}{q}} \geq C^{\frac{p + r}{q}}\).

**Proof.** The hypothesis \( B \geq C \) ensures \( B^r \geq C^r \) for \( r \in (0, 1] \) by Löwner-Heinz theorem. By Douglas' theorem [4], there exists an operator \( X \) such that

\[ B^\frac{r}{2} X = X^* B^\frac{r}{2} = C^\frac{r}{2} \]

and \( \|X\| \leq 1 \). Then we have

\[ (C^\frac{r}{2} A^p C^\frac{r}{2})^{\frac{1}{q}} = (X^* B^\frac{r}{2} A^p B^\frac{r}{2} X)^{\frac{1}{q}} \]

\[ \geq X^* (B^\frac{r}{2} A^p B^\frac{r}{2})^{\frac{1}{q}} X \quad \text{by Hansen's inequality [16]} \]

\[ \geq X^* B^{\frac{p + r}{q}} X \quad \text{by the hypothesis} \]

\[ = C^\frac{r}{2} B^{\frac{p + r}{q}} C^\frac{r}{2} \quad \text{by (3.4) since } \frac{p + r}{q} - r \in [0, 1] \]

\[ \geq C^{\frac{p + r}{q}} \quad \text{by Löwner-Heinz theorem.} \]
Hence the proof is complete. □

Proof of Theorem 3.2. Let $T$ be a class $F(p, r, q)$ operator for $0 < p \leq 1$, $0 \leq r \leq 1$ and $q \geq 1$ with $rq \leq p + r$, i.e.,

$$((|T^n|^r |T|^{2p} |T^n|^r)^{\frac{1}{q}} \geq |T^n|^{\frac{2(p+r)}{q}}). \quad (1.1)$$

Class $F(p, r, q)$ operator $T$ for $0 < p \leq 1$, $0 \leq r \leq 1$ and $q \geq 1$ with $rq \leq p + r$ belongs to class $F(1, 1, 2)$, i.e., class $A$ by (ii) of Theorem 2.B and Theorem 3.1, and also

$$|T^n|^\frac{2}{n} \geq |T|^2 \quad (3.5)$$

and

$$|T^*|^2 \geq |T^n|^\frac{2}{n} \quad (3.6)$$

hold for all positive integer $n$ by Theorem 2.1. By applying Lemma 3.3 to (1.1) and (3.6), we have

$$((|T^n|^r |T|^{2p} |T^n|^r)^{\frac{1}{q}} \geq |T^n|^{\frac{2(p+r)}{q}} \quad (3.7)$$

for $0 < p \leq 1$, $0 \leq r \leq 1$ and $q \geq 1$ with $rq \leq p + r$ since $p + r \leq (1 + r)q$ always holds. Hence we obtain

$$((|T^n|^r |T|^{2p} |T^n|^r)^{\frac{1}{q}} \geq |T^n|^{\frac{2(p+r)}{q}} \quad (3.7)$$

for all positive integer $n$, that is, $T^n$ belongs to class $F(\frac{p}{n}, \frac{r}{n}, q)$ for all positive integer $n$. □

4 Properness of class $F(p, r, \frac{p+r}{\delta+r})$ and $(p, r, \frac{p+r}{\delta+r})$-paranormality

In this section, we shall show the results on inclusion relation among the families of $p$-quasihyponormality, class $F(p, r, q)$ and $(p, r, q)$-paranormality.

Theorem 4.1. For each $p_0 > 0$, there exists a $p_0$-quasihyponormal operator $T$ such that $T$ is not $(p, r, \frac{p+r}{\delta+r})$-paranormal for any $p > 0$, $r > 0$ and $\delta > -r$ such that $\delta \leq p < p_0$.

Theorem 4.2. For each $p_0 > 0$, $r_0 > 0$ and $-r_0 < \delta \leq p_0$,

(i) There exists a $p_0$-quasihyponormal operator $T$ such that $T$ is not $p$-quasihyponormal for any $p > 0$ such that $0 < p < p_0$. 
There exists a class $F(p_0, r_0, \frac{p_0 + r_0}{\delta + r_0})$ operator $T$ such that $T$ does not belong to class $F(p, r, \frac{p + r}{\delta + r})$ for any $p > 0$ and $r > 0$ such that $-r < \delta \leq p < p_0$.

There exists a $(p_0, r_0, \frac{p_0 + r_0}{\delta + r_0})$-paranormal operator $T$ such that $T$ is not $(p, r, \frac{p + r}{\delta + r})$-paranormal for any $p > 0$ and $r > 0$ such that $-r < \delta \leq p < p_0$.

In Theorem 4.2, (i) has been obtained in [24], and also (ii) and (iii) asserts that the families of class $F(p, r, \frac{p + r}{\delta + r})$ and $(p, r, \frac{p + r}{\delta + r})$-paranormality are proper on $p$. Moreover we remark that these properness on $p$ has no connection with $r$, and also we have the following corollary by putting $r = r_0$ in Theorem 4.2.

**Corollary 4.3.** For each $p_0 > 0$, $r_0 > 0$ and $-r_0 < \delta \leq p_0$,

(i) There exists a class $F(p_0, r_0, \frac{p_0 + r_0}{\delta + r_0})$ operator $T$ such that $T$ does not belong to class $F(p, r_0, \frac{p + r_0}{\delta + r_0})$ for any $p > 0$ such that $\delta \leq p < p_0$.

(ii) There exists a $(p_0, r_0, \frac{p_0 + r_0}{\delta + r_0})$-paranormal operator $T$ such that $T$ is not $(p, r_0, \frac{p + r_0}{\delta + r_0})$-paranormal for any $p > 0$ such that $\delta \leq p < p_0$.

Here we shall show two propositions as a preparation of the proof of Theorem 4.1. We remark that these propositions are similar arguments to [2], [15], [20] and so on.

Firstly we shall give a characterization of $(p, r, q)$-paranormal operators.

**Proposition 4.4.** For each $p > 0$, $r > 0$ and $-r < \delta \leq p$, an operator $T$ is $(p, r, \frac{p + r}{\delta + r})$-paranormal if and only if

\[
(\delta + r)|T^*|^r |T^2p| T^*|^r - (p + r)\lambda^{p - \delta} |T^*|^2(\delta + r) + (p - \delta)\lambda^{p + r} \geq 0 \quad \text{for all } \lambda > 0.
\]

**Proof.** Suppose that $T$ is $(p, r, \frac{p + r}{\delta + r})$-paranormal for $p > 0$, $r > 0$ and $-r < \delta \leq p$, i.e.,

\[
|||T^p|T^*|^r x||^{\frac{\delta + r}{p + r}} \geq |||T^*|^\delta x|| \quad \text{for every unit vector } x \in H.
\]

(1.3) holds iff

\[
|||T^p|T^*|^r x||^{\frac{\delta + r}{p + r}} ||x||^{\frac{p - \delta}{p + r}} \geq |||T^*|^\delta x|| \quad \text{for all } x \in H
\]

iff

\[
(|T^*|^r |T^p|^r x, x)^{\frac{\delta + r}{p + r}} (x, x)^{\frac{p - \delta}{p + r}} \geq (|T^*|^2(\delta + r)x, x) \quad \text{for all } x \in H.
\]

(4.1)
By arithmetic-geometric mean inequality,

\[
(\langle |T^*|^r |T|^{2p} |T^*|^r x, x \rangle)^{\frac{\delta + r}{p + r}} \cdot \langle \lambda^{\delta+r}(x, x) \rangle^{\frac{p-r}{p+r}} \leq \frac{\delta + r}{p + r} \frac{1}{\lambda^{p-r}} \langle |T^*|^r |T|^{2p} |T^*|^r x, x \rangle + \frac{p-r}{p+r} \lambda^{\delta+r}(x, x)
\]

(4.2)

for all \( x \in H \) and all \( \lambda > 0 \), so (4.1) ensures the following (4.3) by (4.2).

\[
\frac{\delta + r}{p + r} \frac{1}{\lambda^{p-r}} \langle |T^*|^r |T|^{2p} |T^*|^r x, x \rangle + \frac{p-r}{p+r} \lambda^{\delta+r}(x, x) \geq \langle |T^*|^{2(\delta+r)} x, x \rangle
\]

(4.3)

for all \( x \in H \) and all \( \lambda > 0 \).

Conversely, (4.1) follows from (4.3) by putting \( \lambda = \left\{ \frac{\langle |T^*|^r |T|^{2p} |T^*|^r x, x \rangle}{(x, x)} \right\}^{\frac{1}{p+r}} \). (In case \( \langle |T^*|^r |T|^{2p} |T^*|^r x, x \rangle = 0 \), let \( \lambda \rightarrow +0 \).) Hence (4.3) holds if and only if

\[
(\delta + r)\langle |T^*|^r |T|^{2p} |T^*|^r - (p + r)\lambda^{p-r}|T^*|^{2(\delta+r)} + (p - \delta)\lambda^{p+r} \geq 0 \quad \text{for all } \lambda > 0,
\]

so that the proof is complete. \( \square \)

Secondly we shall give the following Proposition 4.5. But we omit to describe these calculation because it is obtained by easy calculation.

**Proposition 4.5.** Let \( K = \bigoplus_{n=-\infty}^{\infty} H_n \) where \( H_n \cong H \). For given positive operators \( A, B \) on \( H \), define the operator \( T_{A,B} \) on \( K \) as follows:

\[
T_{A,B} = \begin{pmatrix}
\cdots & 0 & 0 & 0 \\
B^\frac{1}{2} & B^\frac{1}{2} & A^\frac{1}{2} & 0 \\
0 & 0 & A^\frac{1}{2} & 0 \\
& & & \cdots
\end{pmatrix},
\]

(4.4)

where \( \square \) shows the place of the \( (0, 0) \) matrix element.

(i) For each \( p > 0 \), \( T_{A,B} \) is \( p \)-quasihyponormal if and only if

\[
B^\frac{1}{2} A^p B^\frac{1}{2} \geq B^{p+1}.
\]
(ii) For each $p > 0$, $r \geq 0$ and $\delta \geq -r$, $T_{A,B}$ belongs to class $F(p, r, \frac{p+r}{\delta+r})$ if and only if

$$(B^{\frac{i}{2}}A^{p}B^{\frac{i}{2}})^{\delta+r} \geq B^{\delta+r}.$$ 

(iii) For each $p > 0$, $r > 0$ and $-r < \delta \leq p$, $T_{A,B}$ is $(p, r, \frac{p+r}{\delta+r})$-paranormal if and only if

$$(\delta + r)B^{\frac{i}{2}}A^{p}B^{\frac{i}{2}} - (p + r)\lambda^{p-r}B^{\delta+r} + (p - \delta)\lambda^{p+r}I \geq 0 \quad \text{for all } \lambda > 0.$$ 

Proof of Theorem 4.1. Let

$$A = U\Lambda U^{*} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

where $U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} (2 - e^{-p_0})^{\frac{1}{2}} & 0 \\ 0 & e^{-2} \end{pmatrix},$ (4.5)

and also let $K = \bigoplus_{n=-\infty}^{\infty} H_n$ where $H_n \cong \mathbb{R}^2$. For positive matrices $A, B$ on $\mathbb{R}^2$ given in (4.5), define the operator $T_{A,B}$ on $K$ as (4.4) in Proposition 4.5. By (i) of Proposition 4.5, $T_{A,B}$ is $p$-quasihyponormal for $p > 0$ if and only if

$$B^{\frac{i}{2}}A^{p}B^{\frac{i}{2}} - B^{p+1} \geq 0$$

if and only if

$$f(p) \equiv \frac{1}{2}\{(2 - e^{-p_0})^{p_0} + e^{-2p}\} - 1 \geq 0.$$ 

On the other hand, let $X_p(\lambda)$ as

$$X_p(\lambda) \equiv (\delta + r)B^{\frac{i}{2}}A^{p}B^{\frac{i}{2}} - (p + r)\lambda^{p-r}B^{\delta+r} + (p - \delta)\lambda^{p+r}I$$

$$= \begin{pmatrix} \frac{1}{2}(2 - e^{-p_0})^{\frac{1}{2}} + e^{-2p} & 0 \\ 0 & (p - \delta)\lambda^{p+r} \end{pmatrix}.$$ 

By (iii) of Proposition 4.5, $T_{A,B}$ is $(p, r, \frac{p+r}{\delta+r})$-paranormal for $p > 0$, $r > 0$ and $-r < \delta \leq p$ if and only if $X_p(\lambda) \geq 0$ for all $\lambda > 0$ if and only if

$$g_p(\lambda) \equiv \frac{1}{2}(\delta + r)\{(2 - e^{-p_0})^{\frac{1}{2}} + e^{-2p}\} - (p + r)\lambda^{p-r} + (p - \delta)\lambda^{p+r} \geq 0 \quad \text{for all } \lambda > 0 (4.6)$$

since $(p - \delta)\lambda^{p+r} \geq 0$ for all $\lambda > 0$. Since $g_p(\lambda) = (p + r)(p - \delta)\lambda^{p-r} + (\delta + r)\lambda^{p+r} + (p - \delta)\lambda^{\delta+r}$, we get that

$$\min_{\lambda > 0} g_p(\lambda) = g_p(1) = \frac{1}{2}(\delta + r)\{(2 - e^{-p_0})^{\frac{1}{2}} + e^{-2p}\} - (\delta + r) = (\delta + r)f(p),$$
so that (4.6) holds if and only if \( f(p) \geq 0 \).

\[ f(p) \] is a convex function for \( p > 0 \) since

\[ f''(p) = \frac{1}{2} \left( 2 - e^{-p_0} \right) \frac{p}{p_0} \left\{ \log(2 - e^{-p_0}) \frac{2}{p_0} \right\}^2 + 4e^{-2p} > 0 \quad \text{for all } p > 0, \]

and also \( f(p) = 0 \) if \( p = 0, p_0 \). So we have \( f(p_0) = 0 \) but \( f(p) < 0 \) for \( 0 < p < p_0 \). Therefore \( g_p(1) < 0 \), that is \( X_p(1) \notin \emptyset \) for any \( p > 0, r > 0 \) and \( \delta > -r \) such that \( \delta \leq p < p_0 \).

Hence \( T \) is \( p_0 \)-quasihyponormal but non-(\( p, r, \frac{p+r}{\delta+r} \))-paranormal for any \( p > 0, r > 0 \) and \( \delta > -r \) such that \( \delta \leq p < p_0 \), so the proof is complete.

**Proof of Theorem 4.2.** Let \( p_0 > 0, r_0 > 0 \) and \( -r_0 < \delta \leq p_0 \).

**Proof of (i).** By (i-1) of Theorem 2.C, \( T \) is \( p \)-quasihyponormal if and only if \( T \) is \( (p, r, 1) \)-paranormal for some \( p > 0 \) and \( r > 0 \). Therefore there exists a \( p_0 \)-quasihyponormal operator \( T \) such that \( T \) is not \( p \)-quasihyponormal for any \( 0 < p < p_0 \) by putting \( \delta = p \) in Theorem 4.1.

**Proof of (ii).** By (i-1) of Theorem 2.C and (ii) of Theorem 2.B, every \( p_0 \)-quasihyponormal operator belongs to class \( F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0}) \). And also, by (iv) of Theorem 2.B, \( T \) does not belong to class \( F(p, r, \frac{p+r}{\delta+r}) \) if \( T \) is not \( (p, r, \frac{p+r}{\delta+r}) \)-paranormal for each \( p > 0, r > 0 \) and \( -r < \delta \leq p \). Therefore there exists a \( F(p_0, r_0, \frac{p_0+r_0}{\delta+r_0}) \) operator \( T \) such that \( T \) does not belong to class \( F(p, r, \frac{p+r}{\delta+r}) \) for any \( p > 0 \) and \( r > 0 \) such that \( -r < \delta \leq p < p_0 \) by Theorem 4.1.

**Proof of (iii).** By (i-1) of Theorem 2.C and (iii) of Theorem 2.B, every \( p_0 \)-quasihyponormal operator is \( (p_0, r_0, \frac{p_0+r_0}{\delta+r_0}) \)-paranormal. Therefore there exists a \( (p_0, r_0, \frac{p_0+r_0}{\delta+r_0}) \)-paranormal operator \( T \) such that \( T \) is not \( (p, r, \frac{p+r}{\delta+r}) \)-paranormal for any \( p > 0 \) and \( r > 0 \) such that \( -r < \delta \leq p < p_0 \) by Theorem 4.1.

**Remark 1.** In [15], we introduced two families of classes of operators based on class A and paranormality as follows: An operator \( T \) belongs to class \( A(k) \) for \( k > 0 \) if \( (T^*|T|^{2k}T)^{1/k} \geq |T|^2 \), and also an operator \( T \) is absolute-\( k \)-paranormal for \( k > 0 \) if \( \| |T|^kT|^{1/k} \geq \| |T|^{k+1} \) for every unit vector \( x \in H \). In [7], Fujii-Izumino-Nakamoto introduced \( p \)-paranormality for \( p > 0 \) defined by \( \| |T|^pU|T|^pTx \| \geq \| |T|^pT^*|T|^p|x \|^2 \) for every unit vector \( x \in H \), where \( T = U|T| \) is the polar decomposition of \( T \). It was pointed out in [27] that class \( A(k) \) equals class \( A(k, 1) \), and also it was shown in [28] that absolute-\( k \)-paranormality equals absolute-(\( k, 1 \))-paranormality and \( p \)-paranormality equals absolute-(\( p, p \))-paranormality. We remark that \( p \)-paranormality corresponds to class \( A(p, p) \). We shall also get the results on inclusion relation among the families of these classes.
Corollary 4.6.

(i) For each $k_0 > 0$, there exists a class $A(k_0)$ operator $T$ such that $T$ does not belong to class $A(k)$ for any $0 < k < k_0$.

(ii) For each $k_0 > 0$, there exists an absolute-$k_0$-paranormal operator $T$ such that $T$ is not absolute-$k$-paranormal for any $0 < k < k_0$.

(iii) For each $p_0 > 0$, there exists a class $A(p_0, p_0)$ operator $T$ such that $T$ is not class $A(p, p)$ for any $0 < p < p_0$.

(iv) For each $p_0 > 0$, there exists a $p_0$-paranormal operator $T$ such that $T$ is not $p$-paranormal for any $0 < p < p_0$.

Proof of Corollary 4.6.

Proofs of (i) and (ii). By putting $p_0 = k_0$, $r_0 = 1$, $\delta = 0$ and $p = k$ in Corollary 4.3, we have (i) and (ii) since class $A(k)$ equals class $F(k, 1, k+1)$ and absolute-$k$-paranormality equals $(k, 1, k+1)$-paranormality.

Proofs of (iii) and (iv). By putting $p_0 = r_0$, $\delta = 0$ and $p = r$ in (ii) and (iii) of Theorem 4.2, we have (iii) and (iv) since class $A(p, p)$ equals class $F(p, p, 2)$ and $p$-paranormality equals $(p, p, 2)$-paranormality. \qed

Remark 2. For each $p > 0$, we can obtain an example of non-class $A(p, p)$ and $p$-paranormal operators by using essentially the same example as [15, (2) of Example 8] as follows: Let $p > 0$ and

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 2\sqrt{23} \end{pmatrix}^{\frac{2}{p}} \quad \text{and} \quad B = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}^{\frac{2}{p}}.$$

Then

$$\left(B^{\frac{2}{p}} A^{p} B^{\frac{2}{p}} \right)^{\frac{1}{2}} - B^{p} = \begin{pmatrix} 0.17472 \ldots & -3.1798 \ldots \\ -3.1798 \ldots & 11.770 \ldots \end{pmatrix}.$$ 

Eigenvalues of $(B^{\frac{2}{p}} A^{p} B^{\frac{2}{p}})^{\frac{1}{2}} - B^{p}$ are 12.585 \ldots and -0.64001 \ldots, so that $(B^{\frac{2}{p}} A^{p} B^{\frac{2}{p}})^{\frac{1}{2}} \not\geq B^{p}$. So $T_{A,B}$ is a non-class $A(p, p)$ operator by (ii) of Proposition 4.5.

On the other hand, for $\lambda > 0$, define $X(\lambda)$ as follows:

$$X(\lambda) \equiv B^{\frac{2}{p}} A^{p} B^{\frac{2}{p}} - 2\lambda B^{p} + \lambda^{2} I = \begin{pmatrix} 404 - 26\lambda + \lambda^{2} & -576 + 24\lambda \\ -576 + 24\lambda & 844 - 26\lambda + \lambda^{2} \end{pmatrix}.$$ 

Put $p(\lambda) = \text{tr} X(\lambda)$ and $q(\lambda) = \det X(\lambda)$, where $\text{tr} X$ denotes the trace of a matrix $X$ and $\det X$ denotes the determinant of a matrix $X$. Then

$$p(\lambda) = 2\lambda^{2} - 52\lambda + 1248 = 2(\lambda - 13)^{2} + 910 > 0.$$
\[ q(\lambda) = (404 - 26\lambda + \lambda^2)(844 - 26\lambda + \lambda^2) - (-576 + 24\lambda)^2 \\
= \lambda^4 - 52\lambda^3 + 1348\lambda^2 - 4800\lambda + 9200. \]

By calculation,

\[ q'(\lambda) = 4\lambda^3 - 156\lambda^2 + 2696\lambda - 4800 \\
= 4(\lambda - 2)(\lambda^2 - 37\lambda + 600) \\
= 4(\lambda - 2)\left\{ \left( \lambda - \frac{37}{2} \right)^2 + \frac{1031}{4} \right\}. \]

So \( q'(\lambda) = 0 \) iff \( \lambda = 2 \), that is, \( q(\lambda) \geq q(2) = 4592 > 0 \) for all \( \lambda > 0 \). Hence \( X(\lambda) \geq 0 \) for all \( \lambda > 0 \) since \( \text{tr} \, X(\lambda) = p(\lambda) > 0 \) and \( \det X(\lambda) = q(\lambda) > 0 \) for all \( \lambda > 0 \). Therefore \( T_{A,B} \) is a p-paranormal operator since \( T_{A,B} \) is p-paranormal if and only if

\[ pB^\frac{r}{2}A^pB^\frac{r}{2} - 2\mu^pB^p + \mu^{2p}I \geq 0 \quad \text{for all } \mu > 0 \]

if and only if

\[ B^\frac{r}{2}A^pB^\frac{r}{2} - 2\lambda B^p + \lambda^2 I \geq 0 \quad \text{for all } \lambda > 0. \]

by (iii) of Proposition 4.5.

5 Relations between Furuta-type inequalities

In this section, we shall show a generalization of Theorem 3.A which plays an important role in the proofs of the results in Section 3. Here we recall Theorem 3.A.

**Theorem 3.A** ([19, Theorem 1]). Let \( A \) and \( B \) be positive operators. Then for each \( p \geq 0 \) and \( r \geq 0 \),

(i) If \( (B^\frac{r}{2}A^pB^\frac{r}{2})^{\frac{r}{p+r}} \geq B^r \), then \( A^p \geq (A^\frac{r}{2}B^rA^\frac{r}{2})^{\frac{p}{p+r}} \).

(ii) If \( A^p \geq (A^\frac{r}{2}B^rA^\frac{r}{2})^{\frac{p}{p+r}} \) and \( N(A) \subseteq N(B) \), then \( (B^\frac{r}{2}A^pB^\frac{r}{2})^{\frac{r}{p+r}} \geq B^r \).

For positive invertible operators \( A \) and \( B \), it was shown in [13] that

\[ (B^\frac{r}{2}A^pB^\frac{r}{2})^{\frac{r}{p+r}} \geq B^r \iff A^p \geq (A^\frac{r}{2}B^rA^\frac{r}{2})^{\frac{p}{p+r}} \]  \( (5.1) \)

for fixed positive numbers \( p \geq 0 \) and \( r \geq 0 \), and Theorem 3.A is a general result for a relation between two inequalities in (5.1). We remark that it was shown in [6] and [13]
(see also [3][8][25]) as an application of Theorem F that for positive invertible operators $A$ and $B$,

$$\log A \geq \log B \iff (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \text{ for all } p \geq 0 \text{ and } r \geq 0,$$

$$\iff A^p \geq (A^\frac{p}{2} B^r A^{\frac{p}{2}})^{\frac{p}{p+r}} \text{ for all } p \geq 0 \text{ and } r \geq 0. \tag{5.2}$$

As an extension of (5.2) and an immediate corollary of results on operator-valued functions in [6] and [13], we have that for positive invertible operators $A$ and $B$,

$$\log A \geq \log B \iff (B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^r \text{ for all } p \geq \gamma \geq 0 \text{ and } r \geq 0,$$

$$\iff A^\frac{p}{2} B^\delta A^{\frac{p}{2}} \geq (A^\frac{p}{2} B^r A^{\frac{p}{2}})^{\frac{\delta}{p+r}} \text{ for all } p \geq 0 \text{ and } r \geq \delta \geq 0. \tag{5.3}$$

We remark that inequalities of type (5.3) were initiated in [21].

Here we shall show a generalization of Theorem 3.1 on inequalities in (5.3).

**Theorem 5.1.** Let $A$ and $B$ be positive operators. Then the following assertions hold, where $S^0$ means the projection onto $N(S)^\perp$ for a positive operator $S$:

(i) For each $r \geq \delta \geq 0$ and $p \geq 0$,

(i-1) $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^{r-\delta}$ ensures $A^\frac{p}{2} B^\delta A^{\frac{p}{2}} \geq (A^\frac{p}{2} B^r A^{\frac{p}{2}})^{\frac{\delta}{p+r}},$

(i-2) $A^\frac{p}{2} B^\delta A^{\frac{p}{2}} \geq (A^\frac{p}{2} B^r A^{\frac{p}{2}})^{\frac{\delta}{p+r}}$ and $N(AB^{\frac{r}{2}}) = N(B)$ ensure $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{r}{p+r}} \geq B^{r-\delta}$.

(ii) For each $p \geq \gamma \geq 0$ and $r \geq 0$,

$A^{p-\gamma} \geq (A^\frac{p}{2} B^r A^{\frac{p}{2}})^{\frac{\gamma+r}{p+r}}$ is equivalent to $(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{\gamma+r}{p+r}} \geq B^{r-\delta} A^\gamma B^{\frac{r}{2}}.$

We remark that two inequalities in (i) and (ii) of Theorem 5.1 are mutually equivalent in case $A$ and $B$ are both invertible [22].

We use the following lemma in order to give a proof of Theorem 5.1. Throughout this section, $P_M$ denotes the projection onto a closed subspace $M$, and also $S^0 = P_{N(S)^\perp}$ for a positive operator $S$.

**Lemma 5.2.** Let $A$ and $B$ be positive operators. Then the following assertions hold:

(i) $\lim_{\epsilon \to +0} A^{\frac{1}{2}}(A + \epsilon I)^{-1} A^{\frac{1}{2}} = \lim_{\epsilon \to +0} (A + \epsilon I)^{-1} A = P_{N(A)^\perp}$.

(ii) $\lim_{\epsilon \to +0} A^{\frac{1}{2}} B^{\frac{1}{2}} \{((B^{\frac{1}{2}} A B^{\frac{1}{2}})^\alpha + \epsilon I)^{-1} B^{\frac{1}{2}} A^{\frac{1}{2}} = (A^{\frac{1}{2}} B A^{\frac{1}{2}})^{1-\alpha} \text{ for } \alpha \in (0, 1]\}$. Particularly, in case $\alpha = 1$,

$\lim_{\epsilon \to +0} A^{\frac{1}{2}} B^{\frac{1}{2}} (B^{\frac{1}{2}} A B^{\frac{1}{2}} + \epsilon I)^{-1} B^{\frac{1}{2}} A^{\frac{1}{2}} = P_{N(B^{\frac{1}{2}} A^{\frac{1}{2}})^\perp}.$
For positive invertible operators $A$ and $B$, equivalence between two inequalities in (i) or (ii) of Theorem 5.1 can be easily proved by applying the following Lemma 5.A.

**Lemma 5.A ([14]).** Let $A$ be a positive invertible operator and $B$ be an invertible operator. Then

$$(BAB^*)^\lambda = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*$$

holds for any real number $\lambda$.

We remark that for non-invertible operators $A$ and $B$, Lemma 5.A is valid in case $\lambda \geq 1$ but cannot be applied in case $\lambda \in [0,1)$. For positive invertible operators $A$ and $B$, Lemma 5.A can be rewritten as

$$A^{\frac{1}{2}}B^{\frac{1}{2}}(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{-\alpha}B^{\frac{1}{2}}A^{\frac{1}{2}} = (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{1-\alpha}$$

for any real number $\alpha$, so that we can regard (ii) of Lemma 5.2 as a non-invertible version of Lemma 5.A for $\alpha \in (0,1]$.

**Proof of Lemma 5.2.** (i) is well known and a proof was given in [19], for example.  
**Proof of (ii).** Let $A^{\frac{1}{2}}B^{\frac{1}{2}} = U|A^{\frac{1}{2}}B^{\frac{1}{2}}|$ be the polar decomposition. For $\alpha \in (0,1]$, we have

$$\lim_{\epsilon \to 0^+}A^{\frac{1}{2}}B^{\frac{1}{2}}\{(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\alpha} + \epsilon I\}^{-1}B^{\frac{1}{2}}A^{\frac{1}{2}}$$

for $\epsilon > 0$, so that we can regard (ii) of Lemma 5.2 as a non-invertible version of Lemma 5.A for $\alpha \in (0,1]$.

We remark that in case $\alpha = 1$ particularly,

$$U|A^{\frac{1}{2}}B^{\frac{1}{2}}|U^* = UP_{N(|A^{\frac{1}{2}}B^{\frac{1}{2}}|)}U^* = UU^*U^* = UU^* = P_{N(B^{\frac{1}{2}}A^{\frac{1}{2}})} = (A^{\frac{1}{2}}BA^{\frac{1}{2}})^0.$$

Hence the proof is complete. \(\square\)

**Proof of Theorem 5.1.**

**Proof of (i).** Let $r > \delta \geq 0$ since the case $r = \delta$ is obvious. If $(B^{\frac{r}{2}}APB^{\frac{r}{2}})^{\frac{\delta}{r-\delta}} \geq B^{r-\delta}$, then

$$A^{\frac{r}{2}}B^{\frac{r}{2}}(B^{\frac{r}{2}}B^{\frac{r}{2}})^{-\delta}B^{\frac{r}{2}}A^{\frac{r}{2}} \geq A^{\frac{r}{2}}B^{\frac{r}{2}}\{(B^{\frac{r}{2}}APB^{\frac{r}{2}})^{\frac{\delta}{r-\delta}} + \epsilon I\}^{-1}B^{\frac{r}{2}}A^{\frac{r}{2}}$$

for $\epsilon > 0$, so that

$$A^{\frac{r}{2}}B^{\frac{r}{2}}A^{\frac{r}{2}} = A^{\frac{r}{2}}B^{\frac{r}{2}}P_{N(B^{\frac{1}{2}}A^{\frac{1}{2}})}B^{\frac{r}{2}}A^{\frac{r}{2}} \geq (A^{\frac{r}{2}}B^{r}A^{\frac{r}{2}})^{\frac{\delta}{r-\delta}}.$$
by tending $\varepsilon \to +0$ and Lemma 5.2, hence we obtain (i-1). On the other hand, if $A^\varepsilon B^\delta A^\varepsilon \geq (A^\varepsilon B^r A^\varepsilon)^{\frac{\delta+r}{\delta+r}}$, then

$$B^\delta A^\varepsilon \{ (A^\varepsilon B^r A^\varepsilon)^{\frac{\delta+r}{\delta+r}} + \varepsilon I \}^{-1} A^\varepsilon B^\delta \geq B^\frac{\delta+r}{2} A^\varepsilon B^\delta (A^\varepsilon B^r A^\varepsilon + \varepsilon I)^{-1} A^\varepsilon B^\delta B^r$$

for $\varepsilon > 0$, so that

$$(B^\delta A^\varepsilon B^\delta)^{\frac{\delta+r}{\delta+r}} \geq B^\frac{\delta+r}{2} P_{N(A^\varepsilon B^\delta)\perp} B^\frac{\delta+r}{2}$$

by tending $\varepsilon \to +0$ and (ii) of Lemma 5.2

$$= B^\frac{\delta+r}{2} P_{N(B)\perp} B^\frac{\delta+r}{2}$$

by $N(AB) = N(B)$

$$= B^\frac{r-\delta}{2},$$

hence we obtain (i-2).

**Proof of (ii).** Let $p > \gamma \geq 0$ since the case $p = \gamma$ is obvious. If $A^{p-\gamma} \geq (A^\varepsilon B^r A^\varepsilon)^{\frac{p+r}{p+r}}$, then

$$B^\delta A^\varepsilon \{ (A^\varepsilon B^r A^\varepsilon)^{\frac{p+r}{p+r}} + \varepsilon I \}^{-1} A^\varepsilon B^\delta \geq B^\delta A^\varepsilon (A^{p-\gamma} + \varepsilon I)^{-1} A^\varepsilon B^\delta$$

for $\varepsilon > 0$, so that

$$(B^\delta A^\varepsilon B^\delta)^{\frac{\delta+r}{\delta+r}} \geq B^\delta A^\varepsilon P_{N(A^\varepsilon B^\delta)\perp} A^\varepsilon B^\delta = B^\delta A^\gamma B^\delta$$

by tending $\varepsilon \to +0$ and Lemma 5.2, hence we obtain ($\Rightarrow$). On the other hand, if $(B^\delta A^\varepsilon B^\delta)^{\frac{\delta+r}{\delta+r}} \geq B^\delta A^\gamma B^\delta$, then

$$A^\frac{\delta+r}{2} B^\delta (B^\frac{r-\delta}{2} A^\varepsilon B^\delta + \varepsilon I)^{-1} B^\delta A^\frac{\delta+r}{2} \geq A^\delta B^\delta \{ (B^\delta A^\varepsilon B^\delta)^{\frac{\delta+r}{\delta+r}} + \varepsilon I \}^{-1} B^\delta A^\delta$$

for $\varepsilon > 0$, so that

$$A^{p-\gamma} \geq A^\frac{\delta+r}{2} P_{N(B^\delta A^\delta)\perp} A^\frac{\delta+r}{2} \geq (A^\delta B^r A^\delta)^{\frac{\delta+r}{\delta+r}}$$

by tending $\varepsilon \to +0$ and (ii) of Lemma 5.2, hence we obtain ($\Leftarrow$).

Theorem 3.1 can be obtained as a corollary of Theorem 5.1 as follows.

**Alternative proof of Theorem 3.1.** Put $\delta = 0$ in (i-1) of Theorem 5.1, then $(B^\delta A^\varepsilon B^\delta)^{\frac{\delta+r}{\delta+r}} \geq B^r$ ensures

$$A^p \geq A^\delta P_{N(B)\perp} A^\delta \geq (A^\delta B^r A^\delta)^{\frac{\delta+r}{\delta+r}},$$

hence we obtain (i). On the other hand, put $\gamma = 0$ in (ii) of Theorem 5.1, then $A^p \geq (A^\delta B^r A^\delta)^{\frac{\delta+r}{\delta+r}}$ ensures

$$(B^\delta A^\varepsilon B^\delta)^{\frac{\delta+r}{\delta+r}} \geq B^\delta P_{N(A)\perp} B^\delta \geq B^\delta P_{N(B)\perp} B^\delta = B^r$$

since $N(A) \subseteq N(B)$ is equivalent to $P_{N(A)\perp} \geq P_{N(B)\perp}$, hence we obtain (ii).
References


[11] T.Furuta, *A \( A \geq B \geq 0 \) assures \( (B^rA^pB^r)^{1/q} \geq B^{(p+2r)/q} \) for \( r \geq 0 \), \( p \geq 0 \), \( q \geq 1 \) with \( (1+2r)q \geq p + 2r \)*, Proc. Amer. Math. Soc., 101 (1987), 85–88.


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