KANTOROVICH TYPE OPERATOR INEQUALITIES VIA THE SPECHT RATIO (II)

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KANTOROVICH TYPE OPERATOR INEQUALITIES
VIA THE SPECHT RATIO II

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ABSTRACT. Yamazaki [14] showed new order preserving operator inequalities on the usual order and the chaotic order by estimating the lower bound of the difference. Mond and Shisha [7, 8] gave an estimate of the difference of the arithmetic mean to the geometric one, as a converse of the arithmetic-geometric mean inequality. In this report, we shall present other order preserving operator inequalities on the usual order and the chaotic one via the Mond-Shisha difference. Among others, as an application of the Furuta inequality, we show that if $A$ and $B$ are positive operators on a Hilbert space $H$ and $k \geq B \geq 1/k$ for some $k \geq 1$, then for a given $\delta \in [0, 1]$, $A^k \geq B^k$ implies

$$A^p + 2k^{p-2\delta}L(1,k^{2p-2\delta}) \log M_k(p-\delta)I \geq B^p,$$

where the case $\delta = 0$ means the chaotic order and the Specht ratio $M_k(r)$ is defined for each $r > 0$ as

$$M_k(r) = \frac{(k^r - 1)k^{r-1}}{re \log k} \quad (k > 0, k \neq 1) \quad \text{and} \quad M_1(r) = 1.$$

1. INTRODUCTION

We shall consider a bounded linear operator on a complex Hilbert space $H$. An operator $A$ is said to be positive ( in symbol: $A \geq 0$) if $(Ax, x) \geq 0$ for all $x \in H$. The Löwner-Heinz theorem asserts that $A \geq B \geq 0$ ensures $A^p \geq B^p$ for all $p \in [0, 1]$. However $A \geq B$ does not always ensure $A^p \geq B^p$ for $p > 1$ in general. Yamazaki [13] showed that $t^2$ is order preserving in the following sense:

$$A \geq B \geq 0 \quad \text{and} \quad M \geq B \geq m > 0 \quad \text{imply} \quad A^2 + \frac{(M-m)^2}{4}I \geq B^2.$$

Moreover, he showed the following order preserving operator inequality as an extension of (1):

Theorem A. Let $A$ and $B$ be positive operators on a Hilbert space $H$ satisfying $M \geq B \geq m > 0$. If $A \geq B \geq 0$, then

$$A^p + M(M^{p-1} - m^{p-1})I \geq A^p + C(m, M, p)I \geq B^p$$

for all $p \geq 1$,

where

$$C(m, M, p) = \frac{mM^p - Mm^p}{M - m} + (p - 1) \left( \frac{M^p - m^p}{p(M - m)} \right)^{\frac{1}{p-1}}.$$

For positive invertible operators $A$ and $B$ on a Hilbert space $H$, the order defined by $\log A \geq \log B$ is called the chaotic order. Since $\log t$ is an operator monotone function, the chaotic order is weaker than the usual one $A \geq B$. J.I.Fujii and the author [1] showed
the following order preserving operator inequalities on the chaotic order which is parallel to Theorem A.

**Theorem B.** Let \( A \) and \( B \) be positive invertible operators on a Hilbert space \( H \) satisfying \( M \geq B \geq m > 0 \). If \( \log A \geq \log B \), then

\[
A^p + \frac{M}{m}(M^p - m^p)I \geq A^p + \frac{1}{m}C(m, M, p + 1)I \geq B^p \quad \text{for all} \quad p \geq 0,
\]

In fact, \( \log A \geq \log B \) does not always ensure \( A \geq B \) in general. However, by Theorem B, it follows that

\[
\log A \geq \log B \quad \text{and} \quad M \geq B \geq m > 0 \quad \text{imply} \quad A + \frac{(M-m)^2}{4m}I \geq B.
\]

On the other hand, Specht [9] estimated the upper bound of the arithmetic mean by the geometric one for positive numbers: For \( x_1, \cdots, x_n \in [m, M] \) with \( M \geq m > 0 \),

\[
M_h(1) \sqrt[n]{x_1 \cdots x_n} \geq \frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n},
\]

where \( h = \frac{M}{m} (\geq 1) \) is a generalized condition number in the sense of Turing [12] and the Specht ratio \( M_h(1) \) is defined for \( h \geq 1 \) as

\[
M_h(1) = \frac{(h-1)h^\frac{1}{h-1}}{e \log h} \quad (h > 1) \quad \text{and} \quad M_1(1) = 1.
\]

Yamazaki [14] showed a new characterization of chaotic order as follows:

**Theorem C.** Let \( A \) and \( B \) be positive invertible operators on a Hilbert space \( H \) satisfying \( M \geq B \geq m > 0 \). Then \( \log A \geq \log B \) is equivalent to

\[
A^p + L(m^p, M^p) \log M_h(p)I \geq B^p \quad \text{holds for all} \quad p > 0,
\]

where \( h = \frac{M}{m} > 1 \), the logarithmic mean \( L(m, M) = \frac{M-m}{\log M - \log m} \) and a generalized Specht ratio \( M_h(p) \) is defined as

\[
M_h(p) = \frac{(h^p - 1)h^\frac{p}{h^p-1}}{p \log h} \quad (h > 0, h \neq 1) \quad \text{and} \quad M_1(p) = 1.
\]

What is the meaning of the constant \( L(m^p, M^p) \log M_h(p) \) in Theorem C? Mond and Shisha [7, 8] made an estimate of the difference between the arithmetic mean and the geometric one: For \( x_1, \cdots, x_n \in [m, M] \) with \( M \geq m > 0 \),

\[
\sqrt[n]{x_1 \cdots x_n} + D(m, M) \geq \frac{x_1 + \cdots + x_n}{n},
\]

where \( h = \frac{M}{m} (\geq 1) \) and

\[
D(m, M) = \theta M + (1 - \theta)m - M^\theta m^{1-\theta} \quad \text{and} \quad \theta = \log \left( \frac{h - 1}{\log h} \right) \frac{1}{\log h}
\]

which we call the *Mond-Shisha difference*. As a matter of fact, J.I.Fujii and the author [1] showed that the Mond-Shisha difference exactly coincides with the constant in Theorem C via the Specht ratio: If \( M > m > 0 \), then

\[
D(m^p, M^p) = L(m^p, M^p) \log M_h(p)
\]
where \( h = \frac{M}{m} \).

Comparing Theorem A with Theorem B, we observe the difference between \( p \) and \( p - 1 \) in the power of the constant. Hence one might expect that the following result holds under the usual order as a parallel result to Theorem C via the Mond-Shisha difference: Let \( A \) and \( B \) be positive invertible operators satisfying \( M \geq B \geq m > 0 \). Then

\[
A \geq B \quad \text{implies} \quad A^p + mL(m^{p-1}, M^{p-1}) \log M_h(p-1)I \geq B^p \quad \text{for all } p \geq 2,
\]

where \( h = \frac{M}{m} \geq 1 \). However, we have a counterexample to this conjecture. Put

\[
A = \begin{pmatrix} 3 & 1 \\ 1 & \frac{3}{2} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}
\]

then \( A \geq B \geq 0 \) and \( 2I \geq B \geq \frac{1}{2}I \). Then we have \( mL(m^1, M^1) \log M_h(1) = 0.126638 \). On the other hand, \( A^2 + \alpha I \geq B^2 \) holds if and only if \( \alpha \geq \frac{-35 + \sqrt{1465}}{8} = 0.409415 \). Therefore \( A^2 + mL(m^1, M^1) \log M_h(1)I \not\geq B^2 \).

We collect the difference between the usual order and the chaotic one in the following table.

**TABLE 1. The difference between the usual order and the chaotic order**

<table>
<thead>
<tr>
<th>( A \geq B )</th>
<th>( \log A \geq \log B )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A^p + M(M^{p-1} - m^{p-1})I \geq B^p ) for ( p \geq 1 )</td>
<td>( A^p + \frac{M}{m}(M^p - m^p)I \geq B^p ) for ( p &gt; 0 )</td>
</tr>
<tr>
<td>( A^p + C(m, M, p)I \geq B^p ) for ( p \geq 1 )</td>
<td>( A^p + \frac{1}{m}C(m, M, p+1)I \geq B^p ) for ( p &gt; 0 )</td>
</tr>
<tr>
<td>( A^p + \frac{1}{m^r}C(m^r, M^r, 1 + \frac{p-1}{r})I \geq B^p ) for ( p, r \geq 1 )</td>
<td>( A^p + \frac{1}{m^r}C(m^r, M^r, 1 + \frac{p}{r})I \geq B^p ) for ( p, r &gt; 0 )</td>
</tr>
<tr>
<td>( A^p + \frac{(M^{p-1} - m^{p-1})^2}{4m^p}I \geq B^p ) for ( p \geq 2 )</td>
<td>( A^p + \frac{(M^p - m^p)^2}{4m^p}I \geq B^p ) for ( p &gt; 0 )</td>
</tr>
<tr>
<td>( \ldots )</td>
<td>( A^p + L(m^p, M^p) \log M_h(p)I \geq B^p ) for ( p &gt; 0 )</td>
</tr>
</tbody>
</table>

In this report, we shall present order preserving operator inequalities on the usual order and the chaotic one in terms of the Mond-Shisha difference. As an application of the Furuta inequality, we show that if \( A \) and \( B \) are positive operators and \( k \geq B \geq 1/k \) for some \( k \geq 1 \), then for a given \( \delta \in [0, 1] \) \( A^\delta \geq B^\delta \) implies

\[
A^p + 2k^{p-2\delta}L(1, k^{2p-2\delta}) \log M_{k\delta}(p - \delta)I \geq B^p \quad \text{holds for all } p \geq 2\delta.
\]
2. Main results

First of all, we present other characterizations of the chaotic order via the Mond-Shisha difference.

**Theorem 1.** Let $A$ and $B$ be positive invertible operators on a Hilbert space $H$ satisfying $k \geq B \geq \frac{1}{k}$ for some $k \geq 1$. Then the following are mutually equivalent:

(i) $\log A \geq \log B$

(ii) $(A^\frac{1}{2} B^p A^\frac{1}{2})^s + q k^r L(1, k^{\frac{(p+t)s+r}{4}}) \log M_k((p+t)s + r) I \geq B^{(p+t)s + r}$

holds for $p \geq 0$, $t \geq 0$, $s \geq 0$, $q \geq 1$ with $(t + r)q \geq (p + t)s + r$.

(iii) $(A^\frac{1}{2} B^p A^\frac{1}{2})^s + 2k^{(p+t)s-2t} L(1, k^{2(p+t)s-2t}) \log M_k(2(p+t)s - 2t) I \geq B^{(p+t)s}$

holds for $p \geq 0$, $t \geq 0$, $s \geq 0$ with $(p + t)s \geq 2t$.

(iv) $A^p + 2k^p L(1, k^{2p}) \log M_k(2p) I \geq B^p$ holds for $p > 0$,

where $M_k(r)$ is defined as (3).

Let $A$ and $B$ be positive invertible operators on a Hilbert space $H$. We consider an order $A^\delta \geq B^\delta$ for $\delta \in (0,1]$ which interpolates the usual order $A \geq B$ and the chaotic one $\log A \geq \log B$ continuously, where the case of $\delta = 0$ means the chaotic order. By virtue of the Furuta inequality, we show the following order preserving operator inequality associated with the Mond-Shisha difference.

**Theorem 2.** Let $A$ and $B$ be positive invertible operators on a Hilbert space $H$ satisfying $k \geq B \geq \frac{1}{k} > 0$. If $A^\delta \geq B^\delta$ for some $\delta \in [0,1]$, then

$A^p + 2k^{p-2\delta} L(1, k^{2p-2\delta}) \log M_{k^2}(p - \delta) I \geq B^p$

holds for all $p \geq 2\delta$,

where $M_k(r)$ is defined as (3).

If we put $\delta = 1$ in Theorem 2, then we have a usual order version via the Mond-Shisha difference.

**Theorem 3.** Let $A$ and $B$ be positive invertible operators on a Hilbert space $H$ satisfying $k \geq B \geq \frac{1}{k} > 0$. If $A \geq B$, then

$A^p + 2k^{p-2} L(1, k^{2p-2}) \log M_{k^2}(p - 1) I \geq B^p$

holds for all $p \geq 2$,

where $M_k(r)$ is defined as (3).

**Remark 4.** Theorem 2 interpolates Theorems 1 (iv) and Theorem 3 by means of the Mond-Shisha difference. Let $A$ and $B$ be positive invertible operators on a Hilbert space $H$ satisfying $k \geq B \geq \frac{1}{k} > 0$. Then the following assertions hold:

(i) $A \geq B$ implies $A^p + 2k^{p-2} L(1, k^{2p-2}) \log M_{k^2}(p - 1) I \geq B^p$ for all $p \geq 2$.

(ii) $A^\delta \geq B^\delta$ implies $A^p + 2k^{p-2\delta} L(1, k^{2p-2\delta}) \log M_{k^2}(p - \delta) I \geq B^p$ for all $p \geq 2\delta$.

(iii) $\log A \geq \log B$ implies $A^p + 2k^p L(1, k^{2p}) \log M_{k^2}(p) I \geq B^p$ for all $p > 0$.

It follows that the Mond-Shisha difference of (ii) interpolates the scalars of (i) and (iii) continuously. In fact, if we put $\delta = 1$ in (ii), then we have (i), also if we put $\delta \to 0$ in (ii), then we have (iii).
TABLE 2. Kantorovich constant

<table>
<thead>
<tr>
<th>$A \geq B$ and $M \geq B \geq m$</th>
<th>$\log A \geq \log B$ and $M \geq B \geq m$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^2 + \frac{(M - m)^2}{4} I \geq B^2$</td>
<td>$A + \frac{(M - m)^2}{4m} I \geq B$</td>
</tr>
</tbody>
</table>

TABLE 3. Mond-Shisha difference

<table>
<thead>
<tr>
<th>$A \geq B$ and $k \geq B \geq 1/k$</th>
<th>$\log A \geq \log B$ and $k \geq B \geq 1/k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^2 + L(1, k^2) \log M_k(1)^2 I \geq B^2$</td>
<td>$A + kL(1, k^2) \log M_k(1)^2 I \geq B$</td>
</tr>
</tbody>
</table>

3. PROOF OF RESULTS

To prove our results, we collect several properties of the Specht ratio, see [11, 15]:

Lemma 5. (i) $M_k(r) = M_{kr}(1)$ for $k > 0$ and $r > 0$.
(ii) $k \to M_k(1)$ is increasing for $k > 1$ and decreasing for $1 > k > 0$.
(iii) $M_k(1) = M_{k^{-1}}(1)$ for $k > 0$.
(iv) For $k > 1$, $M_k(p)^{1/p} \to 1$ as $p \to 0$.

Lemma 6. Let $A$ and $B$ be positive operators on a Hilbert space $H$ satisfying $k \geq B \geq \frac{1}{k}$ for some $k \geq 1$. If $A^p \geq B^p$ for some $p \in (0, 1]$, then

$$A + \frac{1}{p} L(1, k) \log M_k(1) I \geq B,$$

where $M_k(1)$ is defined as (2).

Proof. The following reverse inequality of Young's one is shown in [11]: For a positive operator $A$ satisfying $k \geq A \geq \frac{1}{k}$ for some $k \geq 1$,

$$(4) \quad A^p + L(1, k) \log M_k(1) I \geq pA + (1 - p)I$$

holds for all $1 > p > 0$. Then we have

$L(1, k) \log M_k(1) I + pA + (1 - p)I$

$$\geq L(1, k) \log M_k(1) I + A^p \quad \text{by the Young inequality and } 1 > p > 0$$

$$\geq L(1, k) \log M_k(1) I + B^p \quad \text{by } A^p \geq B^p$$

$$\geq pB + (1 - p)I \quad \text{by (4) and } k \geq B \geq 1/k > 0.$$

The following order preserving operator inequality is our key lemma in this report.
Lemma 7. Let $A$ and $B$ be positive operators on a Hilbert space $H$ satisfying $k \geq B \geq \frac{1}{k} > 0$ for some $k \geq 1$. If $A \geq B$, then

$$A^p + pL(1, k^p) \log M_k(p)I \geq B^p \quad \text{for all } p \geq 1,$$

where $M_k(1)$ is defined as (2).

Proof. Since $(A^p)^{1/p} \geq (B^p)^{1/p}$ for $0 < \frac{1}{p} \leq 1$ and $k^p \geq B^p \geq k^{-p}$, it follows from Lemma 6 that

$$A^p + pL(1, k^p) \log M_k(p)I \geq B^p \quad \text{for all } p \geq 1.$$

To prove Theorem 1, we need the following result [3, Proposition 7]:

Theorem D. Let $A$ and $B$ be positive invertible operators on a Hilbert space $H$. If $\log A \geq \log B$, then

$$\{B^\frac{r}{2} (B^\frac{t}{2} A^p B^\frac{t}{2})^s B^\frac{r}{2}\}^{\frac{1}{q}} \geq B^{(p+q)\frac{r}{q}+t}$$

holds for $p, t, s, r \geq 0$ and $q \geq 1$ with $(t+r)q \geq (p+t)s + r$.

Proof of Theorem 1.

(i) $\Rightarrow$ (ii): By Theorem D, (i) ensures

(5) $$\{B^\frac{r}{2} (B^\frac{t}{2} A^p B^\frac{t}{2})^s B^\frac{r}{2}\}^{\frac{1}{q}} \geq B^{(p+q)\frac{r}{q}+t}$$

holds for $p, t, s, r \geq 0$ and $q \geq 1$ with

(6) $$(t+r)q \geq (p+t)s + r.$$ 

Put $A_1 = A^{(p+t)+r}$ and $B_1 = (A^\frac{t}{2} B^p A^\frac{t}{2})^{1/q}$, then $A_1 \geq B_1$ by (5) and $k \geq A \geq 1/k > 0$ assures $k^{\frac{(p+t)+r}{q}} \geq (p+t+s+r).$ By applying Lemma 7 to $A_1$ and $B_1$, we have

$$A_1^q + qL(1, k^{(p+t)s+r}) \log M_{k(p+t)s+r}(1)I \geq B_1^q.$$ 

Multiplying $B^{-\frac{r}{2}}$ on both sides, we have (ii).

(ii) $\Rightarrow$ (iii): Put $r = (p+t)s - 2t \geq 0$ and $q = 2$ in (ii). Then the condition (6) is satisfied and $(p+t)s \geq 2t$, so we have (iii).

(iii) $\Rightarrow$ (iv): If we put $t = 0$ and $s = 1$ in (iii), then we have (iv).

(iv) $\Rightarrow$ (i): If we put $p \rightarrow 0$ in (iv), then we have (i) by (iv) of Lemma 5. □

Related to the extension of the Löwner-Heinz theorem, Furuta [4] established the following ingenious order preserving inequality which is now called the Furuta inequality.
Theorem F (Furuta inequality)
If \( A \geq B \geq 0 \), then for each \( r \geq 0 \)

(i) \( \left( B^\frac{r}{2} A^p B^\frac{r}{2} \right)^{\frac{1}{q}} \geq \left( B^\frac{r}{2} B^p B^\frac{r}{2} \right)^{\frac{1}{q}} \)

and

(ii) \( \left( A^\frac{r}{2} A^p A^\frac{r}{2} \right)^{\frac{1}{q}} \geq \left( A^\frac{r}{2} B^p A^\frac{r}{2} \right)^{\frac{1}{q}} \)

hold for \( p \geq 0 \) and \( q \geq 1 \) with \((1+r)q \geq p+r\).

Alternative proofs of Theorem F have been given in [2], [6], and one-page proof in [5]. The domain drawn for \( p, q \) and \( r \) in Figure is the best possible one [10] for Theorem F.

To prove Theorem 2, we need the following Furuta inequality:

**Theorem F'**. Let \( A \) and \( B \) be positive invertible operators on a Hilbert space \( H \) and \( \delta \in [0,1] \). Then the following properties are mutually equivalent:

(i) \( A^\delta \geq B^\delta \)

(ii) \( \left( B^\frac{r}{2} A^p B^\frac{r}{2} \right)^{\frac{\delta+r}{p+r}} \geq B^{\delta+r} \)

for \( p \geq \delta \) and \( r \geq 0 \).

**Proof of Theorem 2.**

Suppose that \( A^\delta \geq B^\delta \) for some \( \delta \in [0,1] \). By the Furuta inequality, we have

\[
\left( B^\frac{r}{2} A^p B^\frac{r}{2} \right)^{\frac{\delta+r}{p+r}} \geq B^{\delta+r}
\]

for \( p \geq \delta \) and \( r \geq 0 \).

and \( k^{\delta+r} \geq B^{\delta+r} \geq k^{-\delta-r} \).

By Lemma 7 and \( \frac{\delta+r}{p+r} \geq 1 \), it follows that

\[
B^\frac{r}{2} A^p B^\frac{r}{2} + \frac{p+r}{\delta+r} L(1, k^{p+r}) \log M_k(p+r) I \geq B^{p+r}.
\]

Hence we have

\[
A^p + \frac{p+r}{\delta+r} k^r L(1, k^{2p-2\delta}) \log M_k(2p-2\delta) I \geq B^p
\]

for \( p \geq \delta \) and \( r \geq 0 \). Put \( r = p - 2\delta \geq 0 \), then

\[
A^p + 2k^{2p-2\delta} L(1, k^{2p-2\delta}) \log M_k(2p-2\delta) I \geq B^p
\]

for all \( p \geq 2\delta \). □
REFERENCES

[4] T.Furuta, $A \geq B \geq 0$ assures $(B^rA^pB^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0$, $p \geq 0$, $q \geq 1$ with $(1+2r)q \geq p+2r$, Proc. Amer. Math. Soc., 101(1987), 85–88.