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Some Operator Functions
Implying Order Preserving Inequalities

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This paper is a resume based on my talk at “Structure of operators and related recent topics” which has been held at RIMS on January 24, 2003 and also this is early announcement of [9].

As an application of our previous result [Theorem 1, 11], we show a simple proof of the following result:

If $A \geq B \geq C \geq 0$ with $A > 0$ and $B > 0$, then for each $t \in [0, 1]$, and $p \geq t$, the following (i) and (ii) hold for a fixed real number $q$ and they are mutually equivalent:

(i) if $q \geq 0$, then

$$G_{p,q,t}(A, B, C, r, s) = A^{\frac{r}{2}} \{A^{\frac{r}{2}}(B^\frac{-t}{2}C^pB^\frac{-t}{2})^sA^\frac{r}{2}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing function for $r \geq t$ and $s \geq 1$ such that $(p - t)s \geq q - t$.

(ii) if $p \geq q$, then

$$G_{p,q,t}(A, B, C, r, s) = A^{\frac{r}{2}} \{A^{\frac{r}{2}}(B^\frac{-t}{2}C^pB^\frac{-t}{2})^sA^\frac{r}{2}\}^{\frac{q-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing function for $s \geq 1$ and $r \geq \max\{t, t - q\}$.

This result is further extension of our previous paper [Theorem 2, 11]. On the other hand, M.Uchiyama [17] shows the following interesting result

(iii) If $A \geq B \geq C \geq 0$ with $B > 0$, then for each $t \in [0, 1]$ and $p \geq 1$,

$$A^{1-t+r} \geq \{A^\frac{r}{2}(B^\frac{-t}{2}C^pB^\frac{-t}{2})^sA^\frac{r}{2}\}^{\frac{q-t+r}{(p-t)s+r}}$$

holds for $r \geq t$ and $s \geq 1$.

We show that (i) is equivalent to (iii), that is, follows from each other and also as an application of our previous result [Theorem 1, 11], we give a simple proof of M.Uchiyama’s result [Theorem 3.4, 17].
1 Introduction.

A capital letter means a bounded linear operator on a Hilbert space.

**Theorem L-H (Löwner-Heinz inequality)** [13][15].

\[ A \geq B \geq 0 \text{ ensures } A^\alpha \geq B^\alpha \text{ for all } \alpha \in [0, 1]. \]

Theorem L-H is very useful, but the condition " \( \alpha \in [0, 1] \) " is too restrictive to calculate operator inequalities, the following result has been obtained from this point of view.

**Theorem F (Furuta inequality)** [4].

If \( A \geq B \geq 0 \), then for each \( r \geq 0 \),

(i) \( (B^{\frac{t}{2}} A^p B^{\frac{t}{2}})^{\frac{1}{t}} \geq (B^{\frac{t}{2}} B^p B^{\frac{t}{2}})^{\frac{1}{t}} \)

and

(ii) \( (A^{\frac{t}{2}} A^p A^{\frac{t}{2}})^{\frac{1}{t}} \geq (A^{\frac{t}{2}} B^p A^{\frac{t}{2}})^{\frac{1}{t}} \)

hold for \( p \geq 0 \) and \( q \geq 1 \) with \( (1 + r)q \geq p + r \).

Alternative proofs are in [14][1] and one page proof in [5]. It is proved in [16] that The domain drawn for \( p, q \) and \( r \) in Figure is the best possible one for Theorem F. The following Theorem G is an extension of Theorem F.

**Theorem G** [6][2]. If \( A \geq B \geq 0 \) with \( A > 0 \), then for \( t \in [0, 1] \) and \( p \geq 1 \)

\[ A^{1-t+r} \geq \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1}{(1-t)r+p}} \text{ holds for } s \geq 1 \text{ and } r \geq t. \]

Very recently M. Uchiyama shows the following interesting extension of Theorem G.

**Theorem U** [17]. If \( A \geq B \geq C \geq 0 \) with \( B > 0 \), then for \( t \in [0, 1] \) and \( p \geq 1 \)

\[ A^{1-t+r} \geq \{A^{\frac{r}{2}} (B^{\frac{t}{2}} C^p B^{\frac{t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1}{(1-t)r+p}} \text{ holds for } s \geq 1 \text{ and } r \geq t. \]

We show that Theorem U is equivalent to (i) of Theorem 1 under below, that is, follows from each other and also as an application of our previous result [Theorem 1, 11], we give a simple proof of M. Uchiyama’s result [Theorem 3.4, 17].
Operator Functions Implying Theorem U.

**Theorem 1.** If $A \geq B \geq C \geq 0$ with $A > 0$ and $B > 0$, then for each $t \in [0, 1]$, and $p \geq t$, the following (i) and (ii) hold for a fixed real number $q$ and they are mutually equivalent:

(i) if $q \geq 0$, then

$$G_{p,q,t}(A, B, C, r, s) = A^{\frac{-r}{2}} \{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^{p}B^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)r+t}}A^{\frac{-r}{2}}$$

is decreasing function for $r \geq t$ and $s \geq 1$ such that $(p-t)s \geq q-t$.

(ii) if $p \geq q$, then

$$G_{p,q,t}(A, B, C, r, s) = A^{\frac{-r}{2}} \{A^{\frac{r}{2}}(B^{\frac{-t}{2}}C^{p}B^{\frac{-t}{2}})^{s}A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)r+t}}A^{\frac{-r}{2}}$$

is decreasing function for $s \geq 1$ and $r \geq \max\{t, t-q\}$.

We need the following results to prove Theorem 1.

**Theorem A** [11]. Let $A$ and $B$ be positive invertible operators on a Hilbert space satisfying

$$A \geq (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}}$$

for fixed $\alpha_0 \geq 0$ and $\beta_0 \geq 0$ with $\alpha_0 + \beta_0 > 0$.

Then the following (i) and (ii) hold and they are mutually equivalent:

(i) for any fixed $\delta \geq -\beta_0$,

$$f(\lambda, \mu) = A^{\frac{-\mu}{2}}(A^{\frac{\mu}{2}}B^{\lambda}A^{\frac{\mu}{2}})^{\frac{\lambda+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}}A^{\frac{-\mu}{2}}$$

is decreasing function for $\mu \geq 1$ and $\lambda \geq 1$ such that $\alpha_0\lambda \geq \delta$.

(ii) for any fixed $\delta \leq \alpha_0$,

$$f(\lambda, \mu) = A^{\frac{-\mu}{2}}(A^{\frac{\mu}{2}}B^{\lambda}A^{\frac{\mu}{2}})^{\frac{\lambda+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}}A^{\frac{-\mu}{2}}$$

is decreasing function for $\lambda \geq 1$ and $\mu \geq 1$ such that $\beta_0\mu \geq -\delta$.

**Lemma B** [6]. Let $X$ be a positive invertible operator and $Y$ be an invertible operator. For any real number $\lambda$,

$$(YXY^*)^\lambda = YX^{\frac{1}{2}}(X^{\frac{1}{2}}YX^{\frac{1}{2}})^{\lambda-1}X^{\frac{1}{2}}Y^*.$$
3 Equivalence Relation Associated with Theorem 1.

We show the following equivalence relation between Theorem 1 and related operator inequalities.

**Theorem 2.** The following (i), (ii), (iii) and (iv) hold and follow from each other.

(i) If $A \geq B \geq C \geq 0$ with $A > 0$ and $B > 0$, then for each $t \in [0, 1]$ and $p \geq 1$,

$$A^{1-t+r} \geq \{A^{\frac{r}{2}} (B^{\frac{1}{2}} C^{\frac{1}{2}} B^{\frac{1}{2}})^{s} A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for $r \geq t$ and $s \geq 1$.

(ii) If $A \geq B \geq C \geq 0$ with $A > 0$ and $B > 0$, then for each $1 \geq q \geq t \geq 0$ and $p \geq q$,

$$A^{q-t+r} \geq \{A^{\frac{r}{2}} (B^{\frac{1}{2}} C^{\frac{1}{2}} B^{\frac{1}{2}})^{s} A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-q)s+r}}$$

holds for $r \geq t$ and $s \geq 1$.

(iii) If $A \geq B \geq C \geq 0$ with $A > 0$ and $B > 0$, then for each $t \in [0, 1]$ and $p \geq 1$,

$$F_{p,t}(A, B, C, r, s) = A^{\frac{-r}{2}} \{A^{\frac{r}{2}} (B^{\frac{1}{2}} C^{\frac{1}{2}} B^{\frac{1}{2}})^{s} A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

is decreasing function for $r \geq t$ and $s \geq 1$.

(iv) If $A \geq B \geq C \geq 0$ with $A > 0$ and $B > 0$, then for each $t \in [0, 1], q \geq 0$ and $p \geq t$,

$$G_{p,q,t}(A, B, r, s) = A^{\frac{-r}{2}} \{A^{\frac{r}{2}} (B^{\frac{1}{2}} C^{\frac{1}{2}} B^{\frac{1}{2}})^{s} A^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-q)s+r}}$$

is decreasing function for $r \geq t$ and $s \geq 1$ such that $(p-t)s \geq q-t$.

We remark that Theorem 2 is an extension of [10], a proof of (i) of Theorem 2 is in [Proposition 4.1, 17], one page proof of (i) by using Theorem G itself is in [8], and also mean theoretic proof of (i) is in [3].

4 Satellite Inequalities.

As simple applications of Theorem 1 and Theorem 2, we show the following satellite inequalities.
Theorem 3. If $A \geq B \geq C > 0$, then the following inequalities (i) and (ii) hold for each $t \in [0,1]$, $p \geq 1$, $r \geq t$ and $s \geq 1$:

(i) \[
B^\frac{t}{2}C^\frac{t}{2}\{C^\frac{t}{2}(B^\frac{-t}{2}A^{p}B^\frac{-t}{2})^sC^\frac{t}{2}\}^\frac{1+r-t}{(p-t)s+t}C^\frac{-t}{2}B^\frac{t}{2} \geq B^\frac{t}{2}C^\frac{t}{2}\{C^\frac{t}{2}(B^\frac{-t}{2}A^{p}B^\frac{-t}{2})^sC^\frac{t}{2}\}^\frac{1}{(p-t)s+t}C^\frac{-t}{2}B^\frac{t}{2} \geq A \geq B \geq C \geq B^\frac{t}{2}A^\frac{-t}{2}\{A^\frac{t}{2}(B^\frac{-t}{2}C^{p}B^\frac{-t}{2})^sA^\frac{t}{2}\}^\frac{1+r-t}{(p-t)*+r}A^\frac{-t}{2}B^\frac{t}{2}.
\]

(ii) \[
B^\frac{t}{2}C^\frac{t}{2}\{C^\frac{t}{2}(B^\frac{-t}{2}A^{p}B^\frac{-t}{2})^sC^\frac{t}{2}\}^\frac{1+r-t}{(p-t)s+r}C^\frac{-t}{2}B^\frac{t}{2} \geq B^\frac{t}{2}C^\frac{t}{2}\{C^\frac{t}{2}(B^\frac{-t}{2}A^{p}B^\frac{-t}{2})^sC^\frac{t}{2}\}^\frac{1}{(p-t)s+r}C^\frac{-t}{2}B^\frac{t}{2} \geq A \geq B \geq C \geq B^\frac{t}{2}A^\frac{-r}{2}\{A^\frac{r}{2}(B^\frac{-t}{2}C^{p}B^\frac{1}{2})^sA^\frac{r}{2}\}^\frac{1+r-t}{(p-t)*+r}A^\frac{-r}{2}B^\frac{t}{2}.
\]

Corollary 4. If $A \geq B > 0$, then the following inequalities (i) and (ii) hold for each $t \in [0,1]$, $p \geq 1$, $r \geq t$ and $s \geq 1$:

(i) \[
B^\frac{-(r-t)}{2}\{B^\frac{r}{2}(B^\frac{-t}{2}A^{p}B^\frac{-t}{2})^sB^\frac{r}{2}\}^\frac{1+r-t}{(p-t)*+r}B^\frac{-(r-t)}{2} \geq \{B^\frac{t}{2}(B^\frac{-t}{2}A^{p}B^\frac{-t}{2})^sB^\frac{t}{2}\}^\frac{1}{(p-t)s+r}B^\frac{-(r-t)}{2} \geq A \geq B \geq \{A^\frac{t}{2}(A^\frac{-t}{2}B^{p}A^\frac{-t}{2})^sA^\frac{t}{2}\}^\frac{1}{(p-t)*+r}A^\frac{-(r-t)}{2}.
\]

(ii) \[
B^\frac{-(r-t)}{2}\{B^\frac{r}{2}(B^\frac{-t}{2}A^{p}B^\frac{-t}{2})^sB^\frac{r}{2}\}^\frac{1+r-t}{(p-t)*+r}B^\frac{-(r-t)}{2} \geq B^\frac{-(r-t)}{2}\{B^\frac{r}{2}(B^\frac{-t}{2}A^{p}B^\frac{-t}{2})^sB^\frac{r}{2}\}^\frac{1}{(p-t)*+r}B^\frac{-(r-t)}{2} \geq A \geq B \geq A^\frac{-(r-t)}{2}\{A^\frac{r}{2}(A^\frac{-t}{2}B^{p}A^\frac{-t}{2})^sA^\frac{r}{2}\}^\frac{1}{(p-t)*+r}A^\frac{-(r-t)}{2}.
\]
Corollary 5. If $A \geq B > 0$, then the following inequality holds for $p \geq 1$ and $r \geq 0$

$$B^{\frac{r}{2}}(B^{\frac{r}{2}}A^p B^{\frac{r}{2}})^{\frac{1+r}{p+r}} B^{\frac{r}{2}} \geq A \geq B \geq A^{\frac{r}{2}}(A^{\frac{r}{2}}B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}} A^{\frac{r}{2}}.$$ 

5 M.Uchiyama’s Result via Theorem A.

The following result is contained in Theorem 3.4 of [17].

**Theorem V.** Let $A$ and $B$ be both positive invertible operators. Also let $a$, $b$, and $c$ be positive real numbers and $d$ a real number. Define $F(r, s)$ and $G(r, s)$ by

$$F(r, s) = A^{\frac{r}{2}}(A^{\frac{r}{2}}B^s A^{\frac{r}{2}})^{\frac{r+s}{r-s}} A^{\frac{r}{2}} \quad \text{for } r > 0, s > 0$$

and

$$G(r, s) = A^{\frac{r}{2}}(A^{\frac{r}{2}}B^s A^{\frac{r}{2}})^{\frac{r+s}{r-s}} A^{\frac{r}{2}} \quad \text{for } r > 0, s > 0 \text{ with } 0 \leq \frac{s+d}{r+s} \leq 1.$$ 

Let $a > 0$, $b > 0$ and $-a \leq d \leq bc$. Then for $r_2 \geq r_1 \geq a$ and $s_2 \geq s_1 \geq b$ the following hold:

(a) if $F(a, b) \leq 1$, then $G(r_2, s_2) \leq G(r_1, s_1)$

(b) if $F(a, b) \geq 1$, then $G(r_2, s_2) \geq G(r_1, s_1)$.

On the other hand, in Theorem A replacing $A$ by $A^{\alpha_0}$ and $B$ by $B^{\alpha_0}$, then we have the following result in [12].

**Corollary C.** Let $A$ and $B$ be positive invertible operators on a Hilbert space satisfying

$$A^{\alpha_0} \geq (A^{\frac{\alpha_0}{2}} B^{\alpha_0} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha_0+\beta_0}} \text{ for fixed } \alpha_0 > 0 \text{ and } \beta_0 > 0.$$ 

Then for any fixed $\delta \geq -\beta_0$, 

$$f(\alpha, \beta) = A^{-\frac{\beta}{2}}(A^{\frac{\beta}{2}} B^\alpha A^{\frac{\beta}{2}})^{\frac{\alpha+\beta}{2+\beta}} A^{-\frac{\beta}{2}}$$

is decreasing function of $\alpha$ and $\beta$ such that $\alpha \geq \max\{\delta, \alpha_0\}$ and $\beta \geq \beta_0$.

We can give a proof of Theorem V via Corollary C.

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REFERENCES.


[4] T. Furuta, $A \geq B \geq 0$ assures $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$ for $r \geq 0, p \geq 0, q \geq 1$ with $(1 + 2r)q \geq p + 2r$, *Proc. Amer. Math. Soc.*, 101 (1987), 85-88.


[12] T. Furuta, T. Yamazaki and M. Yanagida, Order preserving operator function via Furuta inequality " $A \geq B \geq 0$ ensures $(A^\frac{r}{2} A^p A^\frac{r}{2})^{\frac{1+r}{p+r}} \geq (A^\frac{r}{2} B^p A^\frac{r}{2})^{\frac{1+r}{p+r}}$ for $p \geq 1$ and $r \geq 0."$ *Operator Theory:Advances and Applications*, Birkhäuser, 115 (2000),


