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CR manifolds in Grauert tubes

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In this article, we introduce a result on the logarithmic term of the Szegö kernel on the boundary of two-dimensional Grauert tubes. In Section 1, we give the definition of Grauert tube and some examples. In Section 2, we introduce a result in [5]. This result plays very important roles in studying CR manifolds in Grauert tubes. Finally, in Section 3, we state the main theorem in [8] and some remarks.

1 The definition of Grauert tube and examples

Let (X, g) be an n-dimensional complete C^ω Riemannian manifold, and let $\gamma : \mathbb{R} \to X$ be a geodesic. Then we define the mapping $\psi_\gamma : \mathbb{C} \to TX$ by

$$\psi_\gamma(\sigma + i\tau) := \tau\dot{\gamma}(\sigma).$$

Definition 1.1. Let $T^rX := \{v \in TX |\ g(v, v) < r^2\}$, where $0 < r \leq \infty$. A complex structure on T^rX is said to be adapted if ψ_γ is holomorphic for every geodesic γ on X.

If an adapted complex structure exists, then it is uniquely determined (see [9]).

The Grauert tube of radius r over X is the manifold T^rX with the adapted complex structure. X is called the center of the Grauert tube.

Let $r_{\max}(X)$ be the maximal radius r such that the adapted complex structure is defined on T^rX. It is known that $r_{\max}(X) > 0$ if X is compact or X is homogeneous.

Example 1.2. Let $X := \mathbb{R}^n$. Then $T^\infty\mathbb{R}^n$ is biholomorphic to $\mathbb{C}^n$.

Example 1.3. Let $X := S^n$, the unit sphere in $\mathbb{R}^{n+1}$. Then $T^\infty S^n$ is biholomorphic to the manifold $Q^n := \{z = (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} |\ z_1^2 + \cdots + z_{n+1}^2 = 1\}$. We call Q^n the complex quadric.

Example 1.4. Let X be the n-dimensional real hyperbolic space with constant sectional curvature -1. Then $T^{\pi/2}X$ is biholomorphic to B^n, the unit ball in $\mathbb{C}^n$. We note that $r_{\max}(X) = \pi/2$ (see [11, Theorem 2.5]).

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2 CR manifolds in Grauert tubes

Let (X, g) be an n-dimensional compact C^ω Riemannian manifold, and let $T^r X$ be the Grauert tube. We define the mapping $\rho : T^r X \to \mathbb{R}$ by $\rho(v) := 2g(v, v)$ for $v \in T^r X$.

Theorem 2.1 **([5], [9]).** *ρ has the following properties:*

(1) *ρ is strictly plurisubharmonic,*
(2) $X = \rho^{-1}(0)$,
(3) *the metric $ds^2_{T^r X}$ obtained from the Kähler form $i\partial\overline{\partial}\rho/2$ is compatible with g, that is, $ds^2_{T^r X}|_X = g$, and*
(4) $(\partial\overline{\partial}\sqrt{\rho})^n = 0$ *in* $T^r X - X$.

Let $\Omega_\varepsilon := \{\rho < \varepsilon^2\} \subset T^r X$, and let $M_\varepsilon := \partial\Omega_\varepsilon$. Then we see that M_ε is a strongly pseudoconvex CR manifold.

One of the interesting problems on Grauert tube is to study relations between M_ε and (X, g). Several results have been known on this problem. Stenzel [10] studied orbits of the geodesic flow and chains. Kan [7] computed the Burns-Epstein invariant, and showed that M_{ε_1} and M_{ε_2} are not CR equivalent if $\varepsilon_1 \neq \varepsilon_2$ when $\dim X = 2$. This Kan's result is also true for $\dim X \geq 3$ (see [12]). This implies that there exist many CR manifolds in the Grauert tube. This fact is one of the reasons why we are interested in this problem.

3 Result

Let (X, g) be a two-dimensional compact Riemannian manifold, and let $T^r X$ be the Grauert tube. We put $\Omega_\varepsilon := \{\rho < \varepsilon^2\} \subset T^r X$ and $M_\varepsilon := \partial\Omega_\varepsilon$.

Let $\theta := \iota_\varepsilon^*(-i\partial\rho)$, where ι_ε is the embedding of M_ε in the Grauert tube. Then θ defines a pseudo-hermitian structure on M_ε. Let S_ε be the Szegö kernel with respect to the volume element $\theta \wedge d\theta$. Then by [2] and [1], the singularity of S_ε on the diagonal of M_ε is of the form

$$S_\varepsilon(z, \bar{z}) = \varphi(z)\rho_\varepsilon(z)^{-2} + \psi(z)\log\rho_\varepsilon(z),$$

where φ, $\psi \in C^\infty(\overline{\Omega_\varepsilon})$ and ρ_ε is a defining function of Ω_ε with $\rho_\varepsilon > 0$ in Ω_ε.

Theorem 3.1 **([8]).** *The boundary value of the logarithmic term coefficient $\psi_0 = \psi|_{M_\varepsilon}$ has the following asymptotic expansion as $\varepsilon \to +0$:*

$$\psi_0 \sim \frac{1}{24\pi^2}\sum_{l=0}^{\infty} F_l^{\psi_0}\varepsilon^{2l}, \tag{3.1}$$

where $F_l^{\psi_0}(\lambda^2 g) = \lambda^{-2l-4}F_l^{\psi_0}(g)$ for $\lambda > 0$.

In particular, we have

$$F_0^{\psi_0} = -\frac{1}{10}\Delta k - \frac{2}{5}(\varepsilon^2 T^2 k)|_{\varepsilon=0}, \tag{3.2}$$

where k is the scalar curvature, Δ is the Laplacian and T is the unique vector field on M_ε such that $\theta(T) = 1$ and $T\rfloor d\theta = 0$.

We now make two remarks on the term $(\varepsilon^2 T^2 k)|_{\varepsilon=0}$. One is that we can regard this term as a function on the circle bundle over X, and it is not constant on each fiber of the bundle in general (see [8, Lemma 4.5]). This means that the value to which ψ_0 tends as $\varepsilon \to +0$ varies with the way ε goes to $+0$.

The other is that

$$\int_{M_\varepsilon} (\varepsilon^2 T^2 k)\big|_{\varepsilon=0} \theta \wedge d\theta = c\varepsilon^2 \int_X \Delta k dV + O(\varepsilon^3), \tag{3.3}$$

where c is a constant and dV is the volume form on X (see also [7]). It follows from (3.1)–(3.3) and $\int_X \Delta k dV = 0$ that the coefficient of ε^2 in the integral

$$\int_{M_\varepsilon} \psi_0 \theta \wedge d\theta$$

is equal to 0. This is not contradict to the fact that the integral above is eqaul to 0.

Finally, we note that ψ_0 is a constant multiple of the Q-curvature of three-dimensional CR manifolds (see [3], [4] and [6]). In conformal geometry, there has been great progress recently in understanding the Q-curvature and its geometric meaning in low dimensions. However, roles of Q-curvature in CR geometry are not clear. We hope that this result will become an approach to studying CR Q-curvature.

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