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Kyoto University
On Uniqueness of The Solutions of
The Obstacle Problem

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Abstract

In [2], R. Fehlmann and F. P. Gardiner studied an extremal problem for a topologically finite Riemann surface and established the slit mapping theorem by showing existence of a quadratic differential which associated with the solution of the extremal problem. In this article, we give a condition for non-uniqueness of such slit mappings, by using deformation of a Riemann surface using the foliation structure of the differential associated with the solution.

1 Introduction

Suppose \( S \) is a finite bordered Riemann surface with the border \( \Gamma \). In other words, the boundary \( \Gamma \) consists of a finite number of simple closed curves, and the double of \( S \) with respect to the border \( \Gamma \) is of finite analytic type. Let \( T(S) \) be the Teichmüller space of the interior \( S^o \) of \( S \). Let \( A(S) \) be the set of integrable holomorphic quadratic differential \( \varphi \) on \( S \) with the properties that \( \varphi = \varphi(z)dz^2 \) is real along the border \( \Gamma \).

Definition 1.1 Let \( E \) be a compact subset of \( S^o \) which satisfies that \( S \setminus E \) is of finitely connected and of the same genus as \( S \). We say that \( E \) with these properties is an allowable subset of \( S \).

Next fix an element \( \varphi \in A(S) \). If each component of an allowable \( E \) is a horizontal arc of \( \varphi \) or a union of a finite number of horizontal arcs and critical points of \( \varphi \), we say that \( E \) is an allowable slit with respect to \( \varphi \).

Let \( E \) be an allowable subset of \( S \). Let \( \mathcal{F}(S,E) \) be the family of pairs \( (g,S_g) \), where \( g \) is a conformal map of \( S \setminus E \) into another Riemann surface \( S_g \) such that \( g \) maps the border \( \Gamma \) onto the border of \( S_g \) and the puncture of \( S \) onto the puncture of \( S_g \). In particular, \( (g,S_g) \in \mathcal{F}(S,E) \) induces an isomorphism \( \iota_g \) from the fundamental group \( \pi_1(S) \) of \( S \) onto \( \pi_1(S_g) \). Let \( f \) be a quasiconformal map of \( S \) onto \( S_g \) which induces the same isomorphism \( \iota_g \), and \( \mu \) the Beltrami differential of \( f \). We denote \( f \) also by \( f^\mu \) and the Teichmüller (equivalence) class of \( f^\mu \) in \( T(S) \) by \([\mu;g,S_g]\).
Let $\mathcal{S}(S)$ be the family of simple closed curves in $S^\circ$ which is homotopic neither to a point of $S$ nor to a puncture on $S$. Let $\mathcal{S}[S]$ be the set of homotopy class of an element of $\mathcal{S}(S)$. For $\varphi \in A(S)$ and $\gamma \in \mathcal{S}(S)$, we denote the height of $\gamma$ with respect to $\varphi$ by $h_\varphi(\gamma)$, and that the height of homotopy class $[\gamma]$ by $h_\varphi[\gamma]$. For the details, see for instance [4].

Now it is known (cf. [4]) that, for every $(f, S_f) \in \mathfrak{F}(S, E)$ and $\varphi \in A(S) \setminus \{0\}$, there is a holomorphic quadratic differential $\varphi_f$ on $S_f$ whose heights on $S_f$ are equal to the corresponding heights of $\varphi$ on $S$. Fehlmann and Gardiner posed the extremal problem for $(S, \varphi, E)$, of maximizing

$$M_f = \|\varphi_f\|_{L^1(S_f)} = \int_{S_f} |\varphi_f|$$

in $\mathfrak{F}(S, E)$, and showed the following result.

**Theorem 1.2 (Fehlmann-Gardiner)** Suppose that $S$ is a finite bordered Riemann surface, and that $\varphi \in A(S) \setminus \{0\}$. Let $E$ be an allowable subset of $S$. Then there exists a point $[(\mu; g, S_g)] \in T(S)$ associated with an element $(g, S_g) \in \mathfrak{F}(S, E)$ such that $M_g$ attains the maximum

$$M = \max_{(f, S_f) \in \mathfrak{F}(S, E)} M_f.$$ 

Moreover, for this point $[(\mu; g, S_g)] \in T(S)$, $E_g = S_g \setminus g(S \setminus E)$ is an allowable slit with respect to $\varphi_g$.

The point $[(\mu; g, S_g)] \in T(S)$ in Theorem 1.2 is called an extremal point of the extremal problem for $(S, \varphi, E)$, the map $g$ an extremal slit mapping associated with it, and the associated differential $\varphi_g$ the structure differential for $g$.

We show in this note the following theorem which gives a condition for extremal points, and hence extremal slit mappings, not to be unique.

**Theorem 1.3** Suppose $R$ is a finite bordered Riemann surface, and that $\psi \in A(R) \setminus \{0\}$. Let $E_\psi$ be an allowable slit of $R$ with respect to $\psi$ such that

1. there is a component of $E_\psi$ which contains a zero point $p_0$ of $\psi$ of order $m \geq 3$ and at least two of horizontal arcs $\ell_1, \ell_2$ with an end point at $p_0$, and that

2. each of the angles between $\ell_1, \ell_2$ are larger than $\frac{2\pi}{m+2}$.

Then, there is a finite bordered Riemann surface $\tilde{R}$, a pair $(h, \tilde{R}) \in \mathfrak{F}(R, E_\psi)$, and a holomorphic quadratic differential $\tilde{\psi} \in A(\tilde{R}) \setminus \{0\}$, such that
(i) $E_{\tilde{\psi}} = \tilde{R} \setminus h(R \setminus E_{\psi})$ is an allowable slit of $\tilde{R}$ with respect to $\tilde{\psi}$,

(ii) the heights of $\tilde{\psi}$ on $\tilde{R}$ is the same as the corresponding heights of $\psi$ on $R$,

(iii) the point $[(\mu; h, \tilde{R})] \in T(R)$ is different from the origin $[(0; id, R)]$ of $T(R)$.

We call the conditions 1. and 2. for $E_{\psi}$ in the Theorem 1.3 the refolding conditions, and the point $p_0$ a refolding point.

**Corollary 1.4** Suppose $S$ is a finite bordered Riemann surface and that $\varphi \in A(S) \setminus \{0\}$. Let $E$ be an allowable subset of $S$, and $[(\mu; g, S_g)] \in T(S)$ the extremal point of the extremal problem for $(S, \varphi, E)$. If the allowable slit $E_g$ of $S_g$ with respect to the structure differential $\varphi_g$ satisfies the refolding conditions, then there exists another extremal point of the extremal problem for $(S, \varphi, E)$ different from $[(\mu; g, S_g)]$.

**Proof.** Take the triple $(S_g, \varphi_g, E_g)$ as the triple $(R, \psi, E_{\psi})$ in the Theorem 1.3. Then we obtain a finite bordered Riemann surface $\tilde{R}$, a pair $(h, \tilde{R}) \in \mathfrak{F}(S_g, E_g)$, and a holomorphic quadratic differential $\tilde{\psi} \in A(\tilde{R}) \setminus \{0\}$ such that

(i) $E_{\tilde{\psi}}$ is an allowable slit of $\tilde{R}$ with respect to $\tilde{\psi}$,

(ii) the heights of $\tilde{\psi}$ on $\tilde{R}$ is the same as the corresponding heights of $\varphi_g$ on $S_g$ (and hence of $\varphi$ on $S$), and

(iii) the point $[(\mu; h, \tilde{R})] \in T(S_g)$ is different from the origin $[(0; id, S_g)]$ of $T(S_g)$.

Then, we know (cf. [2]) that, from (i) and (ii), the point $[(\mu; h \circ g, \tilde{R})] \in T(S)$ is an extremal point of the extremal problem for $(S, \varphi, E)$. By (iii), the point $[(\mu; g, S_g)]$ is different from the point $[(\mu; h \circ g, \tilde{R})]$. Thus we have the assertion. 

**2 Example**

In this section we give an example of the triple $(S, \varphi, E)$ which satisfies the assumptions of Corollary 1.4.

First take three copies $M_1, M_2, M_3$ of a rectangle

$$M = \{z = x + iy \in \mathbb{C} \mid |x| \leq 2, |y| \leq 1\},$$
and let $z_j$ be the coordinate corresponding to $z$ on each $M_j$. Next on each $M_j$, identify two pair of parallel sides under the translations

$$z_j \rightarrow z_j + 4, \quad z_j \rightarrow z_j + 2i.$$ 

Then we obtain three copies $T_1, T_2, T_3$ of a torus $T$. And the quadratic differential $dz^2$ on $M$ induces the holomorphic quadratic differential $\varphi_0$ on $T$.

Cut $M_j$ along the segment

$$I_j = \{z_j = x_j + iy_j \mid -1 \leq x_j \leq 0, y_j = 0\},$$

and connect them cyclically. More precisely, we paste the upper edge $I_1^+$ of the slit $I_1$ and the lower edge $I_2^-$ of the slit $I_2$, the upper edge $I_2^+$ of the slit $I_2$ and the lower edge $I_3^-$ of the slit $I_3$, and the upper edge $I_3^+$ of the slit $I_3$ and the lower edge $I_1^-$ of the slit $I_1$. Then we obtain a compact Riemann surface $S$ of genus three.

Now let $\Pi$ be the natural projection from $S$ to the torus $T$, and $\varphi$ the pull-back of $\varphi_0$ by $\Pi$. Finally, let $E$ be a subset of $S$, consisting of the arcs $\ell_1$ and $\ell_2$, where each $\ell_i$ is one on $M_i$ corresponding to

$$\{z \mid 0 \leq x \leq 1, y = 0\}.$$

Now we consider the extremal problem for $(S, \varphi, E)$. Then the set $E$ is an allowable slit of $S$ with respect to $\varphi$. Hence we know the identical mapping of $S$ gives the extremal slit map associated with the extremal problem for this triple. Moreover, we can easily see that $E$ satisfies the refolding conditions.

Thus the assumptions in Corollary 1.4 are satisfied, and as a consequence, the extremal points of the extremal problem for $(S, \varphi, E)$ are not uniquely determined in $T(S)$.

## 3 Proof of theorem 1.3

Assume that a component $J$ of $E_\psi$ contains a refolding point $p_0$ of $\psi$ of order $m \geq 3$ and horizontal arcs $\ell_1$ and $\ell_2$, one of whose end point is $p_0$ and an angle between $\ell_1$ and $\ell_2$ is

$$\frac{2k\pi}{m+2} \quad \left(2 \leq k \leq \frac{m+2}{2}\right).$$

Here the arcs $\ell_1, \ell_2$ are segments on the real axis with an endpoint at the origin with respect to the natural parameter $\zeta = \zeta_\psi$ induced from $\psi$. 
We take a subarc $\kappa_j \subset \ell_j$ such that $p_0$ is an endpoint of each $\kappa_j$ and that $\psi$ has no zeros on $\kappa_j \setminus \{p_0\}$. Let $p_j$ be the other endpoint of $\kappa_j$ for each $j$. Also set $K = \kappa_1 \cup \kappa_2$.

Now, cut $R$ along $\kappa_1$ and $\kappa_2$. For each $j$, let $\kappa_j^+$ and $\kappa_j^-$, respectively, the right-side and the left-side edge of the slit $\kappa_j$, with respect to the orientation which corresponds to moving along the slit from $p_0$ to $p_j$. Assume that $\kappa_1^-$ and $\kappa_2^+$, resp. $\kappa_1^+$ and $\kappa_2^-$, makes the angle

$$\frac{2k\pi}{m+2}, \text{ resp. } \left(1 - \frac{k}{m+2}\right)2\pi.$$ 

Paste $\kappa_1^-$ and $\kappa_2^+$ so that points having the same absolute value with respect to $\zeta$ are identified. By the same way, paste $\kappa_1^+$ and $\kappa_2^-$. Then we obtain a finite bordered Riemann surface $\tilde{R}$ and the natural conformal embedding $h : R \setminus K \to \tilde{R}$. This pair $(h, \tilde{R})$ is an element of a family $\mathcal{F}(R, K) \subset \mathcal{F}(R, E_\psi)$.

Moreover, from the construction we can extend $\psi$ restricted on $R \setminus K$ naturally to a holomorphic quadratic differential $\tilde{\psi}$ on $\tilde{R}$, and $E_{\tilde{\psi}} = \tilde{R} \setminus h(R \setminus E_\psi)$ is allowable slit of $\tilde{R}$ with respect to $\tilde{\psi}$.

Now let $f^\mu$ be a quasiconformal map from $R$ onto $\tilde{R}$, which is a representation of the point $[(\mu; h, \tilde{R})] \in T(R)$.

**Lemma 3.1**

$$h_\psi[\tilde{\gamma}] = h_\psi[(f^\mu)^{-1}(\tilde{\gamma})]$$

for every $[\tilde{\gamma}] \in \mathcal{G}[\tilde{R}]$.

**Proof.** We say that a simple closed curve $\tilde{\beta}$ on $\tilde{R}$ is a $\tilde{\psi}$-polygon, if $\tilde{\beta}$ is a union of a finite number of horizontal arcs and vertical arcs of $\tilde{\psi}$. Note that for every $[\tilde{\gamma}] \in \mathcal{G}[\tilde{R}]$

$$h_\tilde{\psi}[\tilde{\gamma}] = \inf_{\tilde{\beta}} h_\tilde{\psi}(\tilde{\beta}),$$

where the infimum is taken over all $\tilde{\psi}$-polygons $\tilde{\beta}$ homotopic to $\tilde{\gamma}$ on $\tilde{R}$.

Now we can deform the pre-image $h^{-1}(\tilde{\beta})$ of such a $\tilde{\psi}$-polygon $\tilde{\beta}$ to a $\psi$-polygon $\beta$ such that $\beta$ is homotopic to $h^{-1}(\tilde{\beta})$ on $R$ and

$$h_\psi(\beta) = h_\tilde{\psi}(\tilde{\beta}).$$

Hence we conclude that

$$h_\psi[(f^\mu)^{-1}(\gamma)] \leq h_\psi(\beta) = h_\tilde{\psi}(\tilde{\beta}).$$
for every $\tilde{\psi}$-polygon $\tilde{\beta}$ which is homotopic to $\tilde{\gamma}$, which in turn implies that

$$h_\psi[(f^\mu)^{-1}(\tilde{\gamma})] \leq h_{\tilde{\psi}}[\tilde{\gamma}]$$

for every $[\tilde{\gamma}] \in \mathfrak{S}[\tilde{\mathcal{R}}]$. On the other hand, we can similarly see as above that

$$h_{\tilde{\psi}}[f^\mu(\gamma)] \leq h_\psi[\gamma]$$

for every $[\gamma] \in \mathfrak{S}[\mathcal{R}]$. Thus we have the assertion.

From Lemma 3.1, we see that the holomorphic quadratic differential $\tilde{\psi} \in A(\tilde{\mathcal{R}}) \setminus \{0\}$ satisfies the condition (ii). Moreover, by definition, $E_{\tilde{\psi}}$ is an allowable slit of $\tilde{\mathcal{R}}$ with respect to $\tilde{\psi}$, and

$$\tilde{\psi} \circ h(h')^2 = \psi \text{ on } \mathcal{R} \setminus E_{\psi}.$$  

**Lemma 3.2** The point $[(\mu; h, \tilde{\mathcal{R}})] \in T(\mathcal{R})$ is different from the origin $[(0; id, \mathcal{R})]$ of $T(\mathcal{R})$.

**Proof.** Assume that

$$[(\mu; h, \tilde{\mathcal{R}})] = [(0; id, \mathcal{R})].$$

Then there would exist a conformal map $\iota : \mathcal{R} \to \tilde{\mathcal{R}}$ such that the induced isomorphism $(\iota)_* : \pi_1(\mathcal{R}) \to \pi_1(\tilde{\mathcal{R}})$ is the same as the one induced by $h$.

Fix a $[\gamma] \in \mathfrak{S}[\mathcal{R}]$ arbitrarily. Then Lemma 3.1 gives that

$$h_{\tilde{\psi}}[\iota(\gamma)] = h_\psi[\gamma].$$

Since $h_{\tilde{\psi}}[\iota(\gamma)] = h_{\tilde{\psi}}[\iota^2(\gamma)]$, we obtain

$$h_{\tilde{\psi}}[\iota^2(\gamma)] = h_\psi[\gamma]$$

for every $[\gamma] \in \mathfrak{S}[\mathcal{R}]$. Hence the heights mapping theorem implies that $\tilde{\psi} \circ \iota^2 = \psi$ on $\mathcal{R}$. In particular, the map $\iota$ maps the zeros of $\psi$ to zeros of $\tilde{\psi}$ including multiplicities.

Now from the construction, the zero $p_0$ of order $m \geq 3$ breaks into two zeros $\tilde{q}_1$ and $\tilde{q}_2$ of $\tilde{\psi}$ of order $k - 2$ and $m - k$, respectively, with $2 \leq k \leq (m + 2)/2$. And the endpoints $p_1$ of $\kappa_1$ and $p_2$ of $\kappa_2$ gather to a zero $\tilde{q}$ of $\tilde{\psi}$ on $\tilde{\mathcal{R}}$ of order 2.

Set $\tilde{K} = \tilde{\mathcal{R}} \setminus h(\mathcal{R} \setminus K)$. Then all zeros $\tilde{q}, \tilde{q}_1$, and $\tilde{q}_2$ of $\tilde{\psi}$ on $\tilde{K}$ have the orders strictly less than $m$. Hence we see that

$$\iota(p_0) \not\in \tilde{K}.$$
Since the conformal embedding $h$ maps $R \setminus K$ onto $\tilde{R} \setminus \tilde{K}$, $h^{-1} \circ \iota(p_0)$ is well defined and $h^{-1} \circ \iota(p_0) \notin K$. In particular,

$$h^{-1} \circ \iota(p_0) \neq p_0.$$  

Next assume that, for a positive integer $n$,

$$(h^{-1} \circ \iota)^n(p_0) \neq (h^{-1} \circ \iota)^k(p_0)$$

for every $k$ with $0 \leq k \leq n - 1$. Then, $\iota \circ (h^{-1} \circ \iota)^n(p_0) \notin \tilde{K}$, for the zero $\iota \circ (h^{-1} \circ \iota)^n(p_0)$ of $\psi$ is of order $m$. Hence similarly as above, $(h^{-1} \circ \iota)^{n+1}(p_0) \notin K$. In particular,

$$(h^{-1} \circ \iota)^{n+1}(p_0) \neq p_0.$$ 

Also by the assumption,

$$(h^{-1} \circ \iota)^{n+1}(p_0) \neq (h^{-1} \circ \iota)^k(p_0)$$

for every $k$ with $1 \leq k \leq n$.

Thus by the induction, we conclude that, for every positive integer $n$, we have

$$(h^{-1} \circ \iota)^n(p_0) \neq (h^{-1} \circ \iota)^k(p_0)$$

for every $k$ with $0 \leq k \leq n - 1$, which implies that $\psi$ has infinitely many distinct zeros. This is absurd, and we have shown that

$$[(\mu; h, \tilde{R})] \neq [(0; id, R)].$$

**Remark** As the example in Section 2, if one can see the widths of $\psi$ on $R$ and that of $\tilde{\psi}$ on $\tilde{R}$, it is easy to show the claim of Lemma 3.2. Because if $[(\mu; h, \tilde{R})] = [(0; id, R)]$ in $T(R)$, then from Lemma 3.1 and the heights theorem we can see that widths of $\psi$ on $R$ is equal to the corresponding widths of $\tilde{\psi}$ on $\tilde{R}$. For example, in the case of Section 2 we denote by $\tilde{S}$ a Riemann surface obtained from $S$ by deformation and denote by $\tilde{\varphi}$ an integrable holomorphic quadratic differential whose heights is the same as the corresponding heights of $\varphi$ on $S$. Let $\gamma \in \mathcal{S}(S)$ be rounding $I_1$ on $M_1$. Then the width of $[\gamma] \in \mathcal{S}[S]$ is equal to 2. On the other hand, for this $[\gamma] \in \mathcal{S}[S]$ the corresponding width of $\tilde{\varphi}$ on $\tilde{S}$ is equal to 4. Therefore the deformation actually change the surface in $T(S)$.

Thus we have completed the proof of Theorem 1.3.
References


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