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Kyoto University
LOG-PLURICANONICAL SYSTEMS OF SMOOTH PROJECTIVE SURFACES

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1 Introduction

Let $X$ be a smooth projective variety and $K_X$ be a canonical divisor of $X$. Then $X$ is called of general type when pluricanonical system $|mK_X|$ defines a birational embedding of $X$ for some positive integer $m$. The behavior of the pluricanonical systems is important to study varieties of general type. For example, there is such a problem:

Problem 1.1 Let $X$ be a smooth projective variety of general type. Find a positive integer $m_0$ such that for every $m \geq m_0$, $|mK_X|$ gives a birational map from $X$ into a projective space. 

In the case $\dim X = 1$, it is well known that $|3K_X|$ gives a projective embedding. In the case $\dim X = 2$, E. Bombieri proved that $|5K_X|$ gives a birational embedding([1]). Recently, H. Tsuji showed that there exists an integer $\nu_n$ which depends only on $n = \dim X$ and satisfies above problem([6,7]). But when $\dim X \geq 4$, effective value of $\nu_n$ is unknown. Even if $\dim X = 3$, the value of $\nu_n$ becomes an astronomical number, and it is supposed that the value computed in [7] is not best-possible.

I am interested in this problem for open surfaces. As long as I know, such a situation is not studied yet.

Definition 1.1 Let $X$ be a surface and $D$ be a divisor with normal closings. Then the pair $(X, D)$ is called log-surface. If the linear system $|K_X + D|$ is big, we say that $(X, D)$ is of log-general type.

Now we state the problem more precisely.
Problem 1.2 Let \((X, D)\) be a smooth projective surface of log-general type. Find a positive integer \(m_0\) such that for every \(m \geq m_0\), \(|m(K_X + D)|\) gives a birational map from \(X\) into a projective space. \(\square\)

The main purpose of this paper is to answer the weaker version of this problem. Our result shows the value of \(m_0\) for a given surface. But it depends on \(X\) and divisor \(D\).

Theorem 1.1 Let \((X, D)\) be a smooth projective surface of log-general type, and let \(K_X + D = P + E\) be a Zariski-decomposition of \(K_X + D\), where \(P\) is a nef part and \(E\) is effective part of the decomposition. Then \(|m(K_X + D)|\) defines a birational map from \(X\) into projective space unless

\[ m \geq \frac{6\sqrt{2}}{\sqrt{P^2}} + 4. \]

\(\square\)

Remark 1.1 In the case \((X, D)\) is log-general type, \(P^2 > 0\) holds ([3]).

2 Terminology

In this section we introduce a singular harmonic metric and some results we use after. See [2] for more details.

Definition 2.1 Let \(L\) be a holomorphic line bundle on \(X\), \(h_0\) be a \(C^\infty\)-hermitian metric on \(L\), and \(\varphi\) be a \(L_{loc}^1\)-function on \(X\). Then we call \(h = e^{-\varphi} \cdot h_0\) a singular hermitian metric with respect to \(\varphi\). \(\varphi\) is called a weight function of \(h\). \(\square\)

Definition 2.2 We define a curvature current \(i\Theta_h\) of a singular hermitian line bundle \((L, h)\) as follows:

\[ i\Theta_h := i\bar{\partial}\varphi + i\Theta_{h_0} \]

where \(\partial\bar{\partial}\) is taken as a distribution and \(i\Theta_{h_0}\) is the curvature form of \((L, h_0)\) in usual sense.

A singular hermitian line bundle is said to be positive, if the curvature current \(i\Theta_h\) becomes a measure which takes values in semipositive-defined hermitian matrix. \(\square\)
Next we introduce a concept of multiplier ideal sheaves. Let $U \subset X$ be an open set and $\mathcal{O}(U)$ be the set of holomorphic functions on $U$. Then

$$I_U(h) := \left\{ f \in \mathcal{O}(U) \left| \int_U e^{-\varphi} |f|^2 dV < +\infty \right. \right\}$$

becomes a presheaf when $U$ runs all open subsets of $X$. We put $I(h)$ as the sheafification of $I_U(h)$. $I(h)$ is called the multiplier ideal sheaf with respect to $h$. The following theorem which is a variant of Kodaira's vanishing theorem is due to A.Nadel([5]).

**Theorem 2.1** Let $(X, \omega)$ be a Kähler manifold and $(L, h)$ be a singular hermitian line bundle on $X$. Assume that $i\Theta_h \geq \epsilon_0 \omega$ for some $\epsilon_0 > 0$. Then

$$H^q(X, \mathcal{O}_X(K_X + L) \otimes I(h)) = 0 \quad (q \geq 1).$$

\[\square\]

3 Proof of Theorem 1.1

In this section, we show the outline of the proof of Theorem 1.1. The proof is made along [6, section2], so please refer to [6] for detail.

Let $(X, D)$ be a smooth projective surface of log-general type and $x, y \in X, x \neq y$ be generic two points. Assume that there exists a singular hermitian metric $h_{x,y}$ on $m(K_x + D) + D$ such that:

1. $x, y \in \text{supp}\mathcal{O}_X/I(h_{x,y})$
2. One of the $x$ or $y$, say $x$, is an isolated point of $\text{supp}\mathcal{O}_X/I(h_{x,y})$
3. $i\Theta_h \geq \epsilon_0 \omega$ for some $\epsilon_0 > 0$.

We consider the long exact sequence:

$$\cdots \rightarrow H^0(X, \mathcal{O}_X((m+1)(K_X + D))) \rightarrow H^0(X, \mathcal{O}_X((m+1)(K_X + D)) \otimes \mathcal{O}/I(h_{x,y})) \rightarrow \cdots.$$

Where $H^1(X, \mathcal{O}_X(K_X + m(K_X + D) + D) \otimes I(h_{x,y})) = 0$ by Nadel's vanishing theorem, hence we get surjection and we can conclude that there exists some $\sigma \in H^0(X, \mathcal{O}_X((m+1)(K_X + D)))$ such that $\sigma(y) = 0$ and $\sigma(x) \neq 0$. This shows that $\Phi_{[(m+1)(K_X + D)]}$ separates $x$ and $y$. Therefore to prove theorem 1.1, we have only to compute the value $m$ such that we can construct a singular hermitian metric $h_{x,y}$ on $(m+1)(K_X + D)$ which satisfies the condition 1, 2 and 3 above for arbitrary distinct two points $x, y \in U$, for some nonempty Zariski open subset $U \subset X$. 
3.1 Construction of $h_{x,y}$

Let $K_{X} + D = P + E$ be a Zariski-decomposition of $K_{X} + D$, where $P$ is the nef part and $E$ is the effective part of the decomposition. We put $X^{\circ}$ as follows:

$$X^{\circ} := \{ \ p \in X \mid p \notin Bs|mP| \ \text{and for some } m, |mP| \ \text{gives biholomorphic near } p \}$$

Then $X^{\circ}$ is a nonempty Zariski open set of $X$.

We take arbitrary $x, y \in X^{\circ}$ and we set $\mathcal{M}_{x,y} := \mathcal{M}_{x} \otimes \mathcal{M}_{y}$, where $\mathcal{M}_{x}$ and $\mathcal{M}_{y}$ are the maximum ideal sheaf of the points $x, y$ respectively.

By considering a cohomology exact sequence and comparing the dimension of $H^{0}(X, O_{X}(mP))$ and $H^{0}(X, O_{X}(mP) \otimes O_{X}/\mathcal{M}_{x,y}^{\lceil \sqrt{\frac{P^{2}}{2}} \cdot (1-\varepsilon)m \rceil})$, we can show the following:

**Proposition 3.1** For arbitrary small $\varepsilon > 0$,

$$\dim H^{0}(X, O_{X}(mP) \otimes O_{X}/\mathcal{M}_{x,y}^{\lceil \sqrt{\frac{P^{2}}{2}} \cdot (1-\varepsilon)m \rceil}) \geq 1$$

holds if we take $m$ sufficiently large. \(\square\)

We take $\sigma_{0} \in H^{0}(X, O_{X}(m_{0}P) \otimes O_{X}/\mathcal{M}_{x,y}^{\lceil \sqrt{\frac{P^{2}}{2}} \cdot (1-\varepsilon_{0})m_{0} \rceil})$ for sufficiently small $\varepsilon_{0}$ and sufficiently large $m_{0}$.

If we set $h_{0} := \frac{1}{|\sigma_{0}|^{2/m_{0}}}$, then $h_{0}$ is a singular hermitian metric on $P$ with positive curvature.

We set $\alpha_{0}$ as follows:

$$\alpha_{0} := \inf \{ \alpha > 0 \mid \text{supp} \ O_{X}/I(h_{0}^{\alpha}) \ni x, y \}.$$

$\sigma_{0}$ has zeros of order at least $\lceil \sqrt{\frac{P^{2}}{2}} \cdot (1-\varepsilon)m \rceil$, so we get $\alpha_{0} \leq \sqrt{\frac{2}{P^{2}}} \cdot \frac{2}{1-\varepsilon_{0}}$.

Next we decrease $\alpha_{0}$ a little bit. Then one of the following two cases occurs.

**Case 1.** supp $O_{X}/I(h_{0}^{\alpha-\delta})$ does not include either $x$ nor $y$.

**Case 2.** supp $O_{X}/I(h_{0}^{\alpha-\delta})$ includes one of $x$ or $y$, say $x$.

In Case 1, we can consider a minimal center of log canonical singularities at $z$. Let $X_{1}$ be a minimal center at $x$. In this case one of following two cases occurs.

**Case 1-1.** supp $O_{X}/I(h_{0}^{\alpha-\delta})$ does not include either $x$ nor $y$.

**Case 1-2.** Otherwise.

We shall explain Case 1-1. (Other cases are easier to prove.)

Note that $(X_{1} \cdot P) > 0$ because $X_{1}$ passes through $x \in X^{\circ}$. 
Proposition 3.2 For arbitrary small $\varepsilon > 0$,
\[
\dim H^0(X_1, \mathcal{O}_{X_1}(mP) \otimes \mathcal{O}_{X}/\mathcal{M}_{x,y}^{\otimes \lceil\frac{(X_1 \cdot P)}{2} \cdot (1-\varepsilon)m\rceil}) \geq 1
\]
holds if we take $m$ sufficiently large. \(\square\)

The proof of Proposition 3.2 is the same as the proof of Proposition 3.1. We take $\tilde{\sigma}_1 \in H^0(X, \mathcal{O}_{X_1}(m_1P) \otimes \mathcal{O}_{X}/\mathcal{M}_{ox,y}^{\otimes \lceil 4^{x_2 \cdot P} \cdot (1-\varepsilon_1)m_1 \rceil})$ for sufficiently small $\varepsilon_1$ and sufficiently large $m_1$. Because $P$ is nef big, $P$ has a decomposition $P = A + \mathcal{E}$ by Kodaira's lemma. Where $A$ is a $\mathbb{Q}$-ample divisor and $\mathcal{E}$ is a $\mathbb{Q}$-effective divisor. We take integer $l_1$ sufficiently large so that $L_1 := l_1 \cdot A$ is $\mathbb{Z}$-very ample. Let $\tau \in H^0(X_1, \mathcal{O}_{X_1}(L_1))$ be a section which is not zero section, then
\[
\tilde{\sigma}_1 \otimes \tau \in H^0(X_1, \mathcal{O}_{X_1}(mP+L_1) \otimes \mathcal{O}_{X}/\mathcal{M}_{ax,y}^{\theta \lceil \cdot (1-\varepsilon)m \rceil})
\]
holds.

Proposition 3.3 For $m \geq 0$,
\[
H^0(X, \mathcal{O}_X(mP+L_1)) \rightarrow H^0(X_1, \mathcal{O}_{X_1}(mP+L_1))
\]
is surjective if we take $l_1$ sufficiently large. \(\square\)

Proof. Set $\varphi = \alpha_0 \log \frac{h_p}{\bar{h}_P}$. Where $h_P$ is arbitrary $C^\infty$-hermitian metric on $P$. We consider $\varphi \cdot h_{L_1} \cdot h_{K_X^{-1}}$. This is a singular hermitian metric on $L_1 - K_X$.

If we take $l_1$ sufficiently large, the curvature is strictly positive and $\mathcal{O}_X/I(\varphi) = \mathcal{O}_{X_1}$. Since $P$ is nef, we get $H^1(X, \mathcal{O}_X(mP+L_1) \otimes I(h_{mP+L_1-K_X}) = 0$.

This completes the proof. \(\blacksquare\)

By using this proposition, we extend $\tilde{\sigma}_1 \otimes \tau$ to
\[
\sigma_1 \in H^0(X, \mathcal{O}_X((m_1+l_1) \cdot A))
\]
Let $\{\rho_j\}$ be generator of $\mathcal{O}_X((m_1+l_1) \cdot A) \otimes \mathcal{I}_X$. We put
\[
h_1 := \frac{1}{(|\sigma_1|^2 + \sum |\rho_j|^2)^{1/(m_1+l_1)}}.
\]
We take $m_1$ sufficiently large so that $\frac{1}{m_1} \leq \delta_0 \frac{(X_1 \cdot P)}{2}$ holds.

Proposition 3.4 Let $\alpha_1 = \inf \{ \alpha > 0 \mid \text{supp} \mathcal{O}_X/I(h_0^{\alpha_0-\delta_0} \cdot h_1^{\alpha}) \ni x, y \}$. Assume $x$ and $y$ be regular points of $X_1$. Then
\[
\alpha_1 \leq \frac{2}{(X_1 \cdot P)} + O(\delta_0)
\]
Proof. We can choose a neighborhood $U$ of $x$ and a local coordinate system $(z_1, z_2)$ on $U$ such that

$$U \cap X_1 = \{ p \in U \mid z_1(p) = 0 \} = \{ (0, z_2) \}$$

holds. Then we get

$$|| \sigma_1 ||^2 + \sum || \rho_j ||^2 \leq C \cdot (|z_1|^2 + |z_2|^2 \cdot \left( \frac{X_1 \cdot P}{1 - \epsilon_1} \right) \cdot m_1)$$

here $|| \cdot ||$ is taken with respect to some $C^\infty$-hermitian metric on $(m_1 + l_1)P$, and $C$ is a constant depending on the norm $|| \cdot ||$. By the construction of $\sigma_0$,

$$|| \sigma_0 ||^{\overline{m}_0} \leq O(|z_1|^{2 - \delta_0})$$

also holds on some neighborhood of generic points of $U \cap X_1$. Hence we get

$$\alpha_1 \leq \frac{(m_1 + l_1)}{m_1} \cdot \frac{2}{(X_1 \cdot P)} + O(\delta_0)$$

From the assumption $\frac{1}{m_1} \leq \delta_0 \cdot \left( \frac{X_1 \cdot P}{2} \right)$, we conclude the statement of the proposition.  

Remark 3.1 Even if $x$ and $y$ are not regular points of $X_1$, we can show above result is true by taking $\hat{x}$ and $\hat{y}$ as regular points of $X_1$ and letting $\hat{x} \rightarrow x$ and $\hat{y} \rightarrow y$.

Lemma 3.1 $| m(K_X + D) |$ separates $x$ and $y$ for $m \geq \lceil \alpha_0 + \alpha_1 \rceil + 1$.  

Proof. By the equation

$$m(K_X + D) = K_X + (m - 1)P + (m - 1)E + D$$

and

$$(m - 1)P = \{ (\alpha_0 - \delta_0) + \alpha_1 \} P + \{ m - 1 - (\alpha_0 - \delta_0 + \alpha_1) \} (A + E),$$

we can equip a singular hermitian metric $h_{x,y}$ by

$$h_{x,y} := h_0^{\alpha_0 - \delta_0} \cdot h_1^{\alpha_1} \cdot h_A^{m - 1 - (\alpha_0 - \delta_0 + \alpha_1)} \cdot h_{\text{eff}},$$

where $h_A$ is a $C^\infty$-hermitian metric of $\mathbb{Q}$-ample divisor $A$ and $h_{\text{eff}}$ is a semipositive singular hermitian metric which comes from the other components. Then by the construction of $h_0$ and $h_1$, $h_{x,y}$ satisfies the following conditions:
1. \(x,y \in \text{supp}\mathcal{O}_X/I(h_{x,y})\)

2. One of the \(x\) or \(y\), say \(x\), is an isolated point of \(\text{supp}\mathcal{O}_X/I(h_{x,y})\).

3. \(i\Theta_h \geq \epsilon_0 \omega\) for some \(\epsilon_0 > 0\).

So there exists some \(\sigma \in H^0(X, m(K_X + D))\) such that \(\sigma(y) = 0\) and \(\sigma(x) \neq 0\), or \(\sigma(x) = 0\) and \(\sigma(y) \neq 0\). This completes the proof. \(\blacksquare\)

Corollary 3.1 \(m(K_X + D)\) separates \(x\) and \(y\) for

\[
m \geq \frac{2\sqrt{2}}{\sqrt{P^2}} + \frac{2}{(X_1 \cdot P)} + 1
\]

\(\Box\)

3.2 Construction of \(X_1\) as a family

Our construction of \(X_1\) is depending on the choice of the points \(x\) and \(y\). Therefore it seems that the value of \((X_1, P)\) is also depending on \(x\) and \(y\). But in fact, \((X_1, P)\) is independent of generic choice of \(x, y \in X\). We explain it in this subsection.

Let \(\Delta_X \subset X \times X\) be a diagonal set. We set \(B \subset X \times X\) and \(Z \subset B \times X\) as follows:

\[
B := X^\circ \times X^\circ - \Delta_X
\]

\[
Z := \{(z_1, z_2, z_3) \mid z_3 = x_1 \text{ or } x_2 = x_1\}
\]

Let \(p : X \times B \to X\) and \(q : X \times B \to B\) be the first and second projection respectively. We consider

\[
q_*(\mathcal{O}_{X \times B}(m_0 p^* P) \otimes \mathcal{I}_Z^{\otimes \lceil \sqrt{P^2} \cdot (1-\epsilon) m \rceil})
\]

instead of

\[
H^0(X, \mathcal{O}_X(mP) \otimes \mathcal{O}_X/M_{x,y}^{\otimes \lceil \sqrt{P^2} \cdot (1-\epsilon) m \rceil}),
\]

where \(\mathcal{I}_Z\) denotes the ideal sheaf of \(Z\). For a sufficiently large integer \(m_0\) and sufficiently small \(\epsilon\), we take \(\tilde{\sigma}_0\) as a nonzero global meromorphic section of

\[
q_*(\mathcal{O}_{X \times B}(m_0 p^* P) \otimes \mathcal{I}_Z^{\otimes \lceil \sqrt{P^2} \cdot (1-\epsilon) m \rceil}).
\]
\[ \tilde{h}_0 := \frac{1}{|\tilde{\sigma}_0|^{2/m_0}}, \]

then \( \tilde{h}_0 \) is a singular hermitian metric on \( P \) (but curvature current of \( \tilde{h}_0 \) may not be positive). We shall replace \( \alpha_0 \) by

\[ \tilde{\alpha}_0 = \inf\{ \alpha > 0 \mid \text{The generic points of } Z \subset \text{Spec}(\mathcal{O}_{X \times B}/\mathcal{I}(\tilde{h}_0^\alpha)) \} \]

Then for every small \( \delta > 0 \), there exists a Zariski open subset \( U \) of \( B \) such that \( \tilde{h}_0|_{X \times \{b\}} \) is well-defined for every \( b \in U \), and

\[ b \notin \text{Spec}(\mathcal{O}_{X \times \{b\}}/\mathcal{I}(\tilde{h}_0^{\tilde{\alpha}_0-\delta})) , \]

where we have identified \( b \) with distinct two points in \( X \). By the construction of \( \alpha_0 \), we can see

\[ b \subseteq \text{Spec}(\mathcal{O}_{X \times \{b\}}/\mathcal{I}(\tilde{h}_0^{\tilde{\alpha}_0})) \]

for every \( b \in B \). Let \( \tilde{X}_1 \) be a minimal center of logcanonical singularities of \((X \times B, \tilde{\sigma}_0(\tilde{\sigma}_0))\) at the generic point of \( Z \) (although \( \tilde{\sigma}_0 \) may not be effective, but this is still meaningful in this case because of our construction of \( \tilde{\sigma}_0 \)). Then \( \tilde{X}_1 \cap q^{-1}(b) \) is almost a minimal center at \( b := \{ \text{Distinct two points in } X^\circ \} \) which we construct in the last subsection. Remark that \( \tilde{X}_1 \cap q^{-1}(b) \) may not be irreducible even for a general \( b \in B \). But if we take a suitable finite cover

\[ \phi_0 : B_0 \rightarrow B \]

on the base change \( X \times_B B_0, \tilde{X}_1 \) defines a family of irreducible subvarieties

\[ f : \tilde{X}_1 \rightarrow U_0 \]

of \( X \) parametrized by a nonempty Zariski open subset \( U_0 \) of \( \phi_0^{-1}(U) \).

From above arguments, we see that \( \{X_1\}'s \) are numerically equivalent to each other when we move \( b = (x, y) \in X^\circ \times X^\circ - \Delta_X \) generically. The intersection number \((X_1, P)\) takes value in \( Q \), therefore \((X_1, P)\) is constant if we choose \( b = (x, y) \) generically. Hence we get:

**Proposition 3.5** \(|m(K_X + D)|\) defines a birational map from \( X \) to a projective space if

\[ m \geq \frac{2\sqrt{2}}{\sqrt{P^2}} + \frac{2}{(X_1 \cdot P)} + 1 . \]
3.3 An estimate of \((X_1 \cdot P)\)

To complete the proof of Theorem 1.1, we have to estimate \((X_1 \cdot P)\).

We consider the self-intersection number \((X_1)^2\). Then there are three possibilities:

Case 1. \((X_1)^2 > 0\)

Case 2. \((X_1)^2 = 0\)

Case 3. \((X_1)^2 < 0\)

Let \((x, y)\) and \((\hat{x}, \hat{y})\) be pair of distinct two points of \(X^o\). We put \(X_1\) and \(\hat{X}_1\) as a minimal center at \((x, y)\) and \((\hat{x}, \hat{y})\) respectively. If we take \((x, y)\) and \((\hat{x}, \hat{y})\) general, \(X_1\) and \(\hat{X}_1\) have no common irreducible components. Since \(X_1\) and \(\hat{X}_1\) are numerically equivalent, we get

\[
(X_1)^2 = (\hat{X}_1)^2 = (X_1, \hat{X}_1) \geq 0.
\]

So we have only to consider the case \((X_1)^2 \geq 0\).

i) In the case \((X_1)^2 > 0\).

By the Hodge index theorem, we get

\[
(X_1, P) \geq \sqrt{(X_1)^2} \cdot \sqrt{(P)^2}.
\]

Since \(X_1\) is an integral divisor, \((X_1)^2\) takes value in \(\mathbb{Z}\). As a consequence we have \((X_1)^2 \geq 1\) and

\[
\frac{1}{(X_1, P)} \leq \frac{1}{\sqrt{P^2}}.
\]

So in this case the proof of Theorem 1.1 is completed.

ii) In the case \((X_1)^2 = 0\).

Let \(N_{X_1}\) be a normal bundle of \(X_1\). Then we have \(N_{X_1} = -X_1|_{X_1}\) and \(\deg_{X_1} N_{X_1} = -(X_1)^2 = 0\). So we see that the normal bundle of \(X_1\) is trivial. Furthermore, \(X_1\) can move. As a consequence, we can conclude existence of a fibration of \(X\):

\[
\pi : X \rightarrow S,
\]
where $S$ denotes some algebraic curve. By the definition of $\alpha_0$, $\alpha_0 P - \pi^*(p_x) - \pi^*(p_y)$ is a pseudoeffective line bundle on $X$. Here $p_x$ and $p_y$ denote the point $\pi(x)$ and $\pi(y)$ respectively. Because $\deg_S K_S = 2g_S - 2 \geq -2$, we have

$$\alpha_0 P \geq \pi^*(2 \text{ points in } S) \geq -\pi^* K_S$$

and

$$H^0(X, \mathcal{O}_X(m(1 + \alpha_0)(K_X + D))) \supset H^0(X, \mathcal{O}_X(m(K_X + D - \pi^* K_S))) \ .$$

Recall that we regard $H^0(X, \mathcal{O}_X(m(K_X + D - \pi^* K_S)))$ as a subset of $H^0(X, \mathcal{O}_X(m(1 + \alpha_0)(K_X + D)))$ by using natural injective map derived from the sheaf exact sequence

$$0 \to \mathcal{O}_X(m(K_X + D - \pi^* K_S)) \to \mathcal{O}_X(m(1 + \alpha_0)(K_X + D)) \ ,$$

and hereafter we will often use such notation. By the definition of Zariski decomposition and above inclusion, we have the natural injection

$$\phi : H^0(X, \mathcal{O}_X(m(K_X + D - \pi^* K_S))) \to H^0(X, \mathcal{O}_X(m(1 + \alpha_0)P)) \ ,$$

if we let $m$ be an integer such that $m(1 + \alpha_0)P$ is a $\mathbb{Z}$-divisor.

The divisor $\pi_*(K_X + D - \pi^* K_S)$ is semipositive by Kawamata's semipositivity theorem[4, theorem 1], hence we get

$$H^0(S, \mathcal{O}_S(m\pi_*(K_X + D - \pi^* K_S))) \to m\pi_*(K_X + D - \pi^* K_S) \otimes \mathcal{O}_S/\mathfrak{m}_p$$

is surjective for sufficiently large $m$. From the above surjection, we get

$$H^0(X, \mathcal{O}_X(m(1 + \alpha_0)P)) \to H^0(\pi^{-1}(p), \mathcal{O}_{\pi^{-1}(p)}(m(K_X + D)|_{\pi^{-1}(p)}))$$

is also surjective. Since $\pi^* K_S|_{\pi^{-1}(p)}$ is trivial bundle, we have a surjection:

$$H^0(X, \mathcal{O}_X(m(K_X + D - \pi^* K_S))) \to H^0(\pi^{-1}(p), \mathcal{O}_{\pi^{-1}(p)}(m(K_X + D)|_{\pi^{-1}(p)})) \ .$$

Let us consider $H^0(X, \mathcal{O}_X(m(1 + \alpha_0)P))|_{\pi^{-1}(p)}$. By the natural injective map $\phi$, we can see

$$H^0(X, \mathcal{O}_X(m(1 + \alpha_0)P))|_{\pi^{-1}(p)} \supset H^0(\pi^{-1}(p), \mathcal{O}_{\pi^{-1}(p)}(m(K_X + D)|_{\pi^{-1}(p)}))$$

holds.

Let $\sigma_1$ and $\sigma_2$ be a global section of $H^0(X, \mathcal{O}_X(m(K_X + D - \pi^* K_S)))$ such that $\sigma_1|_{\pi^{-1}(p)}$ and $\sigma_2|_{\pi^{-1}(p)}$ are linearly independent. Then, if we take a general fiber $\pi^{-1}(p)$, $\phi(\sigma_1)|_{\pi^{-1}(p)}$ and $\phi(\sigma_2)|_{\pi^{-1}(p)}$ are also linearly independent.

Hence we get an inequality on dimensions of holomorphic sections:
\[ \dim H^0 \left( X, \mathcal{O}_X \left( m(1 + \alpha_0)P \right) \right) \bigg|_{\pi^{-1}(p)} \geq \dim H^0 \left( \pi^{-1}(p), \mathcal{O}_{\pi^{-1}(p)} \left( m\pi_* (K_X + D) \right|_{\pi^{-1}(p)} \right) . \]

We know the asymptotic relations:

\[ \dim H^0 \left( X, \mathcal{O}_X \left( m(1 + \alpha_0)P \right) \right) \bigg|_{\pi^{-1}(p)} \sim m(1 + \alpha_0)(P, \pi^{-1}(p)) \]

and

\[ \dim H^0 \left( \pi^{-1}(p), \mathcal{O}_{\pi^{-1}(p)} \left( m\pi_* (K_X + D) \right|_{\pi^{-1}(p)} \right) \sim m(K_X + D, \pi^{-1}(p)) , \]

when we keep \( m(1 + \alpha_0)P \) be integral divisor and letting \( m \) to be sufficiently large. Letting \( m \to \infty \), we see

\[ (1 + \alpha_0)(\pi^{-1}(p), P) \geq (\pi^{-1}(p), K_X + D) . \]

By definition, \( (\pi^{-1}(p), P) = (X_1, P) \) and \( (\pi^{-1}(p), K_X + D) = (X_1, K_X + D) \) holds. Hence we have

\[ (1 + \alpha_0)(X_1, P) \geq (X_1, K_X + D) . \]

If we take a general fiber, \( (K_X + D) \big|_{X_1} \) becomes a big divisor and

\[ \deg_{X_1} (K_X + D) = (X_1, K_X + D) \geq 1 \]

holds. Then we get an estimate for \( (X_1, P) : \)

\[ 1 + \alpha_0 \geq \frac{1}{(X_1, P)} . \]

Since \( \alpha_0 \leq \sqrt{2} \cdot \frac{2}{1 - \varepsilon_0} \), then we have

\[ \frac{2\sqrt{2}}{\sqrt{P^2}} + \frac{2}{(X_1 \cdot P)} + 1 \leq \frac{2\sqrt{2}}{\sqrt{P^2}} + \frac{\sqrt{2}}{\sqrt{P^2}} \cdot \frac{4}{1 - \varepsilon_0} + 3 , \]

and this completes the proof of Theorem 1.1.

References


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