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LOG-PLURICANONICAL SYSTEMS OF SMOOTH PROJECTIVE SURFACES

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1 Introduction

Let \( X \) be a smooth projective variety and \( K_X \) be a canonical divisor of \( X \). Then \( X \) is called of general type when pluricanonical system \( |mK_X| \) defines a birational embedding of \( X \) for some positive integer \( m \). The behavior of the pluricanonical systems is important to study varieties of general type. For example, there is such a problem:

**Problem 1.1** Let \( X \) be a smooth projective variety of general type. Find a positive integer \( m_0 \) such that for every \( m \geq m_0 \), \( |mK_X| \) gives a birational map from \( X \) into a projective space.

In the case \( \dim X = 1 \), it is well known that \( |3K_X| \) gives a projective embedding. In the case \( \dim X = 2 \), E. Bombieri proved that \( |5K_X| \) gives a birational embedding([1]). Recently, H. Tsuji showed that there exists an integer \( \nu_n \) which depends only on \( n = \dim X \) and satisfies above problem([6,7]). But when \( \dim X \geq 4 \), effective value of \( \nu_n \) is unknown. Even if \( \dim X = 3 \), the value of \( \nu_n \) becomes an astronomical number, and it is supposed that the value computed in [7] is not best-possible.

I am interested in this problem for open surfaces. As long as I know, such a situation is not studied yet.

**Definition 1.1** Let \( X \) be a surface and \( D \) be a divisor with normal closings. Then the pair \((X, D)\) is called log-surface. If the linear system \( |K_X + D| \) is big, we say that \((X, D)\) is of log-general type.

Now we state the problem more precisely.
Problem 1.2 Let \((X, D)\) be a smooth projective surface of log-general type. Find a positive integer \(m_0\) such that for every \(m \geq m_0\), \(|m(K_X + D)|\) gives a birational map from \(X\) into a projective space.

The main purpose of this paper is to answer the weaker version of this problem. Our result shows the value of \(m_0\) for a given surface. But it depends on \(X\) and divisor \(D\).

Theorem 1.1 Let \((X, D)\) be a smooth projective surface of log-general type, and let \(K_X + D = P + E\) be a Zariski-decomposition of \(K_X + D\), where \(P\) is a nef part and \(E\) is effective part of the decomposition. Then \(|m(K_X + D)|\) defines a birational map from \(X\) into projective space unless

\[
m \geq \frac{6\sqrt{2}}{\sqrt{P^2}} + 4.
\]

Remark 1.1 In the case \((X, D)\) is log-general type, \(P^2 > 0\) holds ([3]).

2 Terminology

In this section we introduce a singular hermitian metric and some results we use after. See [2] for more details.

Definition 2.1 Let \(L\) be a holomorphic line bundle on \(X\), \(h_0\) be a \(C^\infty\)-hermitian metric on \(L\), and \(\varphi\) be a \(L^1_{loc}\)-function on \(X\). Then we call \(h = e^{-\varphi} \cdot h_0\) a singular hermitian metric with respect to \(\varphi\). \(\varphi\) is called a weight function of \(h\).

Definition 2.2 We define a curvature current \(i\Theta_h\) of a singular hermitian line bundle \((L, h)\) as follows:

\[
i\Theta_h := i\partial\overline{\partial}\varphi + i\Theta_{h_0}
\]

where \(\partial\overline{\partial}\) is taken as a distribution and \(i\Theta_{h_0}\) is the curvature form of \((L, h_0)\) in usual sense.

A singular hermitian line bundle is said to be positive, if the curvature current \(i\Theta_h\) becomes a measure which takes values in semipositive-defined hermitian matrix.
Next we introduce a concept of multiplier ideal sheaves. Let $U \subset X$ be an open set and $\mathcal{O}(U)$ be the set of holomorphic functions on $U$. Then

$$I_{U}(h) := \left\{ f \in \mathcal{O}(U) \mid \int_{U} e^{-\varphi} |f|^{2} dV < +\infty \right\}$$

becomes a presheaf when $U$ runs all open subsets of $X$. We put $I(h)$ as the sheafication of $I_{U}(h)$. $I(h)$ is called the multiplier ideal sheaf with respect to $h$. The following theorem which is a variant of Kodaira’s vanishing theorem is due to A.Nade1([5]).

**Theorem 2.1** Let $(X, \omega)$ be a Kähler manifold and $(L, h)$ be a singular hermitian line bundle on $X$. Assume that $i\Theta_{h} \geq \epsilon_{0} \omega$ for some $\epsilon_{0} > 0$. Then

$$H^{q}(X, \mathcal{O}_{X}(K_{X} + L) \otimes I(h)) = 0 \quad (q \geq 1).$$

\[\square\]

### 3 Proof of Theorem 1.1

In this section, we show the outline of the proof of Theorem 1.1. The proof is made along [6, section2], so please refer to [6] for detail.

Let $(X, D)$ be a smooth projective surface of log-general type and $x, y \in X, x \neq y$ be generic two points. Assume that there exists a singular hermitian metric $h_{x,y}$ on $m(K_{X} + D) + D$ such that:

1. $x, y \in \text{supp}\mathcal{O}_{X}/I(h_{x,y})$
2. One of the $x$ or $y$, say $x$, is an isolated point of $\text{supp}\mathcal{O}_{X}/I(h_{x,y})$
3. $i\Theta_{h} \geq \epsilon_{0} \omega$ for some $\epsilon_{0} > 0$.

We consider the long exact sequence:

$$\cdots \rightarrow H^{0}(X, \mathcal{O}_{X}((m+1)(K_{X} + D))) \rightarrow H^{0}(X, \mathcal{O}_{X}((m+1)(K_{X} + D)) \otimes \mathcal{O}/I(h_{x,y})) \rightarrow$$

$$\rightarrow H^{1}(X, \mathcal{O}_{X}(K_{X} + m(K_{X} + D) + D) \otimes I(h_{x,y})) \rightarrow \cdots.$$

Where $H^{1}(X, \mathcal{O}_{X}(K_{X} + m(K_{X} + D) + D) \otimes I(h_{x,y})) = 0$ by Nadel’s vanishing theorem, hence we get surjection and we can conclude that there exists some $\sigma \in H^{0}(X, \mathcal{O}_{X}((m+1)(K_{X} + D)))$ such that $\sigma(y) = 0$ and $\sigma(x) \neq 0$. This shows that $\Phi_{[(m+1)(K_{X}+D)]}$ separates $x$ and $y$. Therefore to prove theorem 1.1, we have only to compute the value $m$ such that we can construct a singular hermitian metric $h_{x,y}$ on $(m+1)(K_{X} + D)$ which satisfies the condition 1, 2, and 3 above for arbitrary distinct two points $x, y \in U$, for some nonempty Zariski open subset $U \subset X$. 
3.1 Construction of \( h_{x,y} \)

Let \( K_X + D = P + E \) be a Zariski-decomposition of \( K_X + D \), where \( P \) is the nef part and \( E \) is the effective part of the decomposition. We put \( X^\circ \) as follows:

\[
X^\circ := \{ \ p \in X \mid p \notin Bs|mP| \text{ and for some } m, \ |mP| \text{ gives biholomorpic near } p \}.
\]

Then \( X^\circ \) is a nonempty Zariski open set of \( X \).

We take arbitrary \( x,y \in X^\circ \) and we set \( M_{x,y} := M_x \otimes M_y \), where \( M_x \) and \( M_y \) are the maximum ideal sheaf of the points \( x, y \) respectively.

By considering a cohomology exact sequence and comparing the dimension of \( H^0(X, O_X(mP)) \) and \( H^0(X, O_X(mP) \otimes O_X/M_{x,y}^{\otimes \lceil \sqrt{\frac{P^2}{2}} \cdot (1-\varepsilon)m \rceil}) \), we can show the following:

**Proposition 3.1** For arbitrary small \( \varepsilon > 0 \),

\[
\dim H^0(X, O_X(mP) \otimes O_X/M_{x,y}^{\otimes \lceil \sqrt{\frac{P^2}{2}} \cdot (1-\varepsilon)m \rceil}) \geq 1
\]

holds if we take \( m \) sufficiently large. \( \Box \)

We take \( \sigma_0 \in H^0(X, O_X(m_0P) \otimes O_X/M_{x,y}^{\otimes \lceil \sqrt{\frac{P^2}{2}} \cdot (1-\varepsilon_0)m_0 \rceil}) \) for sufficiently small \( \varepsilon_0 \) and sufficiently large \( m_0 \).

If we set \( h_0 := \frac{1}{|\sigma_0|^{2/m_0}} \), then \( h_0 \) is a singular hermitian metric on \( P \) with positive curvature.

We set \( \alpha_0 \) as follows:

\[
\alpha_0 := \inf \{ \alpha > 0 \mid \text{supp } O_X/I(h_0^\alpha) \ni x, y \}.
\]

\( \sigma_0 \) has zeros of order at least \( \lceil \sqrt{\frac{P^2}{2}} \cdot (1-\varepsilon)m \rceil \), so we get \( \alpha_0 \leq \sqrt{\frac{2}{P^2}} \cdot \frac{2}{1-\varepsilon_0} \).

Next we decrease \( \alpha_0 \) a little bit. Then one of the following two cases occurs.

**Case 1.** \( \text{supp } O_X/I(h_0^{\alpha_0-\delta_0}) \) does not include either \( x \) nor \( y \).

**Case 2.** \( \text{supp } O_X/I(h_0^{\alpha_0-\delta_0}) \) includes one of \( x \) or \( y \), say \( x \).

In Case 1, we can consider a minimal center of log canonical singularities at \( x \). Let \( X_1 \) be a minimal center at \( x \). In this case one of following two cases occurs.

**Case 1-1.** \( \text{supp } O_X/I(h_0^{\alpha_0-\delta_0}) \) does not include either \( x \) nor \( y \).

**Case 1-2.** Otherwise.

We shall explain Case 1-1. (Other cases are easier to prove.)

Note that \( (X_1 \cdot P) > 0 \) because \( X_1 \) passes through \( x \in X^\circ \).
Proposition 3.2 For arbitrary small $\varepsilon > 0$,
\[ \dim H^0 \left( X_1, \mathcal{O}_{X_1} (mP) \otimes \mathcal{O}_X / \mathcal{M}_{x,y}^\otimes \left( 1 - \varepsilon \right) \cdot m \right) \geq 1 \]
holds if we take $m$ sufficiently large. \(\square\)

The proof of Proposition 3.2 is the same as the proof of Proposition 3.1.

We take $\tilde{\sigma}_1 \in H^0 \left( X, \mathcal{O}_{X_1} (m_1P) \otimes \mathcal{O}_X / \mathcal{M}_{x,y}^\otimes \left( 1 - \varepsilon_1 \right) \cdot m_1 \right)$ for sufficiently small $\varepsilon_1$ and sufficiently large $m_1$.

Because $P$ is nef big, $P$ has a decomposition $P = A + \mathcal{E}$ by Kodaira's lemma. Where $A$ is a $\mathbb{Q}$-ample divisor and $\mathcal{E}$ is a $\mathbb{Q}$-effective divisor. We take integer $l_1$ sufficiently large so that $L_1 := l_1 \cdot A$ is $\mathbb{Z}$-very ample. Let $\tau \in H^0 \left( X_1, \mathcal{O}_{X_1} (L_1) \right)$ be a section which is not zero section. Then
\[ \tilde{\sigma}_1 \otimes \tau \in H^0 \left( X_1, \mathcal{O}_{X_1} (mP + L_1) \otimes \mathcal{O}_X / \mathcal{M}_{x,y}^\otimes \left( 1 - \varepsilon \right) \cdot m \right) \]
holds.

Proposition 3.3 For $m \geq 0$,
\[ H^0 \left( X, \mathcal{O}_X (mP + L_1) \right) \to H^0 \left( X_1, \mathcal{O}_{X_1} (mP + L_1) \right) \]
is surjective if we take $l_1$ sufficiently large. \(\square\)

Proof. Set $\varphi = \alpha_0 \log \frac{h_0}{h_P}$. Where $h_P$ is arbitrary $C^\infty$-hermitian metric on $P$. We consider $\varphi \cdot h_{L_1} \cdot h_{K_X^{-1}}$. This is a singular hermitian metric on $L_1 - K_X$.

If we take $l_1$ sufficiently large, the curvature is strictly positive and $\mathcal{O}_X / I(\varphi) = \mathcal{O}_{X_1}$. Since $P$ is nef, we get $H^1 \left( X, \mathcal{O}_X (mP + L_1) \otimes I(h_{mP + L_1 - K_X}) \right) = 0$. This completes the proof. \(\square\)

By using this proposition, we extend $\tilde{\sigma}_1 \otimes \tau$ to
\[ \sigma_1 \in H^0 \left( X, \mathcal{O}_X \left( (m_1 + l_1) \cdot P \right) \right) . \]

Let $\{ \rho_j \}$ be generator of $\mathcal{O}_X \left( (m_1 + l_1) \cdot A \right) \otimes \mathcal{I}_X$. We put
\[ h_1 := \frac{1}{(|\sigma_1|^2 + \sum |\rho_j|^2)^{1/(m_1 + l_1)}} . \]

We take $m_1$ sufficiently large so that $\frac{h_1}{m_1} \leq \delta_0 \left( \frac{X_1 \cdot P}{2} \right)$ holds.

Proposition 3.4 Let $\alpha_1 = \inf \{ \alpha > 0 \mid \text{supp} \mathcal{O}_X / I(h_0^{\alpha_0 - \delta_0} \cdot h_1^\alpha) \ni x, y \}$. Assume $x$ and $y$ be regular points of $X_1$. Then
\[ \alpha_1 \leq \frac{2}{(X_1 \cdot P)} + O(\delta_0) \]
Proof. We can choose a neighborhood $U$ of $x$ and a local coordinate system $(z_1, z_2)$ on $U$ such that

$$U \cap X_1 = \{ p \in U \mid z_1(p) = 0 \} = \{ (0, z_2) \}$$

holds. Then we get

$$\| \sigma_1 \|^2 + \sum \| \rho_j \|^2 \leq C \cdot \left( |z_1|^2 + |z_2|^2 \cdot \left[ \frac{(X_1 \cdot P)}{1 - \alpha_1} \cdot m_1 \right] \right),$$

here $\| \cdot \|$ is taken with respect to some $C^\infty$-hermitian metric on $(m_1 + l_1)P$, and $C$ is a constant depending on the norm $\| \cdot \|$. By the construction of $\sigma_0$, $\| \sigma_0 \|^\alpha_{0-\delta_0} \leq O(|z_1|^{2-\delta_0})$ also holds on some neighborhood of generic points of $U \cap X_1$. Hence we get

$$\alpha_1 \leq \frac{(m_1 + l_1)}{m_1} \cdot \frac{2}{(X_1 \cdot P)} + O(\delta_0).$$

From the assumption $\frac{l_1}{m_1} \leq \delta_0 \frac{(X_1 \cdot P)}{2}$, we conclude the statement of the proposition.

Remark 3.1 Even if $x$ and $y$ are not regular points of $X_1$, we can show above result is true by taking $\hat{x}$ and $\hat{y}$ as regular points of $X_1$ and letting $\hat{x} \to x$ and $\hat{y} \to y$.

Lemma 3.1 $| m(K_X + D) |$ separates $x$ and $y$ for $m \geq \lceil \alpha_0 + \alpha_1 \rceil + 1.$ □

Proof. By the equation

$$m(K_X + D) = K_X + (m - 1)P + (m - 1)E + D$$

and

$$(m - 1)P = \{ (\alpha_0 - \delta_0) + \alpha_1 \} P + \{ m - 1 - (\alpha_0 - \delta_0 + \alpha_1) \} (A + \mathcal{E})$$,

we can equip a singular hermitian metric $h_{x,y}$ by

$$h_{x,y} := h_0^{\alpha_0 - \delta_0} \cdot h_1^{\alpha_1} \cdot h_A^{m-1-(\alpha_0 - \delta_0 + \alpha_1)} \cdot h_{\text{eff}}$$

where $h_A$ is a $C^\infty$-hermitian metric of $\mathbb{Q}$-ample divisor $A$ and $h_{\text{eff}}$ is a semipositive singular hermitian metric which comes from the other components. Then by the construction of $h_0$ and $h_1$, $h_{x,y}$ satisfies the following conditions:
1. \( x, y \in \text{supp} \mathcal{O}_X / I(h_{x,y}) \)

2. One of the \( x \) or \( y \), say \( x \), is an isolated point of \( \text{supp} \mathcal{O}_X / I(h_{x,y}) \).

3. \( i \Theta_h \geq \varepsilon_0 \omega \) for some \( \varepsilon_0 > 0 \).

So there exists some \( \sigma \in H^0(X, m(K_X + D)) \) such that \( \sigma(y) = 0 \) and \( \sigma(x) \neq 0 \), or \( \sigma(x) = 0 \) and \( \sigma(y) \neq 0 \). This completes the proof. \( \blacksquare \)

Corollary 3.1 \( |m(K_X + D)| \) separates \( x \) and \( y \) for

\[
m \geq \frac{2\sqrt{2}}{\sqrt{P^2}} + \frac{2}{(X_1 \cdot P)} + 1.
\]

\( \square \)

3.2 Construction of \( X_1 \) as a family

Our construction of \( X_1 \) is depending on the choice of the points \( x \) and \( y \). Therefore it seems that the value of \( (X_1, P) \) is also depending on \( x \) and \( y \). But in fact, \( (X_1, P) \) is independent of generic choice of \( x, y \in X \). We explain it in this subsection.

Let \( \Delta_X \subset X \times X \) be a diagonal set. We set \( B \subset X \times X \) and \( Z \subset B \times X \) as follows:

\[
B := X^o \times X^o - \Delta_X
\]

\[
Z := \{ (z_1, z_2, z_3) \mid z_3 = z_1 \text{ or } z_2 = z_1 \}.
\]

Let \( p : X \times B \rightarrow X \) and \( q : X \times B \rightarrow B \) be the first and second projection respectively. We consider

\[
q_*(\mathcal{O}_{X \times B}(m_0 p^* P) \otimes \mathcal{I}_Z^{\otimes \lceil \sqrt{2^2 \cdot (1 - \varepsilon) m} \rceil})
\]

instead of

\[
H^0(X, \mathcal{O}_X(mP) \otimes \mathcal{O}_X/\mathcal{M}_{x,y}^{\otimes \lceil \sqrt{2^2 \cdot (1 - \varepsilon) m} \rceil}),
\]

where \( \mathcal{I}_Z \) denotes the ideal sheaf of \( Z \). For a sufficiently large integer \( m_0 \) and sufficiently small \( \varepsilon \), we take \( \tilde{\sigma}_0 \) as a nonzero global meromorphic section of

\[
q_*(\mathcal{O}_{X \times B}(m_0 p^* P) \otimes \mathcal{I}_Z^{\otimes \lceil \sqrt{2^2 \cdot (1 - \varepsilon) m} \rceil}).
\]
\[ \tilde{h}_0 := \frac{1}{|\tilde{\sigma}_0|^{2/m_0}} , \]

then \( \tilde{h}_0 \) is a singular hermitian metric on \( P \) (but curvature current of \( \tilde{h}_0 \) may not be positive). We shall replace \( \alpha_0 \) by
\[ \tilde{\alpha}_0 = \inf \{ \alpha > 0 \mid \text{The generic points of } Z \subset \text{Spec}(\mathcal{O}_{X \times B}/\mathcal{I}(\tilde{h}_0^{\tilde{\alpha}_0})) \} . \]

Then for every small \( \delta > 0 \), there exists a Zariski open subset \( U \) of \( B \) such that \( \tilde{h}_0 \mid_{X \times \{b\}} \) is well-defined for every \( b \in U \), and
\[ b \not\in \text{Spec}(\mathcal{O}_{X \times \{b\}}/\mathcal{I}(\tilde{h}_0^{\tilde{\alpha}_0-\delta})) , \]
where we have identified \( b \) with distinct two points in \( X \). By the construction of \( \alpha_0 \), we can see
\[ b \subseteq \text{Spec}(\mathcal{O}_{X \times \{b\}}/\mathcal{I}(\tilde{h}_0^{\tilde{\alpha}_0})) \]
for every \( b \in B \). Let \( \hat{X}_1 \) be a minimal center of logcanonical singularities of \( (X \times B, \frac{\tilde{\sigma}_0}{m_0}(\sigma_0)) \) at the generic point of \( Z \) (although \( (\tilde{\sigma}_0) \) may not be effective, but this is still meaningful in this case because of our construction of \( \tilde{\sigma}_0 \) ). Then \( \hat{X}_1 \cap q^{-1}(b) \) is almost a minimal center at \( b := \{ \text{Distinct two points in } X^\circ \} \) which we construct in the last subsection. Remark that \( \hat{X}_1 \cap q^{-1}(b) \) may not be irreducible even for a general \( b \in B \). But if we take a suitable finite cover
\[ \phi_0 : B_0 \rightarrow B , \]
on the base change \( X \times_B B_0, \hat{X}_1 \) defines a family of irreducible subvarieties
\[ f : \hat{X}_1 \rightarrow U_0 \]
of \( X \) parametrized by a nonempty Zariski open subset \( U_0 \) of \( \phi_0^{-1}(U) \).

From above arguments, we see that \( \{X_1\} 's \) are numerically equivalent to each other when we move \( b = (x,y) \in X^\circ \times X^\circ - \Delta_X \) generically. The intersection number \( (X_1, P) \) takes value in \( Q \), therefore \( (X_1, P) \) is constant if we choose \( b = (x,y) \) generically. Hence we get:

**Proposition 3.5** \(|m(K_X + D)| \) defines a birational map from \( X \) to a projective space if
\[ m \geq \frac{2\sqrt{2}}{\sqrt{P^2}} + \frac{2}{(X_1 \cdot P)} + 1 . \]
3.3 An estimate of \((X_1 \cdot P)\)

To complete the proof of Theorem 1.1, we have to estimate \((X_1 \cdot P)\).

We consider the self-intersection number \((X_1)^2\). Then there are three possibilities:

Case 1. \((X_1)^2 > 0\)
Case 2. \((X_1)^2 = 0\)
Case 3. \((X_1)^2 < 0\)

Let \((x, y)\) and \((\grave{x}, \grave{y})\) be pair of distinct two points of \(X^0\). We put \(X_1\) and \(\hat{X}_1\) as a minimal center at \((x, y)\) and \((\grave{x}, \grave{y})\) respectively. If we take \((x, y)\) and \((\grave{x}, \grave{y})\) general, \(X_1\) and \(\hat{X}_1\) have no common irreducible components. Since \(X_1\) and \(\hat{X}_1\) are numerically equivalent, we get

\[
(X_1)^2 = (\hat{X}_1)^2 = (X_1, \hat{X}_1) \geq 0.
\]

So we have only to consider the case \((X_1)^2 \geq 0\).

i) In the case \((X_1)^2 > 0\).

By the Hodge index theorem, we get

\[
(X_1, P) \geq \sqrt{(X_1)^2} \cdot \sqrt{(P)^2}.
\]

Since \(X_1\) is an integral divisor, \((X_1)^2\) takes value in \(\mathbb{Z}\). As a consequence we have \((X_1)^2 \geq 1\) and

\[
\frac{1}{(X_1, P)} \leq \frac{1}{\sqrt{P^2}}.
\]

So in this case the proof of Theorem 1.1 is completed.

ii) In the case \((X_1)^2 = 0\).

Let \(N_{X_1}\) be a normal bundle of \(X_1\). Then we have \(N_{X_1} = -X_1|_{X_1}\) and \(\deg_{X_1} N_{X_1} = -(X_1)^2 = 0\). So we see that the normal bundle of \(X_1\) is trivial. Furthermore, \(X_1\) can move. As a consequence, we can conclude existence of a fibration of \(X\):

\[
\pi : X \rightarrow S,
\]
where $S$ denotes some algebraic curve. By the definition of $\alpha_0$, $\alpha_0 P - \pi^*(p_x) - \pi^*(p_y)$ is a pseudoeffective line bundle on $X$. Here $p_x$ and $p_y$ denote the point $\pi(x)$ and $\pi(y)$ respectively. Because $\deg_S K_S = 2g_S - 2 \geq -2$, we have

$$\alpha_0 P \geq \pi^*(2 \text{ points in } S) \geq -\pi^* K_S$$

and

$$H^0(X, \mathcal{O}_X (m(1 + \alpha_0)(K_X + D))) \supset H^0(X, \mathcal{O}_X (m(K_X + D - \pi^* K_S)))$$

Recall that we regard $H^0(X, \mathcal{O}_X (m(K_X + D - \pi^* K_S)))$ as a subset of $H^0(X, \mathcal{O}_X (m(1 + \alpha_0)(K_X + D)))$ by using natural injective map derived from the sheaf exact sequence

$$0 \rightarrow \mathcal{O}_X (m(K_X + D - \pi^* K_S)) \rightarrow \mathcal{O}_X (m(1 + \alpha_0)(K_X + D))$$

and hereafter we will often use such notation. By the definition of Zariski decomposition and above inclusion, we have the natural injection

$$\phi : H^0(X, \mathcal{O}_X (m(K_X + D - \pi^* K_S))) \hookrightarrow H^0(X, \mathcal{O}_X (m(1 + \alpha_0)P))$$

if we let $m$ be an integer such that $m(1 + \alpha_0)P$ is a $\mathbb{Z}$-divisor.

The divisor $\pi_* (K_X + D - \pi^* K_S)$ is semipositive by Kawamata's semipositivity theorem[4, theorem 1], hence we get

$$H^0(S, \mathcal{O}_S (m\pi_*(K_X + D - \pi^* K_S))) \hookrightarrow m\pi_*(K_X + D - \pi^* K_S) \otimes \mathcal{O}_S/\mathfrak{m}_p$$

is surjective for sufficiently large $m$. From the above surjection, we get

$$H^0(X, \mathcal{O}_X (m(K_X + D - \pi^* K_S)))$$

$$\rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow$$

$$H^0(\pi^{-1}(p), \mathcal{O}_{\pi^{-1}(p)}(m(K_X + D - \pi^* K_S)|_{\pi^{-1}(p)}))$$

is also surjective. Since $\pi^* K_S|_{\pi^{-1}(p)}$ is trivial bundle, we have a surjection :

$$H^0(X, \mathcal{O}_X (m(K_X + D - \pi^* K_S)))$$

$$\rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow \quad \rightarrow$$

$$H^0(\pi^{-1}(p), \mathcal{O}_{\pi^{-1}(p)}(m(K_X + D)|_{\pi^{-1}(p)}))$$

Let us consider $H^0(X, \mathcal{O}_X (m(1 + \alpha_0)P))|_{\pi^{-1}(p)}$. By the natural injective map $\phi$, we can see

$$H^0(X, \mathcal{O}_X (m(1 + \alpha_0)P))|_{\pi^{-1}(p)} \supset H^0(\pi^{-1}(p), \mathcal{O}_{\pi^{-1}(p)}(m(K_X + D)|_{\pi^{-1}(p)}))$$

holds.

Let $\sigma_1$ and $\sigma_2$ be a global section of $H^0(X, \mathcal{O}_X (m(K_X + D - \pi^* K_S)))$ such that $\sigma_1|_{\pi^{-1}(p)}$ and $\sigma_2|_{\pi^{-1}(p)}$ are linearly independent. Then, if we take a general fiber $\pi^{-1}(p)$, $\phi(\sigma_1)|_{\pi^{-1}(p)}$ and $\phi(\sigma_2)|_{\pi^{-1}(p)}$ are also linearly independent.

Hence we get an inequality on dimensions of holomorphic sections:
dim \( H^0(X, \mathcal{O}_X(m(1 + \alpha_0)P)) \big|_{\pi^{-1}(p)} \geq \dim H^0(\pi^{-1}(p), \mathcal{O}_{\pi^{-1}(p)}(m\pi_*(K_X + D)|_{\pi^{-1}(p)})) \).

We know the asymptotic relations:

\[
\dim H^0(X, \mathcal{O}_X(m(1 + \alpha_0)P)) \big|_{\pi^{-1}(p)} \sim m(1 + \alpha_0)(P, \pi^{-1}(p))
\]

and

\[
\dim H^0(\pi^{-1}(p), \mathcal{O}_{\pi^{-1}(p)}(m\pi_*(K_X + D)|_{\pi^{-1}(p)})) \sim m(K_X + D, \pi^{-1}(p)),
\]

when we keep \( m(1 + \alpha_0)P \) be integral divisor and letting \( m \) to be sufficiently large. Letting \( m \rightarrow \infty \), we see

\[
(1 + \alpha_0)(\pi^{-1}(p), P) \geq (\pi^{-1}(p), K_X + D).
\]

By definition, \((\pi^{-1}(p), P) = (X_1, P)\) and \((\pi^{-1}(p), K_X + D) = (X_1, K_X + D)\) holds. Hence we have

\[
(1 + \alpha_0)(X_1, P) \geq (X_1, K_X + D).
\]

If we take a general fiber, \((K_X + D)|_{X_1}\) becomes a big divisor and

\[
\deg_{X_1}(K_X + D) = (X_1, K_X + D) \geq 1
\]

holds. Then we get an estimate for \((X_1, P)\):

\[
1 + \alpha_0 \geq \frac{1}{(X_1, P)}.
\]

Since \( \alpha_0 \leq \sqrt{\frac{2}{P^2}} \cdot \frac{2}{1 - \epsilon_0} \), then we have

\[
\frac{2\sqrt{2}}{\sqrt{P^2}} + \frac{2}{(X_1, P)} + 1 \leq \frac{2\sqrt{2}}{\sqrt{P^2}} + \frac{\sqrt{2}}{\sqrt{P^2}} \cdot \frac{4}{1 - \epsilon_0} + 3,
\]

and this completes the proof of Theorem 1.1.

References


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