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Perturbations of selfadjoint operators with periodic classical flow

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Abstract

We consider non-selfadjoint perturbations of a self-adjoint h-pseudodifferential operator in dimension 2. In the present work we treat the case when the classical flow of the unperturbed part is periodic and the strength ϵ of the perturbation satisfies $h^{\delta_0} < \epsilon \leq \epsilon_0$ for some $\delta_0 \in]0,1/2[$ and a sufficiently small $\epsilon_0 > 0$. We get a complete asymptotic description of all eigenvalues in certain rectangles $[-1/C, 1/C] + i\epsilon[F_0 - 1/C, F_0 + 1/C]$. In particular we are able to treat the case when $\epsilon > 0$ is small but independent of h. [1][2]

1 Introduction

This paper is a continuation of [14], where A. Melin and the author observed that for a wide and stable class of non-selfadjoint operators in dimension 2 and in the semi-classical limit ($h \to 0$), it is possible to describe all eigenvalues individually in an h-independent domain in C, by means of a Bohr-Sommerfeld quantization condition. Notice that the corresponding conclusion in the selfadjoint case seems to be possible only in dimension 1, or in higher dimensions under strong (and unstable) assumptions of complete integrability. In [14] we exploited the absence of small denominators to get a geometric analogue of the KAM-theorem via methods of non-linear Cauchy–Riemann equations and got a corresponding result at the level of operators

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In the present work we make another step by studying small perturbations, roughly of the form $P+i\epsilon Q$, of a selfadjoint h-pseudodifferential operator P whose associated classical flow is periodic. We will here be particularly interested in the case of a small but fixed ϵ, but our methods allow us to let ϵ vary in an interval $[h^{\delta_0},\epsilon_0]$ where $\epsilon_0>0$ is sufficiently small and $\delta_0\in]0,1/2[$ is arbitrary.

From the point of view of applications, it is clear that even smaller perturbations are of a considerable interest and as another step, Hitrik and the author [9] studied the same problem as in the present paper, but in the parameter range $h\ll\epsilon\le h^\delta$ for every fixed $\delta>0$. When the subprincipal symbol vanishes we could even treat the range $h^2\ll\epsilon\le h^\delta$. Actually with M. Hitrik, we plan a whole series of works devoted to small perturbations of non-selfadjoint operators in two dimensions. Among other things we plan to treat the case when the classical flow of the unperturbed operator admits certain invariant Lagrangian torii with a diophantine condition. (Another work ([16]) deals with resonances generated by a closed hyperbolic trajectory and can be viewed as descendant of the pioneering work of M. Ikawa [10] about scattering poles for two strictly convex obstacles.)

The methods in [9] are partly more traditional and rely on reduction by averaging to a one-dimensional problem in the spirit of [21, 5, 4, 6, 11]. Such a reduction does not seem possible here and the problem remains 2-dimensional. In general, we have been motivated by recent progress around the damped wave-equation ([13], [2], [17], [8]), as well as the problem of barrier top resonances for the semi-classical Schrödinger operator ([12]) where more complete results than the corresponding ones for eigenvalues of potential wells ([18], [3], [15]) seem possible. Eventually we also hope to apply our results (though not specifically the ones of the present work) to the distribution of resonances for a strictly convex obstacle in $\mathbf{R}^3$. See [20] and references given there. In the case of analytic obstacles, much more can probably be said, especially in dimension 3 (and 2).

The present work was undertaken before the start of [9], but the latter work is now completed, so we can take advantage of many of the arguments there, even though the main step here will be quite different.

Let M denote $\mathbf{R}^2$ or a compact real-analytic manifold of dimension 2.

When $M=\mathbf{R}^2$, let

$$P_\epsilon=P(x,hD_x,\epsilon;h) \tag{1.1}$$

be the h-Weyl quantization on $\mathbf{R}^2$ of a symbol $P(x,\xi,\epsilon;h)$ depending smoothly on $\epsilon\in\operatorname{neigh}(0,\mathbf{R})$ with values in the space of holomorphic functions of (x,ξ) in a tubular neighborhood of $\mathbf{R}^4$ in $\mathbf{C}^4$, with

$$|P(x,\xi,\epsilon;h)|\le Cm(\operatorname{Re}(x,\xi)) \tag{1.2}$$

there. Here m is assumed to be an order function on $\mathbf{R}^4$, in the sense that $m > 0$ and

$$m(X) \leq C_0 \langle X - Y \rangle^{N_0} m(Y), \; X, Y \in \mathbf{R}^4. \tag{1.3}$$

We also assume that

$$m \geq 1. \tag{1.4}$$

We further assume that

$$P(x, \xi, \epsilon; h) \sim \sum_{j=0}^{\infty} p_{j,\epsilon}(x, \xi) h^j, \; h \to 0, \tag{1.5}$$

in the space of such functions. We make the ellipticity assumption

$$|p_{0,\epsilon}(x, \xi)| \geq \frac{1}{C} m(\mathrm{Re}\,(x, \xi)), \; |(x, \xi)| \geq C, \tag{1.6}$$

for some $C > 0$.

When M is a compact manifold, we let

$$P_\epsilon = \sum_{|\alpha| \leq m} a_{\alpha,\epsilon}(x; h)(hD_x)^\alpha, \tag{1.7}$$

be a differential operator on M, such that for every choice of analytic local coordinates, centered at some point of M, $a_{\alpha,\epsilon}(x; h)$ is a smooth function of ϵ with values in the space of bounded holomorphic functions in a complex neighborhood of $x = 0$. We further assume that

$$a_{\alpha,\epsilon}(x; h) \sim \sum_{j=0}^{\infty} a_{\alpha,\epsilon,j}(x) h^j, \; h \to 0, \tag{1.8}$$

in the space of such functions. The semi-classical principal symbol in this case is given by

$$p_{0,\epsilon}(x, \xi) = \sum a_{\alpha,\epsilon,0}(x) \xi^\alpha, \tag{1.9}$$

and we make the ellipticity assumption

$$|p_0(x, \xi)| \geq \frac{1}{C} \langle \xi \rangle^m, \; (x, \xi) \in T^*M, \; |\xi| \geq C, \tag{1.10}$$

for some large $C > 0$. (Here we assume that M has been equipped with some Riemannian metric, so that $|\xi|$ and $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ are well-defined.)

Sometimes, we write p_ϵ for $p_{0,\epsilon}$ and simply p for $p_{0,0}$. Assume

$$P_{\epsilon=0} \text{ is formally selfadjoint.} \tag{1.11}$$

In the case when M is compact, we let the underlying Hilbert space be $L^2(M, \mu(dx))$ for some positive real-analytic density $\mu(dx)$ on M.

Under these assumptions P_ϵ will have discrete spectrum in some fixed neighborhood of $0 \in \mathbf{C}$, when $h > 0, \epsilon \geq 0$ are sufficiently small, and the spectrum in this region, will be contained in a band $|\mathrm{Im}\, z| \leq \mathcal{O}(\epsilon)$. The purpose of this work as well as of [9] and later ones in this series, is to give detailed asymptotic results about the distribution of individual eigenvalues inside such a band.

Assume for simplicity that (with $p = p_{\epsilon=0}$)

$$\Gamma_0 := p^{-1}(0) \cap T^*M \text{ is connected.} \tag{1.12}$$

Let $H_p = p'_\xi \cdot \frac{\partial}{\partial x} - p'_x \cdot \frac{\partial}{\partial \xi}$ be the Hamilton field of p. In this work, we will always assume that for $E \in \mathrm{neigh}\,(0, \mathbf{R})$:

$$\begin{gathered}\text{The } H_p\text{-flow is periodic on } \Gamma_E := p^{-1}(E) \cap T^*M \text{ with} \\ \text{a period } T(E) > 0 \text{ depending analytically on } E.\end{gathered} \tag{1.13}$$

(In Section 2 we recall how this assumption follows from a seemingly weaker one.) Let $q = \frac{1}{i}(\frac{\partial}{\partial \epsilon})_{\epsilon=0} p_\epsilon$, so that

$$p_\epsilon = p + i\epsilon q + \mathcal{O}(\epsilon^2 m), \tag{1.14}$$

in the case $M = \mathbf{R}^2$ and $p_\epsilon = p + i\epsilon q + \mathcal{O}(\epsilon^2 \langle \xi \rangle^m)$ in the manifold case. Let

$$\langle q \rangle = \frac{1}{T(E)} \int_{-T(E)/2}^{T(E)/2} q \circ \exp(tH_p)\, dt \text{ on } p^{-1}(E) \cap T^*M. \tag{1.15}$$

Notice that $p, \langle q \rangle$ are in involution; $0 = H_p \langle q \rangle =: \{p, \langle q \rangle\}$. As in [9], we shall see how to reduce ourselves to the case when

$$p_\epsilon = p + i\epsilon \langle q \rangle + \mathcal{O}(\epsilon^2), \tag{1.16}$$

near $p^{-1}(0) \cap T^*M$. An easy consequence of this is that the spectrum of P_ϵ in $\{z \in \mathbf{C}; |\mathrm{Re}\, z| < \delta\}$ is confined to $]-\delta, \delta[+ i\epsilon] \langle \mathrm{Re}\, q \rangle_{\min,0} - o(1), \langle \mathrm{Re}\, q \rangle_{\max,0} + o(1)[$, when $\delta, \epsilon, h \to 0$, where $\langle \mathrm{Re}\, q \rangle_{\min,0} = \min_{p^{-1}(0) \cap T^*M} \langle \mathrm{Re}\, q \rangle$ and similarly for $\langle q \rangle_{\max,0}$. We will mainly think about the case when $\langle q \rangle$ is real-valued but we will work under the more general asumption that

$$\mathrm{Im}\, \langle q \rangle \text{ is an analytic function of } p \text{ and } \mathrm{Re}\, \langle q \rangle, \tag{1.17}$$

in the region of T^*M, where $|p| \le 1/\mathcal{O}(1)$.

Let $F_0 \in [\langle \mathrm{Re}\, q\rangle_{\mathrm{min},0}, \langle \mathrm{Re}\, q\rangle_{\mathrm{max},0}]$. The purpose of the present work is to determine all eigenvalues in a rectangle

$$]-\frac{1}{\mathcal{O}(1)}, \frac{1}{\mathcal{O}(1)}[+i\epsilon]F_0 - \frac{1}{\mathcal{O}(1)}, F_0 + \frac{1}{\mathcal{O}(1)}[, \tag{1.18}$$

for

$$h^{\delta_0} \le \epsilon \le \epsilon_0, \tag{1.19}$$

for $h^{\delta_0} \le \epsilon \le \epsilon_0$ with $0 < \delta_0 < 1/2$ and ϵ_0 sufficiently small but fixed. We assume that

$$T(0) \text{ is the minimal period of every } H_p\text{-trajectory in } \Lambda_{0,F_0}, \tag{1.20}$$

where

$$\Lambda_{0,F_0} := \{\rho \in T^*M;\, p(\rho) = 0,\ \mathrm{Re}\, \langle q\rangle(\rho) = F_0\}, \tag{1.21}$$

We also assume that

$$dp,\ d\mathrm{Re}\, \langle q\rangle \text{ are linearly independent at every point of } \Lambda_{0,F_0}. \tag{1.22}$$

This implies that every connected component of Λ_{0,F_0} is a two-dimensional Lagrangian torus. For simplicity, we shall assume that there is only one such component. Notice that in view of (1.20), the space of closed orbits in $p^{-1}(0) \cap T^*M$;

$$\Sigma := (p^{-1}(0) \cap T^*M)/\sim,$$

where $\rho \sim \mu$ if $\rho = \exp tH_p\mu$ for some $t \in \mathbf{R}$, becomes a 2-dimensional symplectic manifold near the image of Λ_{0,F_0}, and (1.22) simply means that $\mathrm{Re}\, \langle q\rangle$, viewed as a function on Σ, has non-vanishing differential. The image of Λ_{0,F_0} is just a closed curve.

In [9] (for ϵ in the range $h \ll \epsilon \le h^\delta$ and sometimes $h^2 \ll \epsilon \le h^\delta$, $\forall \delta > 0$) we also studied the case when F_0 is a non-degenerate extreme valule of $\langle q\rangle$ on Σ. It would be interesting to see to what extent that can be done for ϵ in the range (1.19).

As in [14], the analyticity assumptions seem to be quite essential at least in the case of fixed ϵ. Indeed one is naturally led to work in modified Hilbert spaces defined by introducing microlocal exponential weights in the spirit of [19, 7, 14, 9], and there are closely related Fourier integral operators with complex phase some of which have associated complex canonical transformations that are ϵ-perturbations of the identity.

The plan of the paper is the following:

In section 2, we make the geometrical work and construct invariant torii close to the real domain. This allows us to construct a complex canonical transformation which reduces p to a function on the cotangent space on the standard 2-torus, which is independent of the space-variables.

In section 3 we perform further reductions for the whole operator and obtain a complete asymptotic description of all the eigenvalues of P_ϵ in a rectange of the form (1.18). This is still somewhat formal, but

in section 4, we introduce a global Grushin problem, and verify that the formal eigenvalues of the preceding section coincide modulo $\mathcal{O}(h^\infty)$ with the actual eigenvalue in a rectangle (1.18).

2 Geometric reductions

We use the notation and general set-up of the introduction. Thus let p denote the semi-classical principal symbol of the unperturbed operator. As a warm-up we recall how the assumption (1.13) follows from a seemingly weaker assumption. Thus replace (1.13) by the assumption that for some $\alpha > 0$, every point $\rho \in p^{-1}(]-\alpha, \alpha[)$ belongs to a closed H_p-trajectory $\gamma(\rho)$ with period $T(\rho) > 0$, depending continously on ρ. Also assume $dp \neq 0$ on Γ_0. Then,

1) If $\gamma(\rho) \in \Gamma_E$ is the $T(\rho)$-periodic H_p-trajectory passing through $\rho \in p^{-1}(E)$, then the action $I(\gamma(\rho)) = \int_{\gamma(\rho)} \xi dx$ only depends on E but not on ρ.

2) We have the same conclusion for the period $T(\rho)$ and hence (1.13) holds.

Indeed, consider first two trajectories $\gamma(\rho_0), \gamma(\rho_1)$ and take an intermediate family $\gamma(\rho_s)$, $0 \leq s \leq 1$, depending continuously on s, so that the union of the $\gamma(\rho_s)$ is a two-dimensional manifold $\Gamma \subset p^{-1}(E)$. Notice that $\sigma_{|_\Gamma} = 0$, since H_p is tangent to Γ and belongs to the radical of the restriction of σ to $p^{-1}(E)$. Hence by Stokes' formula,

$$\int_{\gamma(\rho_1)} \xi dx - \int_{\gamma(\rho_0)} \xi dx = \int_\Gamma \sigma = 0.$$

This shows 1). As for 2), let $\gamma_E \subset \Gamma_E$ be a smooth family of H_p-periodic curves with period $= T(\gamma_E)$. Let $\Gamma = \cup_{E_0 \leq E \leq E_1} \gamma_E$ and let ν be a vector field on Γ, with $\nu(p) = 1$. Let t be a multivalued time variable on Γ, so that $H_p t = 1$. Then we claim that

$$\sigma_{|_\Gamma} = dp \wedge dt = d(pdt):$$

On the one hand, $\langle \sigma, \nu \wedge H_p \rangle = \langle dp, \nu \rangle = 1$ and on the other hand

$$\langle dp \wedge dt, \nu \wedge H_p \rangle = \det \begin{pmatrix} \langle dp, \nu \rangle & 0 \\ \langle dt, \nu \rangle & \langle dt, H_p \rangle \end{pmatrix} = 1,$$

since the diagonal elements of the matrix are equal to 1, and the claim follows.

By Stokes' formula,

$$\begin{aligned} \int_{\gamma(E_1)} \xi dx - \int_{\gamma(E_0)} \xi dx &= \int_\Gamma \sigma = \int_\Gamma d(pdt) = -\int_{\widetilde{\Gamma}} d(tdp) \\ &= -\int_\alpha t(\rho) dp(\rho) + \int_\alpha (t(\rho) + T(\rho)) dp = \int_\alpha T(\rho) dp \\ &= T(E_0)(E_1 - E_0) + \mathcal{O}((E_1 - E_0)^2), \end{aligned}$$

where $\widetilde{\Gamma}$ is the "rectangular domain" obtained by placing a "cut" α from $\gamma(E_0)$ to $\gamma(E_1)$, and we get the (well-known) formula,

$$\frac{d}{dE} I(\gamma(E)) = T(\gamma(E)).$$

Since $I(\gamma(E))$ only depends on E and not on the choice of $\gamma(E)$, we get 2).

Let p_ϵ be as in the introduction, and let q be defined in (1.16). Let $G(x, \xi)$ be an analytic function defined in a neighborhood of $p^{-1}(0)$, such that

$$H_p G = q - \langle q \rangle, \tag{2.1}$$

where we recall that $\langle q \rangle$ is the trajectory average, defined in (1.15).

We will replace T^*M by the new IR-manifold $\Lambda_{\epsilon G} = \exp(i\epsilon H_G)(T^*M)$ (defined in a complex neighborhood of Γ_0). Writing $\Lambda_{\epsilon G} \ni (x, \xi) = \exp(i\epsilon H_G)(y, \eta)$, we use $\rho = (y, \eta)$ as real symplectic coordinates on $\Lambda_{\epsilon G}$. By Taylor expansion, we get

$$\begin{aligned} p_\epsilon(\exp(iH_G(\rho))) &= (p + i\epsilon q)(\exp(i\epsilon H_G(\rho)) + \mathcal{O}(\epsilon^2) = \\ & p(\rho) + i\epsilon(q - H_p G)(\rho) + \mathcal{O}(\epsilon^2) = p + i\epsilon \langle q \rangle + \mathcal{O}(\epsilon^2). \end{aligned} \tag{2.2}$$

Recall the assumptions (1.17), (1.22), where we shall assume for simplicity that $F_0 = 0$. (This is no real restriction, since we can always replace p_ϵ by $p_\epsilon - i\epsilon F_0$.) Since the Poisson bracket $\{p, \mathrm{Re}\langle q \rangle\}$ is zero, we see that every component of the set $\Lambda_{0,0} = \{p = 0,\ \langle q \rangle = 0\}$ is a smooth Lagrangian torus. Assume for simplicity (as in the introduction), that we only have one such component. Near this torus, $p, \mathrm{Re}\langle q \rangle$ form an integrable system, so we can find a real and analytic canonical transformation κ^{-1} from a neighborhood of $\Lambda_{0,0}$ to a neighborhood of $\xi = 0$ in

$T^*\mathbf{T}^2$, so that $p\circ\kappa$ and $\mathrm{Re}\,\langle q\rangle\circ\kappa$ (and hence also $\langle q\rangle\circ\kappa$ because of (1.17)) become functions of ξ only. Here $\mathbf{T}^2=(\mathbf{R}/2\pi\mathbf{Z})^2$.

We can do this in the following way: Let $\Lambda_{E,F}$ be the Lagrangian torus given by $p=E, \mathrm{Re}\,\langle q\rangle=F$, for $(E,F)\in\mathrm{neigh}\,(0,\mathbf{R}^2)$. Let $\gamma_1(E,F)$ be the cycle in $\Lambda_{E,F}$ corresponding to a closed H_p-trajectory with minimal period, and let $\gamma_2(E,F)$ be a second cycle so that γ_1,γ_2 form a fundamental system of cycles on the torus $\Lambda_{E,F}$. Necessarily γ_2 maps to the simple loop given by $\langle q\rangle=F$ in the abstract quotient manifold $p^{-1}(E)/\mathbf{R}H_p$. Now it is classical (see Arnold [Ar]) that we can find a real analytic canonical transformation $\kappa:\mathrm{neigh}\,(\eta=0,T^*\mathbf{T}^2)\ni(y,\eta)\mapsto(x,\xi)\in\mathrm{neigh}\,(\Lambda_{0,0},T^*\mathbf{R}^2)$, $\mathbf{T}^2:=(\mathbf{R}/2\pi\mathbf{Z})^2$ such that

$$\eta_j=\frac{1}{2\pi}\Big(\int_{\gamma_j(E,F)}\xi dx-\int_{\gamma_j(0,0)}\xi dx\Big), \tag{2.3}$$

where E,F depend on (x,ξ) and are determined by $(x,\xi)\in\Lambda_{E,F}$, i.e. by $E=p(x,\xi), F=\mathrm{Re}\,\langle q\rangle(x,\xi)$. (See also [9].)

In the following we sometimes write p instead of $p\circ\kappa$ and similarly for $\langle q\rangle$ (cf (1.17)):

$$p=p(\xi),\ \langle q\rangle=\langle q\rangle(\xi).$$

Then $H_p=\sum_1^2\frac{\partial p}{\partial\xi_j}\frac{\partial}{\partial x_j}$. From (2.3) and the discussion at the beginning of this section, we see that $p=p(\xi_1)$ in the new coordinates, so

$$H_p=c(\xi_1)\frac{\partial}{\partial x_1},\ p=p(\xi_1),\ c=\frac{\partial p}{\partial\xi_1}\neq 0. \tag{2.4}$$

The assumption (1.22) implies:

$$\frac{\partial p}{\partial\xi_1}\neq 0,\ \frac{d\mathrm{Re}\,\langle q\rangle}{\partial\xi_2}\neq 0. \tag{2.5}$$

Thus

$$p_\epsilon=p(\xi_1)+i\epsilon\langle q\rangle(\xi)+r_\epsilon(x,\xi), \tag{2.6}$$

where $r_\epsilon=\mathcal{O}(\epsilon^2)$ and p, $\langle q\rangle$ satisfy (2.5).

Now look for a "Lagrangian" torus Γ in the complex domain of the form

$$\xi=\phi'(x),\ x\in\mathbf{T}^2, \tag{2.7}$$

with ϕ grad-periodic (in the sense that $\nabla\phi$ is single-valued on $\mathbf{T}^2$) and complex-valued, $\phi'=\mathcal{O}(\widetilde{\epsilon})$, $\epsilon\ll\widetilde{\epsilon}\ll 1$, such that $p_{\epsilon|_\Gamma}=0$. We get the eiconal equation

$$p(\frac{\partial\phi}{\partial x_1})+i\epsilon\langle q\rangle(\phi'_x)+r_\epsilon(x,\phi'_x)=0, \tag{2.8}$$

where $r_\epsilon = \mathcal{O}(\epsilon^2)$. Write $p(\xi_1) = c\xi_1 + \mathcal{O}(\xi_1^2)$, $\langle q\rangle(\xi) = a\xi_1 + b\xi_2 + \mathcal{O}(\xi^2)$, $c \in \mathbf{R}$, $c, \operatorname{Re} b \neq 0$ so that:

$$((c+ia\epsilon)\frac{\partial}{\partial x_1} + i\epsilon b\frac{\partial}{\partial x_2})\phi + F_\epsilon(x, \phi'_x) = 0,$$

where

$$F_\epsilon(x,\xi) = \mathcal{O}(\epsilon^2 + \epsilon\xi^2 + \xi_1^2).$$

For notational convenience, assume that $a = 0$, $b, c = 1$. Look for $\phi = \widetilde{\epsilon}\psi$, with $\psi' = \mathcal{O}(1)$, $\epsilon \ll \widetilde{\epsilon} \ll 1$. Then we get

$$(\frac{\partial}{\partial x_1} + i\epsilon\frac{\partial}{\partial x_2})\psi + G_{\epsilon,\widetilde{\epsilon}}(x, \psi'_x) = 0, \tag{2.9}$$

with

$$G_{\epsilon,\widetilde{\epsilon}}(x,\xi) = \frac{1}{\widetilde{\epsilon}}F_\epsilon(x, \widetilde{\epsilon}\xi) = \mathcal{O}(\epsilon(\frac{\epsilon}{\widetilde{\epsilon}} + \widetilde{\epsilon}) + \widetilde{\epsilon}\xi_1^2).$$

Let $H^m(\mathbf{T}^2)$ denote the standard Sobolev space of order m. In the following estimates, $m > 1$ is fixed. Using a standard result about non-linear functions of Sobolev class functions (see [1]), we get

1) If $(\epsilon^{-1}\partial_{x_1}, \partial_{x_2})\psi = \mathcal{O}(1)$ in H^m, then $G_{\epsilon,\widetilde{\epsilon}}(x, \psi'_x) = \mathcal{O}(\epsilon(\frac{\epsilon}{\widetilde{\epsilon}} + \widetilde{\epsilon}))$ in H^m.

2) If $(\epsilon^{-1}\partial_{x_1}, \partial_{x_2})\psi_j = \mathcal{O}(1)$ in H^m for $j = 0, 1$, then,

$$\|[G_{\epsilon,\widetilde{\epsilon}}(x, \psi'_j)]_0^1\|_{H^m} = \mathcal{O}(\epsilon(\frac{\epsilon}{\widetilde{\epsilon}} + \widetilde{\epsilon}))\|(\frac{1}{\epsilon}\partial_{x_1}, \partial_{x_2})(\psi_1 - \psi_0)\|_{H^m}.$$

3) If

$$(\frac{\partial}{\partial x_1} + i\epsilon\frac{\partial}{\partial x_2})u = v, \text{ with } u, v \text{ periodic,}$$

then

$$\|(\epsilon^{-1}\partial_{x_1}, \partial_{x_2})u\|_{H^m} \leq \frac{C}{\epsilon}\|v\|_{H^m}.$$

We shall find solutions to (2.9) that are grad-periodic functions of the form

$$\psi = \psi_{\text{per}} + a(\epsilon x_1 + ix_2) + b(\epsilon x_1 - ix_2), \tag{2.10}$$

with a given complex constant $a = \mathcal{O}(1)$, and where the periodic function ψ_{per} and the complex constant b will depend on a. If $\mathcal{F}u(k)$ denotes the Fourier coefficient of u at k, we get the system:

$$\begin{cases} 2\epsilon b + \mathcal{F}(G_{\epsilon,\tilde{\epsilon}}(x, \psi_{\text{per}}' + a(\epsilon x_1 + ix_2)' + b(\epsilon x_1 - ix_2)'))(0) = 0 \\ (\frac{\partial}{\partial x_1} + i\epsilon\frac{\partial}{\partial x_2})\psi_{\text{per}} + G_{\epsilon,\tilde{\epsilon}}(x, \psi_{\text{per}}' + a(\epsilon x_1 + ix_2)' + b(\epsilon x_1 - ix_2)') + 2\epsilon b = 0. \end{cases} \tag{2.11}$$

We will find the solution as a limit of a sequence

$$\psi^{(j)} = \psi_{\text{per}}^{(j)} + a(\epsilon x_1 + ix_2) + b^{(j)}(\epsilon x_1 - ix_2),\ j = 0, 1, 2, \ldots$$

with $\psi^{(0)} = a(\epsilon x_1 + ix_2)$, (and $b^{(0)} = 0$, $\psi_{\text{per}}^{(0)} = 0$) where we impose

$$(\frac{\partial}{\partial x_1} + i\epsilon\frac{\partial}{\partial x_2})\psi^{(j+1)} + G_{\epsilon,\tilde{\epsilon}}(x, \psi^{(j)\prime}) = 0.$$

The last equation gives the following system analogous to (2.11) that we label (S_j):

$$\begin{cases} 2\epsilon b^{(j+1)} + \mathcal{F}(G_{\epsilon,\tilde{\epsilon}}(x, \psi_{\text{per}}^{(j)\prime} + a(\epsilon x_1 + ix_2)' + b^{(j)}(\epsilon x_1 - ix_2)'))(0) = 0 \\ (\frac{\partial}{\partial x_1} + i\epsilon\frac{\partial}{\partial x_2})\psi_{\text{per}}^{(j+1)} + G_{\epsilon,\tilde{\epsilon}}(x, \psi_{\text{per}}^{(j)\prime} + a(\epsilon x_1 + ix_2)' + b^{(j)}(\epsilon x_1 - ix_2)') + 2\epsilon b^{(j+1)} = 0. \end{cases}$$

From (S_0), and the facts 1), 3), we get

$$|b^{(1)}| = \mathcal{O}(1)(\frac{\epsilon}{\tilde{\epsilon}} + \tilde{\epsilon}),$$

$$\|(\frac{1}{\epsilon}\partial_{x_1}, \partial_{x_2})\psi_{\text{per}}^{(1)}\|_{H^m} \leq \mathcal{O}(1)(\frac{\epsilon}{\tilde{\epsilon}} + \tilde{\epsilon}).$$

For $j \geq 1$, we consider $(\mathrm{S}_j) - (\mathrm{S}_{j-1})$ and get, using also 2),

$$|b^{(j+1)} - b^{(j)}| \leq \mathcal{O}(1)(\frac{\epsilon}{\tilde{\epsilon}} + \tilde{\epsilon})(\|(\frac{1}{\epsilon}\partial_{x_1}, \partial_{x_2})(\psi_{\text{per}}^{(j)} - \psi_{\text{per}}^{(j-1)})\|_{H^m} + |b^{(j)} - b^{(j-1)}|),$$

$$\|(\frac{1}{\epsilon}\partial_{x_1}, \partial_{x_2})(\psi_{\text{per}}^{(j+1)} - \psi_{\text{per}}^{(j)})\|_{H^m} \leq$$
$$\mathcal{O}(1)(\frac{\epsilon}{\tilde{\epsilon}} + \tilde{\epsilon})(\|(\frac{1}{\epsilon}\partial_{x_1}, \partial_{x_2})(\psi_{\text{per}}^{(j)} - \psi_{\text{per}}^{(j-1)})\|_{H^m} + |b^{(j)} - b^{(j-1)}|).$$

This implies that

$$|b^{(j+1)} - b^{(j)}| + \|(\frac{1}{\epsilon}\partial_{x_1}, \partial_{x_2})(\psi_{\text{per}}^{(j+1)} - \psi_{\text{per}}^{(j)})\|_{H^m} \leq (\mathcal{O}(1)(\frac{\epsilon}{\tilde{\epsilon}} + \tilde{\epsilon}))^{j+1},$$

and since $\epsilon \ll \widetilde{\epsilon} \ll 1$, we see that the schema converges towards a solution to (2.9) of the form (2.10), with $a = \mathcal{O}(1)$ (given), and

$$|b| + \|(\frac{1}{\epsilon}\partial_{x_1}, \partial_{x_2})\psi'_{\mathrm{per}}\|_{H^m} = \mathcal{O}(1)(\frac{\epsilon}{\widetilde{\epsilon}} + \widetilde{\epsilon}). \tag{2.12}$$

For ϕ we then have

$$\phi = \phi_{\mathrm{per}} + \widetilde{\epsilon}a(\epsilon x_1 + ix_2) + \widetilde{\epsilon}b(\epsilon x_1 - ix_2), \tag{2.13}$$

with $\widetilde{\epsilon}a = \mathcal{O}(\widetilde{\epsilon})$ given, $|\widetilde{\epsilon}b| + \|(\epsilon^{-1}\partial_{x_1}, \partial_{x_2})\phi_{\mathrm{per}}\|_{H^m} = \mathcal{O}(1)(\epsilon + \widetilde{\epsilon}^2)$. In particular,

$$\frac{\partial\phi}{\partial x_1} = a\widetilde{\epsilon}\epsilon + \mathcal{O}(\epsilon(\epsilon + \widetilde{\epsilon}^2)),\ \frac{\partial\phi}{\partial x_2} = ia\widetilde{\epsilon} + \mathcal{O}(\epsilon + \widetilde{\epsilon}^2). \tag{2.14}$$

In this discussion m is fixed and the estimates are uniform with respect to ϵ. Clearly ϕ only depends on the choice of $\widetilde{\epsilon}a$ (with m being fixed). As in [14], we see that ϕ depends holomorphically on $\widetilde{\epsilon}a$, and extends holomorphically in x to some $(\epsilon, \widetilde{\epsilon})$-dependent domain in such a way that the dependence of $\widetilde{\epsilon}a$ is still holomorphic. In the preceding constructions, everything works the same way, if we replace $\mathbf{T}^2$ by $\mathbf{T}^2 + iy$, $|y| < 1/C$, so it follows that ϕ extends in x to a complex neighborhood of the real torus, which is independent of $\epsilon, \widetilde{\epsilon}$, and that the preceding estimates remain valid here.

Write $\phi = \phi_a$, when $\widetilde{\epsilon}$ is fixed. Let Γ_ϕ: $\xi = \phi'(x)$, $x \in \mathbf{T}^2$. Let $I_j(\Gamma_\phi)$, $j = 1, 2$ be the corresponding actions with respect to $\xi_1 dx_1 + \xi_2 dx_2$. From (2.12), (2.13) (or simply (2.14)) we get:

$$\begin{aligned} I_1(\Gamma_\phi) &= 2\pi\widetilde{\epsilon}\epsilon(a + b) = 2\pi\widetilde{\epsilon}\epsilon(a + \mathcal{O}(\frac{\epsilon}{\widetilde{\epsilon}} + \widetilde{\epsilon})), \\ I_2(\Gamma_\phi) &= 2\pi i\epsilon(a - b) = 2\pi\widetilde{\epsilon}(ia + \mathcal{O}(\frac{\epsilon}{\widetilde{\epsilon}} + \widetilde{\epsilon})), \end{aligned} \tag{2.15}$$

We are interested in finding a such that both actions are real. This leads to

$$\mathrm{Im}\,(a + \mathcal{O}(\frac{\epsilon}{\widetilde{\epsilon}} + \widetilde{\epsilon})) = 0,\ \mathrm{Im}\,(ia + \mathcal{O}(\frac{\epsilon}{\widetilde{\epsilon}} + \widetilde{\epsilon})) = 0,$$

i.e.

$$\begin{cases} \mathrm{Re}\, a + \mathcal{O}(\frac{\epsilon}{\widetilde{\epsilon}} + \widetilde{\epsilon}) = 0 \\ \mathrm{Im}\, a + \mathcal{O}(\frac{\epsilon}{\widetilde{\epsilon}} + \widetilde{\epsilon}) = 0. \end{cases} \tag{2.16}$$

Here the $\mathcal{O}$-terms are real parts of holomorphic functions, so they remain $\mathcal{O}(\widetilde{\epsilon} + \epsilon/\widetilde{\epsilon})$ after derivation with respect to $\mathrm{Re}\, a$, $\mathrm{Im}\, a$. By the implicit function theorem, we therefore have a unique solution to (2.16), which is $\mathcal{O}(\widetilde{\epsilon}+\epsilon/\widetilde{\epsilon})$, and correspondingly $\widetilde{\epsilon}a=\mathcal{O}(\epsilon+\widetilde{\epsilon}^2)$. Recall that $\widetilde{\epsilon}a$ is independent of the choice of $\widetilde{\epsilon}$, so if we take $\widetilde{\epsilon}=\sqrt{\epsilon}$, we get

$$\widetilde{\epsilon}a=\mathcal{O}(\epsilon). \tag{2.17}$$

For this particular ϕ, we have

$$\partial_{x_1}\phi=\mathcal{O}(\epsilon^2),\ \partial_{x_2}\phi=\mathcal{O}(\epsilon)\text{ in }H^m. \tag{2.18}$$

If we do not make the simplifying assumption that $\frac{\partial p}{\partial\xi_1}(0)=1$, $\frac{\partial\langle q\rangle}{\partial\xi_1}(0)=0$, $\frac{\partial\langle q\rangle}{\partial\xi_2}(0)=1$, then the earlier discussion tells us that we have solutions of the type $\phi=\widetilde{\epsilon}\psi$, $\psi=\psi_{\mathrm{per}}(x)+a\alpha(x)+b\beta(x)$, with

$$\alpha(x)=\epsilon\frac{\partial\langle q\rangle}{\partial\xi_2}(0)x_1+i\frac{\partial(p+i\epsilon\langle q\rangle)}{\partial\xi_1}(0)x_2,$$

$$\beta(x)=\epsilon\frac{\partial\langle q\rangle}{\partial\xi_2}(0)x_1-i\frac{\partial(p+i\epsilon\langle q\rangle)}{\partial\xi_1}(0)x_2.$$

Observe that if we put,

$$Z:=\frac{\partial(p+i\epsilon\langle q\rangle)}{\partial\xi_1}(0)\frac{\partial}{\partial x_1}+\frac{\partial i\epsilon\langle q\rangle}{\partial\xi_2}(0)\frac{\partial}{\partial x_2},$$

then

$$Z\alpha=0,\ Z\beta=2\epsilon\frac{\partial(p+i\epsilon\langle q\rangle)}{\partial\xi_1}(0)\frac{\partial\langle q\rangle}{\partial\xi_2}(0)\neq 0.$$

The earlier discussion goes through without any changes. Especially, in the case $a=0$, the corresponding ϕ is independent of $\widetilde{\epsilon}$.

Let now ζ vary in neigh $(0,\mathbf{C}^2)$. Put $z(\zeta)=p(\zeta_1)+i\epsilon\langle q\rangle(\zeta)$. Then the discussion above can be applied with $p_\epsilon(x,\xi)$ replaced by

$$p_\epsilon(x,\zeta+\xi)-z(\zeta)=p(\zeta_1+\xi_1)-p(\zeta_1)+i\epsilon(\langle q\rangle(\zeta+\xi)-\langle q\rangle(\zeta))+\mathcal{O}(\epsilon^2).$$

We get a solution to the eiconal equation

$$p_\epsilon(x,\zeta+\psi'_x)-z(\zeta)=0$$

of the form

$$\psi(x,\zeta)=\psi_{\mathrm{per}}(x,\zeta)+b(\zeta)\beta(x,\zeta),$$

where

$$\beta(x,\zeta)=\epsilon\frac{\partial\langle q\rangle}{\partial\xi_2}(\zeta)x_1-i\frac{\partial(p+i\epsilon\langle q\rangle)}{\partial\xi_1}(\zeta)x_2,$$

depending holomorphically on ζ. (So we choose $a = 0$ in the earlier discussion, but compensate for this by introducing a ζ-dependence and even varying the energy level $z(\zeta)$.)

As before, we get

$$\|(\frac{1}{\epsilon}\partial_{x_1}, \partial_{x_2})\psi_{\mathrm{per}}\|_{H^m} = \mathcal{O}(\epsilon),\ |b| = \mathcal{O}(\epsilon), \tag{2.19}$$

uniformly with respect to ζ. Moreover, since the problem depends holomorphically on ζ, it is easy to see (for instance by working in a space of holomorphic functions of ζ with values in H^m) that $\nabla_x\psi, b$ depend holomorphically on ζ. Notice that

$$\widetilde{\psi}(x,\zeta) := x\cdot\zeta + \psi(x,\zeta) \tag{2.20}$$

solves the eiconal equation

$$p_\epsilon(x, \partial_x\widetilde{\psi}(x,\zeta)) - z(\zeta) = 0. \tag{2.21}$$

Write $b(\zeta)\beta(x,\zeta) + x\cdot\zeta = x\cdot\eta$, where $\eta(\zeta)$ depends holomorphically on ζ and satisfies

$$\eta_1(\zeta) = \zeta_1 + \mathcal{O}(\epsilon^2),\ \eta_2(\zeta) = \zeta_2 + \mathcal{O}(\epsilon).$$

Let $\zeta(\eta)$ with $\zeta_1(\eta) = \eta_1 + \mathcal{O}(\epsilon^2)$, $\zeta_2(\eta) = \eta_2 + \mathcal{O}(\epsilon)$ denote the inverse. Then with $\phi_{\mathrm{per}}(x,\eta) = \psi_{\mathrm{per}}(x,\zeta)$, we have

$$\widetilde{\psi}(x,\zeta) = x\cdot\eta + \phi_{\mathrm{per}}(x,\eta) =: \phi(x,\eta), \tag{2.22}$$

solving

$$p_\epsilon(x, \partial_x\phi(x,\eta)) - \widetilde{p}_\epsilon(\eta) = 0,\ \widetilde{p}_\epsilon(\eta) = z(\zeta(\eta)) = p(\eta_1) + i\epsilon\langle q\rangle(\eta) + \mathcal{O}(\epsilon^2), \tag{2.23}$$

while (2.19) gives

$$|(\frac{1}{\epsilon}\partial_{x_1}, \partial_{x_2})\phi_{\mathrm{per}}(x,\eta)| = \mathcal{O}(\epsilon), \tag{2.24}$$

for x in a fixed complex neighborhood of $\mathbf{T}^2$ and as usual, we get corresponding estimates for $\partial_x^\alpha\partial_\eta^\beta\phi_{\mathrm{per}}(x,\eta)$ from the Cauchy inequalities. We normalize the choice of $\phi_{\mathrm{per}}(x,\eta)$ by requiring that

$$\langle\phi_{\mathrm{per}}(\cdot,\eta)\rangle = \frac{1}{(2\pi)^2}\int_{\mathbf{T}^2}\phi_{\mathrm{per}}(x,\eta)dx = 0.$$

Then

$$\kappa_\epsilon : (\phi'_\eta(x,\eta),\eta) \mapsto (x, \phi'_x(x,\eta)) \tag{2.25}$$

maps a complex (ϵ-independent) neighborhood of the zero section of $T^*\mathbf{T}^2$ onto another neighborhood of the same type (containing an ϵ-independent neighborhood of $\xi = 0$). (2.23) shows that

$$p_\epsilon \circ \kappa_\epsilon = \widetilde{p}_\epsilon. \tag{2.26}$$

By construction, we also know that κ_ϵ conserves actions along closed curves.

Using that $\phi'_\eta(x,\eta) = x + \mathcal{O}(\epsilon)$, $\phi'_x(x,\eta) = \eta + \mathcal{O}(\epsilon^2, \epsilon)$ together with (2.24), which also holds with ϕ_{per} replaced by its gradient, we see that

$$\kappa_\epsilon(y,\eta) = (y + \mathcal{O}(\epsilon); \eta_1 + \mathcal{O}(\epsilon^2), \eta_2 + \mathcal{O}(\epsilon)). \tag{2.27}$$

In particular, we have

$$\operatorname{Im} x = \mathcal{O}(\epsilon),\ \operatorname{Im}\xi_1 = \mathcal{O}(\epsilon^2),\ \operatorname{Im}\xi_2 = \mathcal{O}(\epsilon), \tag{2.28}$$

on the image of $T^*\mathbf{T}^2$. We can therefore represent $\kappa_\epsilon(T^*\mathbf{T}^2)$ by

$$\operatorname{Im} x = G'_\xi(\operatorname{Re}(x,\xi)),\ \operatorname{Im}\xi = -G'_x(\operatorname{Re}(x,\xi)), \tag{2.29}$$

where G is a smooth, a priori grad-periodic function which satisfies,

$$\partial_\xi G, \partial_{x_2} G = \mathcal{O}(\epsilon),\ \partial_{x_1} G = \mathcal{O}(\epsilon^2). \tag{2.30}$$

Since κ_ϵ conserves actions, the actions along closed cycles in $\kappa_\epsilon(T^*\mathbf{T}^2)$ are real and it follows that G is single-valued. We may assume that $G = \mathcal{O}(\epsilon)$. Let $\chi(\xi)$ be a standard cutoff around $\xi = 0$ and let $\widetilde{M}_\epsilon$ be given by

$$\begin{gathered}\operatorname{Im} x = \widetilde{G}'_\xi(\operatorname{Re}(x,\xi)),\ \operatorname{Im}\xi = -\widetilde{G}'_x(\operatorname{Re}(x,\xi)),\\ \text{where } \widetilde{G}(\operatorname{Re}(x,\xi)) = \chi(\operatorname{Re}\xi)G(\operatorname{Re}(x,\xi)).\end{gathered} \tag{2.31}$$

Then $\widetilde{M}_\epsilon$ is an IR-manifold which coincides with $T^*\mathbf{T}^2$ outside a (complex ϵ-independent) neighborhood of $\xi = 0$. Moreover, we know that $\widetilde{M}_\epsilon$ is an ϵ-perturbation of $T^*\mathbf{T}^2$, along which we have

$$\operatorname{Im}\xi_1 = -\chi(\operatorname{Re}\xi)G'_{x_1}(\operatorname{Re}(x,\xi)) = \mathcal{O}(\epsilon^2).$$

It follows that outside the neighborhood of $\xi = 0$, where $\widetilde{M}_\epsilon$ coincides with $\kappa_\epsilon(T^*\mathbf{T}^2)$, we have

$$|\operatorname{Re} p_{\epsilon|_{\widetilde{M}_\epsilon}}| + \frac{1}{\epsilon}|\operatorname{Im} p_{\epsilon|_{\widetilde{M}_\epsilon}}| \geq \frac{1}{C}. \tag{2.32}$$

Now recall the initial global situation, that we simplified the original principal symbol by composing with $\exp i\epsilon H_G$ for the function G in (2.1) and then further by κ, introduced prior to (2.4).

We introduce an IR-deformation M_ϵ of real phase space which is an ϵ-deformation, equal to real phase space away from Γ_0, and equal to $\exp i\epsilon H_G \circ \kappa(\widetilde{M}_\epsilon)$ near $\Gamma_0 = p^{-1}(0) \cap T^*M$. Then we have achieved the following:

Proposition 2.1 *a) There exists an analytic real canonical transformation* κ_ϵ : $\mathrm{neigh}\,(\xi = 0, T^*\mathbf{T}^2) \to \mathrm{neigh}\,(\exp{(i\epsilon H_G)}(\Lambda_{0,0}), M_\epsilon)$, *such that*

$$p_\epsilon \circ \kappa_\epsilon = \widetilde{p}_\epsilon(\eta), \tag{2.33}$$

where $\widetilde{p}_\epsilon$ *is given in (2.23).*
b) Away from the small neighborhood, where (2.33) holds, we have

$$|\mathrm{Re}\, p_{\epsilon|_{M_\epsilon}}| + \frac{1}{\epsilon}|\mathrm{Im}\, p_{\epsilon|_{M_\epsilon}}| \geq \frac{1}{C}. \tag{2.34}$$

Here p_ϵ *denotes the original principal symbol of the perturbed operator.*

It is now clear that the main result of [MeSj] can be applied to give the full asymptotics for all eigenvalues of P_ϵ in a domain $|\mathrm{Re}\, z| < 1/\mathcal{O}(1)$, $|\mathrm{Im}\, z| < \epsilon/\mathcal{O}(1)$, for $\epsilon > 0$ small enough and for $h < h(\epsilon) > 0$ small enough depending on ϵ. It is not apriori clear however what kind of uniformity with respect to ϵ we may have in this result. We shall employ quantum Birkhoff normal forms in the next section and obtain a more uniform result, valid for $\epsilon > h^\delta$ for any fixed $\delta \in]0, \frac{1}{2}[$.

3 Formal spectral asymptotics

As in [9] (see also [14]) we can implement κ_ϵ by an elliptic Fourier integral operator $U = U_\epsilon : L^2_S(\mathbf{T}^2) \to H(M_\epsilon)$ which is microlocally defined from a neighborhood of $\xi = 0$ in $T^*\mathbf{T}^2$ to a neighborhood of $\exp i\epsilon H_G(\Lambda_{0,0})$ in M_ϵ. Here $S = (S_1, S_2) \in \mathbf{R}^2$, with $S_j = \int_{\gamma_j} \xi dx$, and $\gamma_j = \gamma_j(0,0)$ are introduced prior to (2.3). $L_S(\mathbf{T}^2)$ denotes the space of locally L^2-functions u on $\mathbf{R}^2$ satisfying the Floquet periodicity condition:

$$u(x - \gamma) = e^{\frac{i\gamma}{2\pi}\cdot(\frac{1}{h}S + \frac{\pi}{2}\alpha^0)},\ \gamma \in (2\pi\mathbf{Z})^2, \tag{3.1}$$

where $\alpha^0 = (\alpha^0_1, \alpha^0_2) \in \mathbf{Z}^2$ is a Maslov index. By abuse of notation, we still denote by P_ϵ, the conjugated operator $U_\epsilon^{-1} P_\epsilon U_\epsilon$.

We have an analytic h-pseudodifferential operator P_ϵ on $\mathbf{T}^2$ (defined microlocally near $\xi = 0$), of order 0 in h, with leading symbol independent of x:

$$p_\epsilon(\xi) = p(\xi_1) + i\epsilon\langle q\rangle(\xi) + \mathcal{O}(\epsilon^2), \tag{3.2}$$

defined in a fixed complex neighborhood of $\xi = 0$ in $T^*\mathbf{T}^2$, depending holomorphically on $\epsilon \in D(0, \epsilon_0)$. The full symbol is

$$P_\epsilon(x, \xi; h) = \sum_{j=0}^{\infty} h^j p_j(x, \xi, \epsilon), \tag{3.3}$$

with $p_j(x, \xi, \epsilon)$ holomorphic with respect to (x, ξ) in a j-independent complex neighborhood of $\xi = 0$ and C^∞ with respect to $\epsilon \in [0, \epsilon_0[$, with $p_0(x, \xi, \epsilon) = p_\epsilon(\xi)$. Following the standard Birkhoff normal form procedure, we shall remove the x-dependence in the p_j by means of conjugation by an elliptic h-pseudodifferential operator of order 0. Let A be an h-pseudodifferential operator of order 0. Recall that

$$e^A P e^{-A} = e^{\mathrm{ad}_A} P = \sum \frac{1}{k!} \mathrm{ad}_A^k P.$$

Let the full symbol of A be of the form $\sum_{k=0}^{\infty} h^k a_k$. Then on the operator level,

$$\begin{aligned} e^A P e^{-A} &= \sum_{\ell=0}^{\infty} \sum_{k=0}^{\infty} \sum_{j_1=0}^{\infty} \cdots \sum_{j_k=0}^{\infty} \frac{1}{k!} h^{j_1+\ldots+j_k+\ell+k} \left(\frac{1}{h}\mathrm{ad}_{a_{j_1}}\right) \ldots \left(\frac{1}{h}\mathrm{ad}_{a_{j_k}}\right) p_\ell \\ &= \sum_{n=0}^{\infty} h^n s_n, \end{aligned}$$

with $s_0 = p_0$, $s_1 = \frac{1}{i} H_{a_0} p_0 + p_1 = i H_{p_0} a_0 + p_1$, ..., $s_{n+1} = i H_{p_0} a_n + \widetilde{s}_{n+1}$,..., where $\widetilde{s}_{n+1}$ only depends on $a_0, \ldots, a_{n-1}$ and is the sum of the coefficients for h^{n+1} from the terms

$$\frac{1}{k!} h^{j_1+\ldots+j_k+\ell+k} \left(\frac{1}{h}\mathrm{ad}_{a_{j_1}}\right) \ldots \left(\frac{1}{h}\mathrm{ad}_{a_{j_k}}\right)(p_\ell),$$

with

$$j_1 + \ldots + j_k + \ell + k \leq n+1, \; j_1, \ldots, j_k < n, \text{ or } k = 0, \; \ell = n+1.$$

Notice that

$$H_{p_0} = H_{p_\epsilon} = \frac{\partial p(\xi_1)}{\partial \xi_1} \partial_{x_1} + i\epsilon \left(\frac{\partial \langle q \rangle}{\partial \xi} + \mathcal{O}(\epsilon)\right) \cdot \partial_x,$$

and that we can solve

$$H_{p_0} a = b(x, \xi) - \langle b(\cdot, \xi) \rangle, \; x \in \mathbf{T}^2, \tag{3.4}$$

with $\|a\|_{H^{m+1}} \leq \mathcal{O}(1) \epsilon^{-1} \|b\|_{H^m}$. As already noticed in the preceding section, the same equation can be solved in a complex domain $\{x \in \mathbf{T}^2; |\mathrm{Im}\, x| < C_2\}$, and we get

$$\sup_{|\mathrm{Im}\, x| < C_2} |a(x, \xi)| \leq \frac{C(C_1, C_2)}{\epsilon} \sup_{|\mathrm{Im}\, x| < C_1} |b(x, \xi)|, \tag{3.5}$$

if $C_1 < C_2$. The shrinking of the domains in (3.5) is not a problem, since we can take a sequence of such domains with $C_j \searrow C_\infty > 0$.

By solving equations of the type (3.4), we can determine $a_0, a_1, \dots$ successively, so that $s_j = s_j(\xi, \epsilon)$ are independent of x. Assume by induction that $\nabla a_j = \mathcal{O}(\epsilon^{-1-2j})$, for $j \leq n-1$ (in a complex domain, so that we have the same estimates on the derivatives of ∇a_j). Then the general term in $\widetilde{s}_{n+1}$ is

$$\mathcal{O}(1)\epsilon^{-1-2j_1}\dots\epsilon^{-1-2j_k} = \mathcal{O}(1)(\frac{1}{\epsilon})^{2(j_1+..+j_k)+k}.$$

Here,

$$2(j_1+..+j_k)+k = 2(j_1+..+j_k+k)-k \leq 2(n+1-\ell)-k = 2n+2-2\ell-k.$$

So this quantity is $\mathcal{O}(1)(\frac{1}{\epsilon})^{2n}$ except possibly when $2\ell+k<2$, i.e. when $k=\ell=0$ or when $k=1$, $\ell=0$. In the first case we get the coefficient for h^{n+1} in p_0 which is 0. In the second case, we get the coefficient for h^{n+1} in $h^{j_1+1}(\frac{1}{h}\mathrm{ad}_{a_{j_1}})(p_0)$ with $j_1 < n$, which is $\mathcal{O}(1)(\frac{1}{\epsilon})^{1+2j_1}$. Here $1+2j_1 \leq 2n$. Thus $\widetilde{s}_{n+1} = \mathcal{O}(\epsilon^{-2n})$ (in a complex domain). We can choose a_n periodic, with $iH_{p_0}a_n = -\widetilde{s}_{n+1} + \langle \widetilde{s}_{n+1}(\cdot, \xi)\rangle$ and with $\nabla a_n = \mathcal{O}(\epsilon^{-1-2n})$. This completes the induction step and we find a_k with $\nabla a_k = \mathcal{O}(\epsilon^{-1-2k})$ in a fixed complex neighborhood of $\mathbf{T}^2 \times \{\xi = 0\}$ such that if

$$A^{(N)} = \sum_{k=0}^{N-1} h^k a_k,$$

then

$$\widetilde{P}^{(N)} := e^{A^{(N)}} P_\epsilon e^{-A^{(N)}} = \sum_{n=0}^{\infty} h^n \widetilde{p}_n^{(N)}, \tag{3.6}$$

where $\widetilde{p}_n^{(N)}(\xi, \epsilon) = \mathcal{O}(\epsilon^{-2(n-1)_+})$ and $\widetilde{p}_n^{(N)} = \widetilde{p}_n^{(\infty)}$ is independent of x and N, for $n \leq N$. From this we get the following formal spectral result:

Theorem 3.1 *Under the assumptions above, there exists a constant $C > 0$ such that if $\delta > 0$ is fixed and $h^{\frac{1}{2}-\delta} < \epsilon < 1/C$, and $0 < h \leq h(\delta)$ with $h(\delta) > 0$ small enough, then in the region*

$$|\mathrm{Re}\, z| < \frac{1}{C}, \ \frac{|\mathrm{Im}\, z|}{\epsilon} < \frac{1}{C}, \tag{3.7}$$

P has the following quasi-eigenvalues:

$$z_k \sim \sum_{n=0}^{\infty} h^n \widetilde{p}_n^{(\infty)}(h(k - \frac{S}{2\pi h} - \frac{\alpha^0}{4}), \epsilon), \ k \in \mathbf{Z}^2. \tag{3.8}$$

Here, $S \in \mathbf{R}^2$, $\alpha^0 \in \mathbf{Z}^2$ were introduced in the beginning of this section, and $p_0^{(\infty)} = p_\epsilon$ is given in (3.2).

We leave undefined, the notion of quasi-eigenvalue, and interpret the above theorem as the formal consequence of the reductions above and the fact that the functions

$$e_k(x) = e^{ix\cdot(k-\frac{S}{2\pi h}-\frac{\alpha^0}{4})},\ k \in \mathbf{Z}^2,$$

form an orthonormal basis in $L^2_S(\mathrm{T}^2)$.

4 Justification via a global Grushin problem.

In this section we outline how Theorem 3.1 actually gives all eigenvalues in the rectangle (3.7). As in [14], [9] we construct an auxiliary, so called Grushin problem. Actually, this construction is identical with the one in [9], so we shall only recall the main steps.

For $C > 0$ sufficiently large, let $I(C, \epsilon)$ (depending also on h) be the set of all $k \in \mathbf{Z}^2$, for which the values z_k in (3.8) belong to the rectangle (3.7). Recall that z_k correspond to the orthonormal family of functions e_k, defined after Theorem 3.1.

Let $\kappa_\epsilon, M_\epsilon$ be as in Proposition, 2.1 and let U_ϵ be the Fourier integral operator quantization of κ_ϵ introduced in the beginning of Section 3. With $A^{(N)}$ defined there, let A be a natural asymptotic limit. Define

$$R_+ : H(M_\epsilon) \to \mathbf{C}^{I(C,\epsilon)}, \tag{4.1}$$

by

$$R_+u(k) = (e^A U_\epsilon^{-1} u | e_k)_{L^2_S}. \tag{4.2}$$

Notice that R_+ is a globally welldefined operator modulo some indetermination of norm $\mathcal{O}(h^\infty)$, since $e^A U_\epsilon^{-1} u$ is microlocally welldefined in a neighborhood of the zero section in $T^*\mathrm{T}^2$. Similarly, we define $R_- : \mathbf{C}^{I(C,\epsilon)} \to H(M_\epsilon)$, by

$$R_- u_- = \sum_{k\in I(C,\epsilon)} u_-(k) U_\epsilon e^{-A} e_k. \tag{4.3}$$

Then for z in the rectangle (3.7), with an increased value of C, the problem

$$(P - z)u + R_- u_- = v,\ R_+ u = v_+, \tag{4.4}$$

has a unique solution $(u, u_-) \in H(M_\epsilon) \times \mathbf{C}^{I(C,\epsilon)}$ for every $(v, v_+) \in H(M_\epsilon) \times \mathbf{C}^{I(C,\epsilon)}$. (Here we assume for simplicity that P is a bounded operator, otherwise we would have to work with modifications of $H(M_\epsilon)$ of Sobolev type, depending on additional order functions. See the appendix in [9] for more details and further references.) We have the corresponding apriori estmate

$$\|u\| + \|u_-\| \le \frac{C}{\epsilon}(\|v\| + \epsilon\|v_+\|), \tag{4.5}$$

and if we write the solution

$$\begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix} \begin{pmatrix} v \\ v_+ \end{pmatrix}, \tag{4.6}$$

then modulo $\mathcal{O}(h^\infty)$, E_{-+} is the diagonal matrix $((z - z_k)\delta_{j,k})$, where z_k are given in (3.8).

Recall from [9] that the verification of these facts consists of half-estimates away from $\Lambda_{0,0}$ and the exploitation near $\Lambda_{0,0}$ of the reduction to a translation invariant operator on $\mathbf{T}^2$ in the preceding section. Since the eigenvalues of P in our rectangle are precisely the values z for which $E_{-+}(z)$ is non invertible, we get

Theorem 4.1 *Under the assumptions of Theorem 3.1, there exists a constant $C > 0$ such that if $\delta > 0$ is fixed and $h^{\frac{1}{2}-\delta} < \epsilon < 1/C$, and $0 < h \le h(\delta)$ with $h(\delta) > 0$ small enough, then in the region*

$$|\mathrm{Re}\, z| < \frac{1}{C}, \ \frac{|\mathrm{Im}\, z|}{\epsilon} < \frac{1}{C}, \tag{4.7}$$

the eigenvalues of P are simple and given by

$$z_k \sim \sum_{n=0}^{\infty} h^n \widetilde{p}_n^{(\infty)}(h(k - \frac{S}{2\pi h} - \frac{\alpha^0}{4}), \epsilon), \ k \in \mathbf{Z}^2, \tag{4.8}$$

with one eigenvalue for each k such that z_k belongs to (4.7). Here, $S \in \mathbf{R}^2$, $\alpha^0 \in \mathbf{Z}^2$ were introduced in the beginning of Section 3, and the $\widetilde{p}_n^{(\infty)}$ were constructed prior to Theorem 3.1. Further, $p_0^{(\infty)}(\xi, \epsilon) = p(\xi_1) + i\epsilon\langle q\rangle(\xi) + \mathcal{O}(\epsilon^2)$.

5 Application to barrier top resonances.

We extend the domain of validity of one of the results of section 7 in [9], by using Theorem 4.1 as the new ingredient. The discussion that follows will therefore only be a brief recollection of a part of Section 7 in [9], and we refer to that work for more details. Let

$$P = -h^2\Delta + V(x), \ p(x,\xi) = \xi^2 + V(x), \ (x,\xi) \in T^*\mathbf{R}^2 = \mathbf{R}^4, \tag{5.1}$$

satisfy the general conditions for defining resonances near the energy level $E_0 > 0$. Assume that $V(0) = 0$, $\nabla V(0) = 0$, $V''(0) < 0$ and that V is everywhere analytic. After a linear change of x-coordinates, we have near $x = 0$:

$$p(x,\xi) - E_0 = \sum_1^2 \frac{\lambda_j}{2}(\xi_j^2 - x_j^2) + p_3(x) + p_4(x) + \ldots, \tag{5.2}$$

where $\lambda_j > 0$ and p_ν is a homogeneous polynomial of degree ν. Also assume that (0,0) is the only trapped point for the H_p-flow on the real energy surface $p^{-1}(E_0)$.

We assume $\lambda = (\lambda_1, \lambda_2)$ fulfills the resonance condition

$$\lambda \cdot k = 0, \text{ for some } 0 \neq k \in \mathbf{Z}^2. \tag{5.3}$$

Somewhat roughly, the problem of determining the resonances near E_0 is then equivalent to determining the eigenvalues of $P - E_0$ near 0, after the change of variables, $x = e^{i\pi/4}\widetilde{x}$ (and $\xi = e^{-i\pi/4}\widetilde{\xi}$) near 0, and we get a new operator with symbol

$$-i(p_2(\widetilde{x},\widetilde{\xi}) + ie^{3\pi i/4}p_3(\widetilde{x}) + ie^{4\pi i/4}p_4(\widetilde{x}) + \ldots) = -iq(\widetilde{x},\widetilde{\xi}), \tag{5.4}$$

$$p_2(\widetilde{x},\widetilde{\xi}) = \sum_1^2 \frac{\lambda_j}{2}(\widetilde{\xi}_j^2 + \widetilde{x}_j^2).$$

Dropping the tildes for the new variables, we are then interested in eigenvalues E of $Q = q(x, hD_x)$ with $|E| \sim \epsilon^2$, $h^\delta < \epsilon \ll 1$, $0 < \delta < 1/2$. Write $x = \epsilon y$, $\widetilde{h} = h/\epsilon^2$. Then $hD_x = \epsilon\widetilde{h}D_y$ and

$$\epsilon^{-2}q(x,\xi) = \epsilon^{-2}q(\epsilon(y,\eta)) = p_2(y,\eta) + i\epsilon e^{3\pi i/4}p_3(y) + \mathcal{O}(\epsilon^2),$$

in a region $|(y,\eta)| = \mathcal{O}(1)$, where the corresponding eigenfunctions are concentrated.

The resonance condition (5.3) implies that the H_{p_2}-flow is periodic with a period $T > 0$, independent of the energy level. Using Theorem 4.1 in the discussion of section 7 in [9], we get the following variant of Proposition 7.1 of that paper:

Proposition 5.1 *Let $\langle p_3\rangle$ denote the average of p_3 along the trajectories of the Hamilton vector field of p_2 in (5.4), and assume that $\langle p_3\rangle$ is not identically zero. Let $F_0 \in \mathbf{R}$ be a regular value of $\cos(3\pi/4)\langle p_3\rangle$ restricted to $p_2^{-1}(1)$, and assume that T is the minimal period of the H_{p_2}-trajectories in the torus Λ_{1,F_0} given by*

$$\Lambda_{1,F_0} : p_2 = 1, \cos\left(\frac{3\pi}{4}\right)\langle p_3\rangle = F_0.$$

Let ϵ satisfy

$$h^{\delta} \ll \epsilon \leq \epsilon_0,\ 0 < \epsilon_0 \ll 1,\ 0 < \delta < \frac{1}{4}. \tag{5.5}$$

Then for z in the set

$$\left[1 - \frac{1}{\mathcal{O}(1)}, 1 + \frac{1}{\mathcal{O}(1)}\right] + i\epsilon \left[F_0 - \frac{1}{\mathcal{O}(1)}, F_0 + \frac{1}{\mathcal{O}(1)}\right],$$

the resonances of the form $E_0 - i\epsilon^2 z$ are given by

$$z = \widehat{P}\left(\widetilde{h}(k - \frac{\alpha}{4}) - \frac{S}{2\pi}, \epsilon; \widetilde{h}\right) + \mathcal{O}(h^{\infty}),\quad \widetilde{h} = \frac{h}{\epsilon^2},\ \ k \in \mathbf{Z}^2.$$

(with precisely one resonance for every k). Here $\widehat{P}\left(\xi, \epsilon; \widetilde{h}\right)$ has an expansion as $\widetilde{h} \to 0$,

$$\widehat{P}\left(\xi, \epsilon; \widetilde{h}\right) \sim \sum_{n=0}^{\infty} \widetilde{h}^n \widetilde{p}_n^{(\infty)}(\xi, \epsilon),$$

where

$$\widetilde{p}_0(\xi, \epsilon) = p_2(\xi) + i\epsilon e^{3\pi i/4}\langle p_3\rangle(\xi) + \mathcal{O}(\epsilon^2),\ \widetilde{p}_j(\xi, \epsilon) = \mathcal{O}(\epsilon^{-2(j-1)}),\ j \geq 1.$$

The coordinates $\xi_1 = \xi_1(E)$ and $\xi_2 = \xi_2(E, F)$ are the normalized actions of

$$\Lambda_{E,F} : p_2 = E,\ \cos\left(\frac{3\pi}{4}\right)\langle p_3\rangle = F,$$

for $E \in \mathrm{neigh}(1, \mathbf{R})$, $F \in \mathrm{neigh}(F_0, \mathbf{R})$, given by

$$\xi_j = \frac{1}{2\pi}\left(\int_{\gamma_j(E,F)} \eta\, dy - \int_{\gamma_j(1,F_0)} \eta\, dy\right),\quad j = 1, 2, \tag{5.6}$$

with $\gamma_j(E, F)$ being fundamental cycles in $\Lambda_{E,F}$, such that $\gamma_1(E, F)$ corresponds to a closed H_{p_2}-trajectory of minimal period T. Furthermore,

$$S_j = \int_{\gamma_j(1,F_0)} \eta\, dy,\quad j = 1, 2,\ \ S = (S_1, S_2), \tag{5.7}$$

and $\alpha \in \mathbf{Z}^2$ is fixed.

The interest of this result (as well as of Theorem 4.1) compared to the corresponding ones in [9] is that we can reach small but h-independent values of ϵ. On the other hand our method does not immediately seem to be able to hand as small values of ϵ as in [9], and the results there give a desrciption of how the negative powers of ϵ appear in our estimates of the terms in the asymptotic expansion of the symbol $\widehat{P}(\xi, \epsilon; h)$.

References

[1] S. Alinhac, P. Gérard, *Opérateurs pseudo-différentiels et théorème de Nash–Moser*, Savoirs actuels, InterEditions/Editions du CNRS, Paris 1991.

[2] M. Asch, G. Lebeau, *The spectrum of the damped wave equation*, preprint, 1999. See http://www.math.u-psud.fr/~biblio/rt/1999/.

[3] D. Bambusi, S. Graffi and T. Paul, *Normal forms and quantization formulae*, Comm. Math. Phys. **207** (1999), 173–195.

[4] L. Boutet de Monvel, V. Guillemin, *The spectral theory of Toeplitz operators*, Annals of Math. Studies 99, Princeton University Press, 1981.

[5] Y. Colin de Verdière, *Sur le spectre des opérateurs elliptiques a bicaractéristiques toutes periodiques*, Comment Math. Helv. **54** (1979), 508-522.

[6] B. Helffer, D. Robert, *Puits de potentiel généralisés et asymptotique semiclassique*, Ann. Inst. H. Poincaré **41** (1984), 291–331.

[7] B. Helffer, J. Sjöstrand, *Résonances en limite semiclassique*, Mém. Soc. Math. France (N.S.) 24–25 (1986).

[8] M. Hitrik, *Eigenfrequencies for damped wave equations on Zoll manifolds*, Asymptot. Analysis **31** (2002), 265–277.

[9] M. Hitrik, J. Sjöstrand, *Non-selfadjoint perturbations of selfadjoint operators in 2 dimensions I,* Preprint 2003.

[10] M. Ikawa, *On the scattering matrix for two convex obstacles,* Hyperbolic equations and related topics (Katata/Kyoto, 1984), 63–84, Academic Press, Boston, MA, 1986.

[11] V. Ivrii, *Microlocal analysis and precise spectral asymptotics*, Springer-Verlag, Berlin, 1998.

[12] N. Kaidi, P. Kerdelhue, *Forme normale de Birkhoff et résonances*, Asymptot. Analysis **23** (2000), 1–21.

[13] G. Lebeau, *Equation des ondes amorties*, in Algebraic and Geometric Methods of Mathematical Physics (Kaciveli 1993), 73–109, Math. Phys. Stud., 19 Kluwer Acad. Publ., Dordrecht, 1996.

[14] A. Melin, J. Sjöstrand, *Bohr-Sommerfeld quantization condition for non-self-adjoint operators in dimension* 2, Astérisque, to appear.

[15] G. Popov, *Invariant tori, effective stability, and quasimodes with exponentially small error terms.* II. *Quantum Birkhoff normal forms*, Ann. Inst. Henri Poincaré, Phys. Th. **1** (2000), 249–279.

[16] J. Sjöstrand, *Resonances associated to a closed hyperbolic trajectory in dimension 2*, preprint, Autumn 2002.

[17] J. Sjöstrand, *Asymptotic distribution of of eigenfrequencies for damped wave equations*, Publ. Res. Inst. Math. Sci. **36** (2000), 573–611.

[18] J. Sjöstrand, *Semi-excited states in non-degenerate potential wells*, Asymptot. Analysis **6** (1992), 29–43.

[19] J. Sjöstrand, *Singularités analytiques microlocales*, Astérisque 85 (1982).

[20] J. Sjöstrand and M. Zworski, *Asymptotic distribution of resonances for convex obstacles*, Acta Math. **183** (1999), 191–253.

[21] A. Weinstein, *Asymptotics of eigenvalue clusters for the Laplacian plus a potential*, Duke Math. J. **44** (1977), 883–892.