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VORTICITY AND SMOOTHNESS IN INCOMPRESSIBLE VISCOUS FLOWS

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Dedicated to Mitsuru Ikawa and Sadao Miyatake on the occasion of their 60.th birthday

Abstract

A quite simple relation between the direction and the amplitude of the vorticity is shown to be sufficient to guarantee the regularity of the weak solutions to the evolution Navier–Stokes equations in the three-dimensional case. See [3]. The proof is done by applying ideas introduced by Constantin and Fefferman [6] together to improvements due to the author and Berselli [4]. We follow this last reference in a straightforward way.

1 Introduction

Consider the evolution 3-D Navier–Stokes equations

$$(1.1)\qquad \begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)\, u - \nu \Delta u + \nabla p = 0 \quad \text{in } \mathbf{R}^3 \times [0,T], \\ \nabla \cdot u = 0 \quad \text{in } \mathbf{R}^3 \times [0,T], \\ u(x,0) = u_0(x) \quad \text{in } \mathbf{R}^3, \end{cases}$$

and define the vorticity

$$\omega(x,t) = \nabla \times u(x,t),$$

and also the direction of the vorticity

$$\xi(x) = \frac{\omega(x)}{|\omega(x)|}.$$

In this note we exhibit a simple relation between the amplitude $|\omega(x)|$ and the direction of the vorticity $\xi(x)$ which implies the regularity of the solutions to the above evolution Navier–Stokes equations. See [3]. The proof follows the method introduced by Constantin and Fefferman in reference [6] and improvements due to the author and Berselli [4]. Actually, the proofs follow quite directly from that in this last reference.

We denote by $|\cdot|_p$ the canonical norm in the Lebesgue space $L^p := L^p(\mathbf{R}^3)$, $1 \leq p \leq \infty$. $H^s := H^s(\mathbf{R}^3)$, $0 \leq s$, denotes the classical Sobolev spaces. Scalar and vector function spaces are indicated by the same symbol.

It is well known, see Leray [8], that given any fixed $T > 0$ there exists at least a *weak solution*

$$u \in C_w(0,T;L^2) \cap L^2(0,T;H^1),$$

of the system (1.1) in (0,T), where C_w indicates weak continuity. Moreover, the following energy estimate

$$\frac{1}{2}|u(t)|_2^2 + \nu \int_0^t \int_{\mathbf{R}^3} |\nabla u(x,\sigma)|^2 \, dx \, d\sigma \le \frac{1}{2}|u_0|_2^2 \tag{1.2}$$

holds, for each $t \in (0,T)$.

A weak solution such that

$$u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2) \tag{1.3}$$

is called a ***strong solution*** in $[0, T]$. Moreover, we say that u is a strong solution in $[0, T)$ if u is a strong solution in $[0, t]$, for each $t < T$. Strong solutions are regular, unique, and exist at least for some $T^* > 0$.

It is not known wether weak solutions are unique and strong solutions are global in time. Hence many efforts have been done to obtain significant conditions that guarantee the regularity of the weak solutions. For instance, in reference [2] it is proved that if

$$\omega \in L^p(0,T;L^q) \quad \text{for } \frac{2}{p} + \frac{n}{q} \le 2, \quad 1 \le p \le 2, \tag{1.4}$$

then the weak solution is regular. Note that the limit case $p = 1$ in (1.4) corresponds to the regularity condition $u \in L^1(0,T;W^{1,\infty})$, due to Beale, Kato, and Majda [1]. On the other hand, the limit case $p = 2$ is equivalent to $u \in L^2(0,T;W^{1,n})$. Note that this assumption is different from the classical one, namely $u \in L^2(0,T;L^\infty)$.

The sufficient condition (1.4) is an assumption on the amplitude of ω. In contrast, in reference [6], regularity is proved under an hypothesis on the direction of the vorticity. Let $\theta(x, x + y, t)$ denote the angle between the vorticity ω at points x and $x + y$ at time t. In [6] the authors essentially prove that if

$$|\sin\theta(x, x+y, t)| \le c\,|y|,$$

then the solution is necessarily smooth in $(0,T)$.

In [4], we improve this result by showing that

$$|\sin\theta(x, x+y, t)| \le c|y|^{1/2} \tag{1.5}$$

is sufficient to guarantee the regularity of weak solutions. More precisely, we prove that the following condition implies regularity:

For some $\beta \in [1/2, 1]$ and $g \in L^a(0,T;L^b)$, where

$$\frac{2}{a} + \frac{3}{b} = \beta - \frac{1}{2}, \quad a \in \left[\frac{4}{2\beta - 1}, \infty\right],$$

one has

$$|\sin\theta(x, x+y, t)| \le g(t,x)|y|^\beta \tag{1.6}$$

in the region where the vorticity at both points x and $x+y$ is larger than an arbitrary fixed positive constant K.

In this note we consider the case in which $\beta \in [0,1/2]$ and give a sufficient condition for the regularity of weak solutions that involves, simultaneously, the modulus and the direction of the vorticity. We prove the following assertion.

Theorem 1.1. *Let u be a weak solution of* (1.1) *in* $(0,T)$ *with* $u_0 \in H^1$ *and* $\nabla \cdot u_0 = 0$. *Let* $\beta \in [0,1/2]$ *and assume that*

$$|\sin\theta(x, x+y, t)| \leq c|y|^\beta \tag{1.7}$$

in the region where the vorticity at both points x and $x+y$ is larger than an arbitrary fixed positive constant K. Moreover, suppose that

$$\omega \in L^2(0,T; L^r), \tag{1.8}$$

where

$$r = \frac{3}{\beta+1}. \tag{1.9}$$

Then the solution u is strong in $[0,T]$ *and, consequently, is regular.*

The above hypotheses may be relaxed by assuming that (1.6) and (1.7) are satisfied only for $|y| \leq \delta$, for an arbitrary positive constant δ.

It is worth noting that in the two extreme cases, $\beta = 1/2$ and $\beta = 0$, the above result coincides with two already known results. In fact, for $\beta = 1/2$, the assumptions in Theorem 1.1 reduce just to the assumption (1.5) (note that $\beta = 1/2$ implies $r = 2$. Hence (1.8) is satisfied due to (1.2)). Hence, when $\beta = 1/2$, our thesis follows from [4]. On the other hand, if $\beta = 0$, the proof given below fails. However, the statement in Theorem 1.1 still holds as a consequence of the results proved by us in [2]. In fact, the assumption (1.4) is satisfied for $n = q = 3$ and $p = 2$ (since $r = 3$).

For the readers convenience we recall some results proved in [5]. By differentiating the Biot–Savart law

$$u(x) = -\frac{1}{4\pi}\int_{\mathbf{R}^3} \left(\nabla \frac{1}{|y|}\right) \times \omega(x+y)\, dy, \tag{1.10}$$

we obtain the following expression for the strain matrix:

$$S[\omega](x) = \frac{1}{2}[\nabla u(x) + (\nabla u(x))^*] = \frac{3}{4\pi}\,\mathrm{P.V.}\int_{\mathbf{R}^3} M(\widehat{y}, \omega(x+y))\frac{dy}{|y|^3}, \tag{1.11}$$

where $\widehat{y}$ is the unit vector in the direction of y, and

$$M(\widehat{y}, \omega) = \frac{1}{2}\left[\widehat{y} \otimes (\widehat{y} \times \omega) + (\widehat{y} \times \omega) \otimes \widehat{y}\right]$$

is a symmetric traceless matrix that defines a proper singular operator since, for each fixed ω, its mean value on the unit sphere vanishes. Define

$$\alpha(x) = S[\omega](x)\xi(x) \cdot \xi(x)\,, \tag{1.12}$$

on the set $\{x \in \mathbf{R}^3 : |\omega(x)| > 0\}$. From (1.11) it follows that (see [5])

$$\alpha(x) = \frac{3}{4\pi}\,\mathrm{P.V.}\int_{\mathbf{R}^3} D(\widehat{y}, \xi(x+y), \xi(x))|\omega(x+y)|\frac{dy}{|y|^3}, \tag{1.13}$$

where

$$D(a,b,c) = (a \cdot c)\,\mathrm{Determinant}(a,b,c). \tag{1.14}$$

2 Proof of Theorem 1.1

Here we prove the Theorem 1.1. The proof follows that given in reference [3], which, in turn, is an adaptation of that in [4] (to which the reader is referred for details). For convenience we will assume that the constant K in Theorem 1.1 vanishes, since the main difficulties are already present in this case. For the general case we refer the reader to [4].

Since $u_0 \in H^1$, the solution is strong in $[0, \tau)$, for some $\tau > 0$. Let $\tau \leq T$ be the maximum of these values. We will show that, under this hypothesis, u is strong in $[0, \tau]$. Hence, by a continuation principle (note that $u(\tau) \in H^1$), u is strong in $[\tau, \tau + \varepsilon)$. This shows that $\tau = T$.

By taking the curl of the first equation in (1.1) we find

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\,\omega - \nu\Delta\omega = (\omega \cdot \nabla)\,u. \tag{2.1}$$

Multiplication of this equation by ω, and integration by parts, yield

$$\frac{1}{2}\frac{d}{dt}|\omega|_2^2 + \nu|\nabla\omega|_2^2 = \int_{\mathbf{R}^3} S[\omega](x)\,\omega(x)\cdot\omega(x)\,dx. \tag{2.2}$$

By (1.12) we get

$$S[\omega]\omega \cdot \omega = |\omega|^2 \alpha\,\xi \cdot \xi,$$

and, by using (1.13),

$$S[\omega]\omega \cdot \omega = \frac{3}{4\pi}|\omega(x)|^2\,\mathrm{P.V.}\int_{\mathbf{R}^3} D(\widehat{y}, \xi(x+y), \xi(x))|\omega(x+y)|\frac{dy}{|y|^3}.$$

Since, by (1.7),

$$|D(\widehat{y}, \xi(x+y), \xi(x))| \leq c|y|^\beta,$$

it readily follows that

$$|S[\omega](x)\omega(x)\cdot\omega(x)| \leq \frac{3}{4\pi}|\omega(x)|^2 I(x),$$

where

$$I(x) = \int_{\mathbf{R}^3} |\omega(x+y)|\frac{dy}{|y|^{3-\beta}}.$$

Recall that here $\beta > 0$.

The Hardy–Littlewood–Sobolev inequality in R^n, for $n = 3$ (cf. Stein [9, Chap. V, Sec. 1.2]) states that if $f \in L^r$, for $1 < r < 3$, then

$$I(x) = \int_{\mathbf{R}^3} \frac{f(x+y)}{|y|^{3-\beta}}\, dy$$

belongs to L^q, $1 < q < \infty$, for $1/q = 1/r - \beta/3$. Furthermore, the map $f \mapsto I$ is linear and continuous from L^r into L^q. Using this inequality, with β and r as in (1.9) (hence $q = 3$), we get

$$|I(x)|_3 \leq c|\omega|_r.$$

Using Hölder's inequality with exponents 3, 2, and 6, one shows that

$$\left| \int_{\mathbf{R}^3} S[\omega](x)\omega(x) \cdot \omega(x)\, dx \right| \leq c|\omega|_r |\omega|_6 |\omega|_2. \tag{2.3}$$

Since $|\omega|_6 \leq c|\nabla\omega|_2$, from (2.3) it follows that

$$\left| \int_{\mathbf{R}^3} S[\omega](x)\omega(x) \cdot \omega(x)\, dx \right| \leq \frac{\nu}{4}|\nabla\omega|_2^2 + c\nu^{-1}|\omega|_r^2|\omega|_2^2. \tag{2.4}$$

From (2.2) and (2.4) we find

$$\frac{d}{dt}|\omega|_2^2 + \nu|\nabla\omega|_2^2 \leq c\nu^{-1}|\omega|_r^2\, |\omega|_2^2. \tag{2.5}$$

By the assumption (1.8), it follows that $|\omega|_r^2$ is integrable on $(0,\tau)$. Hence, a standard argument show that $\omega \in L^\infty(0,\tau;L^2) \cap L^2(0,\tau;H^1)$. It follows that u satisfies (1.3) on $(0,\tau)$. Hence u is a strong solution in the closed interval $[0, \tau]$. Now, we can extend u, as a strong solution, to some interval $[\tau, \tau+\varepsilon)$, by starting from the "initial" data $u(\tau)$, which belongs to H^1.

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