SMOOTHNESS OF HIGHER ORDER TERMS IN A BACKSCATTERING TRANSFORMATION

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ABSTRACT. The consideration of backscattering data of Schrödinger operators $H_v = |D|^2 - v$ in $\mathbb{R}^n$, when $n \geq 3$ is odd, motivates the introduction of a nonlinear transformation $v \mapsto Bv$ from $L^q_{\text{comp}}(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$ when $q > n$. We define $Bv$ by considering the wave group associated to the equation $(\partial_t^2 - \Delta_v - v(x))K(x,t) = 0$. Simple estimates show that $Bv$ is entire analytic in $v$. When $v$ is sufficiently small and real-valued, $Bv$ is uniquely determined from the backscattering data. If $n = 3$ and $\nabla v$ has a small norm in $L^1$ it is known also that $v$ is uniquely determined by $Bv$. We prove that the $N$-th order term $B_Nv$ in the power series expansion of $Bv$ is $\mu_N$-times continuously differentiable for $N$ large, where $\mu_N/N \to 1 - n/q$ as $N \to \infty$.

1. INTRODUCTION

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces and $B(\mathcal{H}, \mathcal{K})$ be the space of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. Denote by $C^k([0, \infty); B(\mathcal{H}, \mathcal{K}))$ the space of mappings

$$[0, \infty) \ni t \mapsto A(t) \in B(\mathcal{H}, \mathcal{K})$$

which are $k$ times continuously differentiable in the strong sense, i.e. $t \mapsto A(t)f \in \mathcal{K}$ is a $C^k$-mapping for every $f \in \mathcal{H}$. Let $\mathcal{H}_s$ be the standard Sobolev space of functions in $\mathbb{R}^n$ with all derivatives up to order $s$ in $L^2(\mathbb{R}^n)$ so that $\mathcal{H}_0 = L^2(\mathbb{R}^n)$. When $v \in L^q(\mathbb{R}^n)$ and $q \geq n/2$ it follows from the Sobolev embedding theorem that the operator $M_v$, multiplication by $v$, is continuous from $\mathcal{H}_s$ to $\mathcal{H}_0$. The Schrödinger operator $H_v = -\Delta - M_v = H_0 - M_v$ is therefore a continuous linear operator between the same spaces.

Main assumptions: It will be assumed throughout this paper that $n \geq 3$ is odd and that $n < q \leq \infty$.

In Section 2 we shall present a simple proof of the following theorem.

Theorem 1. Assume $v \in L^q(\mathbb{R}^n)$ (with $q$ as above). Then there is a unique

$$K_v \in C^2([0, \infty); B(\mathcal{H}_2, \mathcal{H}_0)) \cap C^0([0, \infty); B(\mathcal{H}_2, \mathcal{H}_2))$$

such that

$$K_v'f + H_vK_vf = 0,$$

and

$$K_v(0)f = 0, \quad K_v'(0)f = f$$

when $f \in \mathcal{H}_2$.

The family of operators $K_v(t)$, $t \geq 0$ will sometimes be referred to as the wave group. We are also going to use the following properties of $K_v$, where $K_v(x,y,t)$ denotes the distribution kernel of $K_v(t)$:

$$|x - y| \leq t \quad \text{in the support of } K_v(x,y,t) \quad \text{with equality when } v = 0.$$

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(4) \[ K_v \in C^0([0, \infty); \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1)), \]

and

(5) \[ K_v \in C^1([0, \infty); \mathcal{B}(\mathcal{H}_0, \mathcal{H}_0)). \]

It follows from Sobolev’s embedding theorem and (4) (with \( v = 0 \)) that \( K_0(t) \) is continuous from \( L^2 \) to \( L^p \) when \( 2 \leq p \leq 2n/(n - 2) \). Hence \( M_v K_0 \in C^0([0, \infty); \mathcal{B}(\mathcal{H}_0, \mathcal{H}_0)) \) by Hölder’s inequality when \( v \in L^2 \), and it follows then from (5) that \( K'_v(t) M_v K_0(t) \) is a strongly continuous family of bounded operators on \( L^2(\mathbb{R}^n) \).

Let \( L_{\text{comp}}^2(\Omega) \) be the space of functions in \( L^p(\mathbb{R}^n) \) with compact support contained in \( \Omega \), where \( \Omega \subset \mathbb{R}^n \) are open bounded sets. Assume that \( v \in L_{\text{comp}}^2(\mathbb{R}^n) \). It follows from property (3) that for every \( \Omega \) there is a constant \( T = T(\Omega, \text{supp}(v)) \) such that \( M_v K_0(t) f = 0 \) when \( f \in L_{\text{comp}}^2(\Omega) \) and \( t \geq T \). Another application of property (3) shows that the union of the supports of the \( K'_v(t) M_v K_0(t) f \) when \( t \) ranges from 0 to \( \infty \) is contained in a compact set which depends on \( \Omega \) and \( \text{supp}(v) \) only. It follows that the operator \( G = G_v \) defined by

(6) \[ G f = \int_0^\infty K'_v(t) M_v K_0(t) f \, dt \]

is a continuous linear operator on \( L_{\text{comp}}^2(\mathbb{R}^n) \). Since \( v \in L^2 \) the operator \( M_v G \) is continuous from \( L_{\text{comp}}^2 \) to \( L^1 \), and hence also from \( G^\infty(\mathbb{R}^n) \) to \( \mathcal{E}'(\mathbb{R}^n) \). Let \((M_v G)(x, y)\) denote its distribution kernel. A linear change of variables in \( \mathbb{R}^n \times \mathbb{R}^n \) allows us to consider the distribution \((M_v G)(y, 2x - y)\). Since this distribution is compactly supported in \( y \), we may define its integral with respect to that variable, formally written as \( \int v(y) G(y, 2x - y) \, dy \). This procedure gives rise to a nonlinear mapping from \( L_{\text{comp}}^2(\mathbb{R}^n) \) to \( \mathcal{D}'(\mathbb{R}^n) \), and we adopt the following definition:

**Definition 2.** The backscattering transform \( B_v \) of \( v \in L_{\text{comp}}^2(\mathbb{R}^n) \) is defined by

\[ B_v(x) = v(x) - 2^n \int v(y) G(y, 2x - y) \, dy, \]

where \( G \) is defined by (6).

Our terminology is motivated by the following. In the case when \( v \) is real-valued, compactly supported and satisfies some weak regularity conditions we have a scattering matrix corresponding to the two unitary groups \( e^{-itH_0} \) and \( e^{-it\theta} \). Its anti-diagonal part is a function depending on the parameters \((k, \theta)\) where \( k \in \mathbb{R}_+ \) and \( \theta \in S^{n-1} \). Viewing these as polar coordinates in frequency space and taking the inverse Fourier transform we get a distribution in \( \mathbb{R}^n \). The real part of that distribution is after suitable normalization equal to the backscattering transform \( B_v \) defined above apart from a smooth term which is due to bound states that may occur when \( v \) becomes large. We refer to Lagergren [L] (in the case when \( n = 3 \) and \( H_v \) has no bound states) and to a forthcoming paper by the author to a proof of these facts in arbitrary odd dimension (see also [M]). The advantage of this approach is that it gives a representation of backscattering data without reference to wave operators, and that there is no need to let the time parameter (in \( K_v(t) \)) tend to infinity when studying the local behaviour of the backscattering transform as long as the potentials are compactly supported. In other words, we take advantage of the finite speed of propagation in the wave equation, and in particular the validity of Huygens’ principle in odd dimension. (For more extensive discussions on an approach to backscattering closely related to Lax-Phillips theory of scattering we refer to Uhlmann [U] and Wang [W].)

Inverse backscattering deals with the recovery of \( v \) from the backscattering data. (See [ER1] and [ER2].) In view of the previous discussions the recovery of \( v \) from \( B_v \) is closely related to the inverse backscattering problem. Since the leading part of \( B_v \) equals \( v \) one is tempted, at least when considering small potentials, to view the backscattering transformation as a nonlinear perturbation of the identity. The problem is then to find suitable spaces of functions to work within. In the case when \( n = 3 \) it turns out (see [L]) that the completion of \( C^\infty \) in the norm
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\[ \|\nabla v\|_{L^1} \] is a space for which \( v \mapsto Bv \) is a homeomorphism in a neighbourhood of the origin. A natural candidate in the \( n \)-dimensional case, when \( n > 3 \) is odd, is the completion of \( C_0^\infty \) in the norm \( \|\nabla^{n-2} v\|_{L^1} \).

A more modest version of the inverse backscattering problem would be to compare the singularities of \( Bv \) with those of \( v \) (see [J] and [OPS]). This paper will focus on some aspects of this question. As we shall see, \( Bv \) is an entire analytic function of \( v \) when viewed as an element of \( \mathcal{D}'(\mathbb{R}^n) \). Thus

\[ Bv = \sum_{1}^{\infty} B_N^v \]

with convergence in \( \mathcal{D}'(\mathbb{R}^n) \), where \( B_N^v \) is the part of \( Bv \) that is homogeneous of degree \( N \) in \( v \). The main result of this paper, Theorem 8, says that the smoothness of \( B_N^v \) increases with \( N \). In fact, we are going to prove that \( B_N^v \in C^{\mu_N}(\mathbb{R}^n) \) for \( N \) large where

\[ \mu_N/N \to 1 - n/q \quad \text{as} \quad N \to \infty. \]

Also, \( \sum_{\mu_N \geq k} B_N^v \) is convergent in \( C^k(\mathbb{R}^n) \) for every \( k \). This means that we may for every \( k \) write \( B \) as a sum of a map which is a polynomial in \( v \) and a map which is continuous from \( L^2_{\text{comp}} \) to \( C^k \). A study of the finer regularity properties of \( Bv \) may therefore be reduced to the individual terms \( B_N^v \). Part of these results, which will be proved in the last section, may be summed up in the following theorem.

**Theorem 3.** The backscattering transformation \( B \) may for any nonnegative integer \( k \) be written as a sum \( B = B_{\text{pol}} + B_{\text{smooth}} \), where \( B_{\text{pol}} \) is a polynomial mapping and \( B_{\text{smooth}} \) is continuous from \( L^2_{\text{comp}}(\mathbb{R}^n) \) to \( C^k(\mathbb{R}^n) \).

2. **Proof of Theorem 1 and properties of the wave group**

**Proof of the uniqueness part of Theorem 1.** We have to prove that \( f(t) \equiv 0 \) if

\[ f \in C^2([0, \infty); \mathcal{H}_0) \cap C^0([0, \infty); \mathcal{H}_2), \quad f(0) = f'(0) = 0, \]

and

\[ f''(t) + H_{0} f(t) = 0. \]

Set

\[ G(t) = \|f'(t)\|^{2} + ((I + H_{0}) f(t), f(t)) \]

and

\[ g_{\epsilon}(t) = ((I + H_{0})(I + \epsilon H_{0})^{-1} f(t), f(t)) \]

when \( 0 \leq \epsilon \). When \( \epsilon \to 0 \) we have

\[ g_{\epsilon}'(t) = 2 \text{Re}((I + H_{0})(I + \epsilon H_{0})^{-1} f(t), f'(t)) \]

which converges in \( L^1_{\text{loc}}(\mathbb{R}_{+}) \) to the continuous function

\[ h(t) = 2 \text{Re}((I + H_{0}) f(t), f'(t)) \]

when \( \epsilon \to 0 \). Since \( g_{\epsilon} \) converges to \( g_0 \) in \( L^1_{\text{loc}}(\mathbb{R}_{+}) \) when \( \epsilon \to 0 \) it follows that \( g_0 \) is a \( C^1 \) function in \( \mathbb{R}_{+} \) and that \( g_0 = h \). Hence \( G \in C^1(\mathbb{R}_{+}) \) and

\[ G'(t) = 2 \text{Re}(f''(t), f'(t)) + h(t) = 2 \text{Re}(f''(t) + (I + H_{0}) f(t), f'(t)) \]

\[ = 2 \text{Re}((1 + v) f(t), f'(t)). \]

Since \( v \in L^q \) and \( (I + H_{0})^{-1/2} \) is continuous from \( L^2 \) to \( L^p \) when \( \frac{1}{2} - \frac{1}{n} \leq \frac{1}{p} \leq \frac{1}{2} \) we may estimate the norm in \( L^2 \) of \( vf \) by a constant times the norm in \( L^2 \) of \( (I + H_{0})^{1/2} f \). Hence, there is a constant \( C \) such that

\[ G'(t) \leq CG(t), \quad t > 0, \]

and since \( G(0) = 0 \) we may conclude that \( G \) vanishes identically. \( \square \)
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We need some simple preparations in order to construct $K_v$. In the case when $v$ is real one must have $K_v(t) = t\sigma(t^2H_v)$, where $\sigma$ is the unique entire analytic function which satisfies $\sigma(t^2) = (\sin t)/t$ when $t \in \mathbb{R}$. Since we allow $v$ to be complex-valued, and since we are going to need rather precise information about $K_v$, we shall construct it by considering convolutions of operator valued functions on $\mathbb{R}_+$. 

**Convolutions of operator valued functions.** Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces and recall that $C^k([0, \infty); B(\mathcal{H}, \mathcal{K}))$ denotes the space of mappings

$$[0, \infty) \ni t \mapsto A(t) \in B(\mathcal{H}, \mathcal{K})$$

which are $k$-times continuously differentiable in the strong sense. We equip this space with the topology defined by the semi-norms

$$\|A\|_{T,f} = \sum_{0 \leq j \leq k} \max_{0 \leq t \leq T} \|A^{(j)}(t)f\|, \quad T \geq 0, f \in \mathcal{H}.$$

Under this topology $C^k([0, \infty); B(\mathcal{H}, \mathcal{K}))$ becomes a Fréchet space. We say that an element $A$ in $C^k([0, \infty); B(\mathcal{H}))$ is simple if $A(t) = f(t)A_0$ where $A_0 \in B(\mathcal{H}, \mathcal{K})$ is independent of $t$ and $f \in C^k([0, \infty))$. The finite linear combinations of simple elements form a dense subspace of $C^k([0, \infty); B(\mathcal{H}, \mathcal{K}))$, and if $A \in C^k([0, \infty); B(\mathcal{H}, \mathcal{K}))$, then the integral $\int_0^t A(s) ds$ is an element in $C^{k+1}((0, \infty); B(\mathcal{H}, \mathcal{K}))$ with derivative $A(t)$.

Assume that $A \in C^0([0, \infty); B(\mathcal{K}, \mathcal{L}))$ and that $B \in C^0([0, \infty); B(\mathcal{H}, \mathcal{K}))$, where $\mathcal{H}, \mathcal{K}$ and $\mathcal{L}$ are Hilbert spaces. Define

$$(A \ast B)(t) = \int_0^t A(t-s)B(s) ds = \int_0^t A(s)B(t-s) ds.$$

Then $A \ast B \in C^0([0, \infty), B(\mathcal{H}, \mathcal{L}))$. The convolution is associative, i.e.

$$(A \ast B) \ast C = A \ast (B \ast C)$$

when $A, B$ and $C$ take values in appropriate spaces so that the convolutions are defined. For reasons of continuity and linearity it suffices to prove this when $A, B$ and $C$ are simple, and then it follows from the corresponding properties for convolution of scalar valued functions. We shall use the fact that if $A \in C^1([0, \infty); B(\mathcal{K}, \mathcal{L}))$ and $B \in C^0([0, \infty); B(\mathcal{H}, \mathcal{K}))$, then $A \ast B \in C^1([0, \infty); B(\mathcal{H}, \mathcal{L}))$ and

$$(A \ast B)' = A' \ast B + A(0)B.$$

When $A \in C^0, B \in C^1$ we have instead $(A \ast B)' = A \ast B' + AB(0)$.

If $f$ and $g$ are locally integrable function on $[0, \infty)$ we define their convolution by

$$(f \ast g)(t) = \int_0^t f(t-s)g(s) ds.$$

In this formula we may replace $g$ by $G$ where $G \in C^0([0, \infty); B(\mathcal{H}, \mathcal{K}))$. Then we get an element $f \ast G$ in the same space of operator valued functions. The obvious laws of associativity hold so that in particular $C^0([0, \infty); B(\mathcal{H}, \mathcal{K}))$ becomes a module with respect to the convolution algebra of locally integrable functions on $[0, \infty)$.

Since it will be important for us also to consider fractional derivatives of operator valued functions we need one more definition. Set

$$B = B(\mathcal{H}_0, \mathcal{H}_0) = B(L^2, L^2)$$

and define

$$X_0 = C^0([0, \infty); B).$$

In order to define $X_a$ when $a > 0$ we introduce

$$X_a(t) = t^{a-1}/\Gamma(a), \quad t > 0.$$
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Then \( \chi_a \ast \chi_b = \chi_{a+b} \). If \( A \in X_0 \) we say that \( A \in X_a \) if \( A = \chi_a \ast B \), where \( B \in X_0 \). If \( a = k \) is a positive integer this implies that \( A \in C^k([0, \infty); B) \) and \( B = A^{(k)} \). If \( a = k + b \) where \( 0 < b < 1 \) then \( A^{(k)} = \chi_b \ast B \) and \( B = C' \) where \( C = \chi_{1-b} \ast A^{(k)} \in X_1 \). It follows that \( B \) is uniquely determined by \( A \) and we write \( B = A^{(a)} \). The following lemma is immediate from the definitions.

**Lemma 4.** Assume \( A \in X_a \) and \( B \in X_b \) then \( A \ast B \in X_{a+b} \) and

\[
(A \ast B)^{(a+b)} = A^{(a)} \ast B^{(b)}.
\]

If \( 0 \leq a \leq b \) then \( X_b \subseteq X_a \), and if \( A \in X_b \) then \( A^{(a)} = \chi_{b-a} \ast A^{(b)} \).

**Mapping properties of \( K_0 \).** It is easily verified that the conditions (1)–(5) are satisfied by

\[
K_0(t) = (\sin t|D|)/|D| \quad \text{where} \quad D = \partial i \text{ and } |D| = H_0^{1/2}.
\]

This is a convolution operator, and its distribution kernel \( k_0(x,t) \) is supported in the wave cone \( |x| = t \). We notice that \( K_0(t) \) extends to a continuous operator on \( S'(\mathbb{R}^n) \). We have \( K_0(t) = \cos(t|D|) \), and \( K_0 \in X_1 \) since \( K_0(0) = 0 \). If \( 0 < a < 1 \) then

\[
K_0^{(a)} = \chi_{1-a} \ast K_0.
\]

From this follows that

\[
K_0^{(a)}(t) = |D|^{a-1} h_a(t|D|),
\]

where

\[
h_a(t) = \int_0^t (t-s)^{-a} \cos s ds/\Gamma(1-a)
\]

is a bounded function. Since \( |D|^{-1} \) is convolution by a constant times \( |x|^{1-n} \) it follows from formula (10) and the Hardy-Littlewood-Sobolev (HLS) inequality (see [H], Sec. 4.5) that \( K_0(t) \) is continuous from \( L^p \) to \( L^2 \) and from \( L^2 \) to \( L^{p'} \) when \( \frac{1}{p} \in \left[ \frac{1}{2}, \frac{1}{2} + \frac{1}{n} \right] \) and \( p' \) is the conjugated exponent, i.e. \( \frac{1}{p} + \frac{1}{p'} = 1 \). It follows from Hölder's inequality then that the operators

\[
Y_-(t) = K_0(t) M_v, \quad Y_+(t) = M_v K_0(t)
\]

are continuous in \( L^2 \), and from some simple estimates one deduces that \( Y_\pm \in C^0([0, \infty); B) = X_0 \).

Define \( \delta = \delta_q \in (0,1] \) by

\[
\delta = 1 - \frac{n}{q}.
\]

**Lemma 5.** We have \( Y_\pm \in X_\delta \), and there is a constant \( C = C_{q,n} \), which depends on \( q \) and \( n \) only, such that

\[
\| Y_\pm^{(\delta)}(t) \| \leq C \|v\|_{L^q}, \quad t \geq 0.
\]

**Proof.** Since \( |D|^{\delta-1} \) is convolution by a constant times \( |x|^{1-\delta-n} \) it follows from (12) and the HLS-inequality that \( K_0^{(\delta)} \) is continuous from \( L^r \) to \( L^2 \) and from \( L^2 \) to \( L^{r'} \), where

\[
\frac{1}{r} = \frac{1}{2} + \frac{1-\delta}{n} = \frac{1}{2} + \frac{1}{q}.
\]

It follows then from Hölder's inequality that \( M_v K_0^{(\delta)}(t) \) and \( K_0^{(\delta)}(t) M_v \) are continuous operators in \( L^2 \), and as such they are strongly continuous in \( t \). The operator norm may be estimated from above by \( C \|v\|_{L^q} \). The lemma follows since

\[
Y_- = \chi_\delta \ast (M_v K_0^{(\delta)}), \quad Y_+ = \chi_\delta \ast (K_0^{(\delta)} M_v).
\]
The construction of $K_v$. Let $q \in (n, \infty]$ and $v \in L^q(\mathbb{R}^n)$ be as before. Define $K_N$ inductively when $N \geq 1$ by
\begin{equation}
K_N = Y_- * K_{N-1}.
\end{equation}
Since $Y_- \in X_\delta$ by Lemma 5, and since $K_0 \in X_1$, it follows by induction over $N$ that
\begin{equation}
K_N \in X_{N\delta+1}, \quad N \geq 1.
\end{equation}
An application of Lemma 4 and Lemma 5 shows that
\begin{align*}
\|K_N^{(1+N\delta)}\| &\leq \|Y_-^{(\delta)}\| \ast \|K_{N-1}^{(1+(N-1)\delta)}\| \\
&\leq C \|v\|_{L^q} \chi_1 \ast \|K_{N-1}^{(1+(N-1)\delta)}\| \\
&= C^2 \|v\|_{L^q} \chi_2 \ast \|K_{N-2}^{(1+(N-2)\delta)}\| \\
&\leq \cdots \leq C^N \|v\|_{L^q}^N \chi_N \ast \|K_0^{(1)}\| \\
&\leq C^N \|v\|_{L^q}^N \chi_N \ast \chi_1 = C^N \|v\|_{L^q \chi N+1}^N.
\end{align*}
Since $K_N^{(a)} = \chi_{1+N\delta-a} \ast K_N^{(1+N\delta)}$, when $0 \leq a < 1+N\delta$, it follows that $K_N \in X_a$ when $0 \leq a \leq 1+N\delta$, and one has the estimate
\begin{equation}
\|K_N^{(a)}(t)\| \leq C^N t^{1+N(1+\delta)-a} \|v\|_{L^q}^N / \Gamma(2+N(1+\delta)-a), \quad 0 \leq a \leq 1+N\delta.
\end{equation}
We now define
\begin{equation}
K_v = \sum_{0}^{\infty} K_N.
\end{equation}
It follows from (17) with $a = 1$ that the sum converges in $C^1([0, \infty); B)$. Hence condition (5) is fulfilled and (3) holds since $|x-y| = t$ in the support of the distribution kernel $K_0(x,y,t)$.

Lemma 6. We have $(K_v - K_0)(I + H_0)^{-1} \in X_2$.

Proof. Set $P = M_v(I + H_0)^{-1}$. Then $P$ is bounded on $L^2(\mathbb{R}^n)$ and
\begin{equation}
M_v K_0(I + H_0)^{-1} = PK_0 \in X_1
\end{equation}
since $K_0 \in X_1$. Since
\begin{equation}
K_1(I + H_0)^{-1} = K_0 \ast (M_v K_0(I + H_0)^{-1}) = K_0 \ast (PK_0),
\end{equation}
it follows from Lemma 4 that
\begin{equation}
K_1(I + H_0)^{-1} \in X_2.
\end{equation}
Let us introduce
\begin{equation}
V_N = Y_- \ast \cdots \ast Y_-, \quad W_N = Y_+ \ast \cdots \ast Y_+,
\end{equation}
where the number of factors equals $N$. Then
\begin{equation}
K_N = V_{N-1} \ast K_1 = K_1 \ast W_{N-1}, \quad N \geq 2.
\end{equation}
It follows from Lemma 4 and Lemma 5 that $V_{N-1}, W_{N-1} \in X_{(N-1)\delta}$. Hence (19) and (21) imply that
\begin{equation}
K_N(I + H_0)^{-1} = V_{N-1} \ast (K_1(I + H_0)^{-1}) \in X_{(N-1)\delta+2}, \quad N \geq 2.
\end{equation}
Arguments similar to those leading to (17) give the estimate
\begin{equation}
\|(K_N(I + H_0)^{-1})^{(N-1)\delta+2}\| \leq C^N \|v\|^N_{L^q \chi N}, \quad N \geq 1.
\end{equation}
The lemma is an immediate consequence of these estimates, since (22) implies that $K_N(I + H_0)^{-1} = \chi_2 \ast Z_N$ when $N \geq 1$, where \(\sum_{1}^{\infty} Z_N\) is convergent in $C^0([0, \infty), B)$. □
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It follows from the previous lemma that

\[ K_v \in C^2([0, \infty); \mathcal{B}(\mathcal{H}_2, \mathcal{H}_0)) \]

and that (2) holds. We need also to verify (1) and that

\[ K_v \in C^0([0, \infty); \mathcal{B}(\mathcal{H}_2, \mathcal{H}_2)), \]

or, equivalently, that

\[ (I + H_0)K_v(I + H_0)^{-1} \in C^0([0, \infty); \mathcal{B}). \]

We notice that

\[ K_{v-1}M_v = V_N \in X_{N\delta}, \quad M_vK_{v-1} = W_N \in X_{N\delta}, \quad N \geq 1, \]

since \( Y_{\pm} \in X \). Hence we have

\[ P_N \in X_{N\delta}, \quad \text{where} \quad P_N = (W_N - V_N)(I + H_0)^{-1}. \]

**Lemma 7.** Assume \( N \geq 1 \). Then

\[ K_N''(t) = V_N - K_NH_0, \quad \text{(on} \mathcal{H}_2) \]

and

\[ (I + H_0)K_N(t)(I + H_0)^{-1} = K_N(t) + P_N(t). \]

**Proof.** The estimates (17) and (22) (and their polarized versions) show that both sides of (26) and (27), viewed as mappings from \( S(\mathbb{R}^n) \) to \( S'(\mathbb{R}^n) \) depend continuously on \( v \in L^q \). It suffices therefore to prove the lemma when \( v \in C_0^\infty(\mathbb{R}^n) \). Consider first \( K_1 = (K_0M_v)*K_0 \). Since \( K_0 \in C^2([0, \infty); \mathcal{B}(\mathcal{H}_2, \mathcal{H}_0)) \), \( K_0(0) = 0, K'_0(0) = I \) and \( K''_0 = -K_0H_0 \), it follows that

\[ K''_1 = K_0M_v - K_1H_0 = V_1 - K_1H_0. \]

If \( N \geq 2 \) we write \( K_N = (K_{N-2}M_v)*K_1 \) and get

\[ K''_N = (K_{N-2}M_v)*K''_1 = (K_{N-2}M_v)*(K_0M_v) - (K_{N-2}M_v)*(K_1H_0) \]

\[ = K_{N-1}M_v - K_NH_0 = V_N - K_NH_0. \]

This proves (26). Since \( K_N \) is its own transpose we also have

\[ K''_N = M_vK_{N-1} - H_0K_N = W_N - H_0K_N. \]

Hence

\[ (H_0 + I)K_N = K_N(H_0 + I) + W_N - V_N \]

from which (27) follows. \( \square \)

We notice that (1) follows from (28). The only remaining part in the proof of Theorem 1 is therefore the assertion (24). The series \( \sum_{1}^{\infty} P_N \) converges in \( C^0([0, \infty); \mathcal{B}) \) and its sum \( (M_vK_v - K_vM_v)(I + H_0)^{-1} \) is an element in \( X_{\delta} \). It follows from Lemma 7 therefore that

\[ (I + H_0)K_v(t)(I + H_0)^{-1} = (M_vK_v(t) - K_v(t)M_v)(I + H_0)^{-1} + K_v(t) \in C^0([0, \infty); \mathcal{B}). \]

This completes the proof of Theorem 1.

We have already verified (3) and (5) and want to prove now that (4) holds. Since

\[ K_N = K_1 \ast W_{N-1}, \quad N \geq 2 \]

a summation over \( N \) gives

\[ K_v = K_0 + K_1 + K_1 \ast W, \]

where \( W = \sum_{1}^{\infty} W_N \in X_{\delta} \). It suffices therefore to observe that

\[ K_1 = K_0 \ast Y_+ \in C^0([0, \infty); \mathcal{B}(\mathcal{H}_0, \mathcal{H}_1)). \]
since $K_0$ is in that space.

3. **The Backscattering Transform**

Let $v \in L^q_{\text{comp}}(\mathbb{R}^n)$ where $q > n$. Define $G = G_v$ as in (6) and recall that the backscattering transformation $B$ was introduced in Definition 2.

Define $B_1 v = v$ and

$$
B_N v(x) = -2^n \int v(y) G_{N-1}(y, 2x - y) \, dy, \quad N > 1,
$$

where

$$
G_{N-1} = \int_0^\infty K'_{N-2}(t) M_v K_0(t) \, dt.
$$

It is a simple consequence from these definitions and the estimates in the previous section that

$$
Bv = \sum_{1}^{\infty} B_N v
$$

with convergence in $\mathcal{D}'(\mathbb{R}^n)$, and also that $Bv$ is entire analytic in $v$ when viewed as an element of that space.

The main result of this paper is a proof for the fact that the smoothness of $B_N v$ increases with $N$. (We shall not discuss the smoothness of the lower order terms in the expansion of $Bv$.) It follows from the theorem below that for every nonnegative integer $k$ there is a positive integer $N_k$ such that $B_N \in C^k$ when $N \geq N_k$, and $\sum_{N \geq N_k} B_N$ is convergent in $C^k(\mathbb{R}^n)$. Moreover, $k/N_k \to \delta = 1 - n/q$ as $k \to \infty$.

**Theorem 8.** Let $n^*$ be the smallest integer $> n/4$ and set $\delta = 1 - n/q$, where $q > n$. Assume $2(n^* + k) < (N - 2)\delta$. Then $\Delta^k B_N v \in L^q_{\text{loc}}(\mathbb{R}^n)$ when $v \in L^q_{\text{comp}}(\mathbb{R}^n)$. Moreover, if $\Omega_1$ and $\Omega_2$ are open bounded sets in $\mathbb{R}^n$ there is a constant $C = C_k$, depending on $k$, $\Omega_1$, $\Omega_2$ and $q$ only such that

$$
\left( \int_{\Omega_1} |\Delta^k B_N v(x)|^2 \, dx \right)^{1/2} \leq C_k^N \|v\|_{L^q_{\text{loc}}} / N!
$$

when $v \in L^q_{\text{comp}}(\Omega_2)$.

We notice that Theorem 3 in the introduction is an immediate consequence of this theorem and its polarized version, which we leave to the reader to formulate.

**Proof of the theorem.** Let $\Omega_1$ and $\Omega_2$ be open bounded sets in $\mathbb{R}^n$ and let $v \in L^q_{\text{comp}}(\Omega_2)$. If $f \in C^\infty(\mathbb{R}^n)$ then $F(t) = M_v K_0(t) f$ is a smooth function of $t$ with values in $L^2_{\text{comp}}(\mathbb{R}^n)$ and $F^{(2k)}(t) = M_v K_0(t) \Delta^k f$. It follows when $N \geq 2$ that

$$
G_{N-1} \Delta^{k+n^*} f = \int_0^\infty K'_{N-2}(t) F^{(2n^*+2k)}(t) \, dt.
$$

Since $2(n^* + k) < (N - 2)\delta$ it follows from (17) that

$$
K_{N-2} \in C^{2k+2n^*}([0, \infty); B),
$$

and its derivatives up to order $2k + 2n^*$ vanish at the origin. Integrating by parts $2k + 2n^*$ times we get

$$
G_{N-1} \Delta^{k+n^*} f = \int_0^\infty K_{N-2}^{(1+2n^*+2k)}(t) F(t) \, dt.
$$

Set $Q_{N-1,k} = G_{N-1} \circ \Delta^k$ and define

$$
G_{N-1,k} = \int_0^\infty K_{N-2}^{(1+2n^*+2k)}(t) M_v K_0(t) \, dt.
$$
SMOOTHNESS OF HIGHER ORDER TERMS IN A BACKSCATTERING TRANSFORMATION

This is a continuous operator on $L^2_{\text{comp}}$. Let $E$ be a properly supported pseudo-differential operator of order $-2n^*$ which is a parametrix of $\Delta^{n^*}$. Since $Q_{N-1,k} \circ \Delta^{n^*} = G_{N-1,k}$ we have

$$Q_{N-1,k} = G_{N-1,k} \circ E + Q_{N-1,k} \circ R$$

where $R$ is an integral operator with a smooth and properly supported kernel (i.e. the projections $\text{supp}(R) \ni (x, y) \to x$ and $\text{supp}(R) \ni (x, y) \to y$ are proper). Let $\varphi \in C_0^\infty(\mathbb{R}^n)$ and choose $\psi \in C_0^\infty(\mathbb{R}^n)$ such that $EM_{\varphi} = M_{\psi}EM_{\varphi}$. Then

$$Q_{N-1,k}M_{\varphi} = (G_{N-1,k}M_{\psi})EM_{\varphi} + Q_{N-1,k}(RM_{\varphi})$$

We notice that $G_{N-1,k}M_{\psi}$ is a continuous linear operator on $L^2_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}^n)$, and its distribution kernel is compactly supported. It follows from (17) that its norm in $B$ can be estimated from above by $C_k^N ||v||_{L^q}^N/N!$, where $C_k$ depends on $\Omega_2$, $q$ and $\psi$ only. Since $EM_{\varphi}$ is a Hilbert-Schmidt operator we get the same kind of estimate for the Hilbert-Schmidt norm of $G_{N-1,k}M_{\psi}EM_{\varphi}$, if let $C_k$ depend on $\varphi$ also. Writing $Q_{N-1,k}RM_{\varphi} = G_{N-1}(\Delta^kRM_{\varphi})$ we may also estimate the second term in the right-hand side of (32) in this way. Since $\varphi \in C_0^\infty$ was arbitrary it follows that $\Delta^kG_{N-1}(x, y) = Q_{N-1,k}(x, y)$ is in $C^{2(n^*+k)}_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}^n)$ and we have the estimates

$$\left( \int_{\mathbb{R}^n \times \Omega_0} |\Delta^kG_{N-1}(x, y)|^2 \, dx \, dy \right)^{1/2} \leq C_k^N ||v||_{L^q}^{N^*+k}/N!$$

when $v$ is supported in $\Omega_2$, $\Omega_0 \subset \mathbb{R}^n$ is an open bounded set and $2(n^*+k) < (N-2)\delta$. Here $C_k$ depends also on $\Omega_0$, $\Omega_2$ and $q$.

It is now a straightforward procedure to deduce the conclusion of the theorem from the inequality above. In fact, if one chooses $\Omega_0 = 2\Omega_1 - \Omega_2$, then Cauchy's inequality and the definition of $B_N$ gives the estimate

$$\int_{\Omega_1} |B_N(x)|^2 \, dx \leq 2^n ||v||_{L^2}^2 \int_{\Omega_0} |G_{N-1}(x, y)|^2 \, dx \, dy,$$

and the estimates for $\Delta^kB_N(x)$ are obtained by replacing $G_{N-1}$ in the right-hand side by $2^{2k} \Delta^kG_{N-1}(x, y)$ and then using (33).

REFERENCES


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