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Kyoto University
On the Number of Poles of the First Painlevé Transcendents and Higher Order Analogues II

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1. Introduction

Let $w(z)$ be an arbitrary solution of the first Painlevé equation

\[(\Pi) \quad w'' = 6w^2 + z\]

\(('=d/dz)\). Then, $w(z)$ is a transcendental meromorphic function, and every pole is double. The counting function for poles is defined by

\[N(r, w) = \int_{0}^{r} \left( n(\rho, w) - n(0, w) \right) \frac{d\rho}{\rho} + n(0, w) \log r,\]

where $n(r, w)$ denotes the number of poles inside the disk $|z| \leq r$, each counted according to its multiplicity. By a well-known argument in the Nevanlinna theory ([4, §2.4]), we have

\[(1.1) \quad \liminf_{r \to \infty} \frac{m(r, w)}{T(r, w)} = 0, \quad \text{namely,} \quad \limsup_{r \to \infty} \frac{N(r, w)}{T(r, w)} = 1,\]

which implies $N(r, w) \to \infty$ as $r \to \infty$. Here, $m(r, w)$ and $T(r, w)$ are, respectively, the proximity and the characteristic functions defined by

\[m(r, w) = \frac{1}{2\pi} \int_{0}^{2\pi} \log^{+} |w(re^{i\phi})| d\phi, \quad \log^{+} x = \max\{0, \log x\},\]

\[T(r, w) = m(r, w) + N(r, w)\]

(for the standard notation and basic facts in the Nevanlinna theory, see [2], [4]). For the magnitude of $N(r, w)$, the following is known ([1], [5], [6], [9]):

\[(1.2) \quad r^{5/2} \log r \ll N(r, w) \ll r^{5/2},\]

which implies that the growth order of $w(z)$

\[\sigma(w) = \limsup_{r \to \infty} \frac{\log T(r, w)}{\log r}\]
is equal to $5/2$. (We write $f(r) \ll g(r)$ (or $g(r) \gg f(r)$) if $f(r) = O(g(r))$ as $r \to \infty$.)

A sequence of higher order analogues of (PI) is given by the following:

$$(PI_{2\nu}) \quad d_{\nu+1}[w] + 4z = 0, \quad \nu \in \mathbb{N}$$

(cf. [1, §16]; [3]). Here, $d_{\nu}[w] \ (\nu = 0, 1, 2, \ldots)$ are differential polynomials in $w$ determined by

\begin{align*}
(1.3) & \quad d_{0}[w] = 1, \\
(1.4) & \quad Dd_{\nu+1}[w] = (D^{3} - 8wD - 4w')d_{\nu}[w], \quad D = d/dz, \quad \nu \in \mathbb{N} \cup \{0\}.
\end{align*}

Since

$$d_{2}[w]/4 = -w'' + 6w^2 + C_{1}w + C_{0},$$

where $C_{j} \in \mathbb{C} \ (j = 0, 1)$ are arbitrary, equation $(PI_{2})$ essentially coincides with (PI). In general, $(PI_{2\nu})$ is a $2\nu$-th order nonlinear equation; e.g. for $\nu = 2, 3$,

\begin{align*}
(PI_{4}) & \quad w^{(4)} = 20ww'' + 10(w')^2 - 40w^3 + z, \\
(PI_{6}) & \quad w^{(6)} = 28ww^{(4)} + 56w'w^{(3)} + 42(w''')^2 - 280(w^2w'' + w(w')^2 - w^4) + z,
\end{align*}

where the arbitrary constants corresponding to $C_{j}$ of $(PI_{2})$ are taken to be 0. Let $w_{\nu}(z)$ be an arbitrary meromorphic solution of $(PI_{2\nu})$. It is interesting to evaluate the growth order of $w_{\nu}(z)$. The following result gives a lower estimate of it:

**Theorem 1.1.** For every $\nu \in \mathbb{N}$,

$$\limsup_{r\to\infty} \frac{\log N(r, w_{\nu})}{\log r} \geq \frac{2\nu + 3}{\nu + 1},$$

namely the growth order of $w_{\nu}(z)$ is not less than $(2\nu + 3)/(\nu + 1)$.

As an immediate consequence, we have

**Corollary 1.2.** Equation $(PI_{2\nu})$ admits no rational solutions.

Viewing Theorem 1.1 combined with (1.2), we pose the following:

**Conjecture.** The growth order of $w_{\nu}(z)$ is equal to $(2\nu + 3)/(\nu + 1)$.

We sketch the proof of Theorem 1.1, illustrating the particular case $\nu = 2$. The full proof is found in [8].

**2. Sketch of the proof of Theorem 1.1 for (PI_{4})**

The basic idea is the same as in the proof for (PI) (cf. [7]). Suppose the contrary:

$$\limsup_{r\to\infty} \frac{\log N(r, w_{2})}{\log r} < \frac{7}{3},$$

$$\limsup_{r\to\infty} \frac{\log N(r, w_{4})}{\log r} < \frac{7}{3},$$
namely, for some $\epsilon > 0$, $N(r, w_2) \ll r^{7/3-\epsilon}$, from which it follows that

$$n(r) = n(r, w_2) \ll r^{7/3-\epsilon},$$

because

$$N(2r, w_2) \geq \int_r^{2r} (n(\rho, w_2) - n(0, w_2)) \frac{d\rho}{\rho} \geq (n(r, w_2) + O(1)) \log 2.$$  

Starting from (2.1), we will derive a contradiction. Let $\{a_j\}_{j=1}^\infty$ (or $\{a_j\}_{j=1}^q$, $q \in \mathbb{N}$) be the sequence of all distinct poles of $w_2(z)$ arranged as $|a_1| \leq \cdots \leq |a_j| \leq \cdots$. It is easy to check that, around each pole $a_j$,

$$w_2(z) = c(j)(z - a_j)^{-2} + O(1),$$

where $c(j) = 1$ or $3$. By this fact combined with (2.2), we write $w_2(z)$ in the form

$$w_2(z) = \Phi(z) + \varphi(z),$$

(2.3)

$$\Phi(z) = \sum_{j=1}^\infty c(j)((z-a_j)^{-2} - a_j^{-2}),$$

(2.4)

where $\varphi(z)$ is an entire function; and in (2.4), if $a_1 = 0$ the term $(z - a_1)^{-2} - a_1^{-2}$ should be replaced by $z^{-2}$. Under (2.2), we have the following lemmas whose proofs are similar to those of [7, Lemmas 1.1 and 1.2].

**Lemma 2.1.** For every $r > 1$, there exists $z_r$ satisfying

$$0.7r \leq |z_r| \leq r, \quad \sum_{|a_j| < 2r} |z_r - a_j|^{-2} \ll r^{1/3-\epsilon/2}, \quad \sum_{|a_j| < 2r} |z_r - a_j|^{-3} \ll r^{1/2-\epsilon}.$$  

**Lemma 2.2.** Let $r$ be an arbitrary number satisfying $r > 1$. Then,

$$\sum_{|a_j| \geq 2r} |(z - a_j)^{-2} - a_j^{-2}| \ll r^{1/3-\epsilon}, \quad \sum_{|a_j| \geq 2r} |z - a_j|^{-3} \ll 1$$

for $|z| \leq r$, and

$$\sum_{0 < |a_j| < 2r} |a_j^{-2}| \ll r^{1/3-\epsilon}.$$  

By a well-known argument of the Nevanlinna theory, it is shown that $\varphi(z)$ is a polynomial. Note that $|\Phi(z)| \leq \sum_{|a_j| < 2r} + |\sum_{|a_j| \geq 2r}|$. By Lemmas 2.1 and 2.2, for every $r > 1$, there exists $z_r$, $0.7r \leq |z_r| \leq r$ satisfying

$$|\Phi(z_r)| \ll r^{1/3-\epsilon/2}, \quad |\Phi'(z_r)| \ll r^{1/2-\epsilon},$$

$$|\Phi''(z_r)| \ll r^{2/3-\epsilon}, \quad |\Phi^{(4)}(z_r)| \ll r^{1-3\epsilon/2}.$$
\[(2.6) \quad w_2(z_r) \ll (|w_2^{(4)}(z_r)| + |w_2(z_r)||w_2''(z_r)| + |w_2'(z_r)|^2 + |z_r|)^{1/3} \]

\[\ll |w_2^{(4)}(z_r)|^{1/3} + |w_2(z_r)||w_2''(z_r)|^{1/3} + |w_2'(z_r)|^{2/3} + |z_r|^{1/3}.\]

Substituting \(w_2^{(k)}(z_r) = \varphi^{(k)}(z_r) + \Phi^{(k)}(z_r)\) (\(k = 0, 1, 2, 4\)) into (2.6) and using \(|\Phi^{(k)}(z_r)| \ll r^{1/3+k/6}\) (cf. (2.5)), we have

\[(1) \quad |\varphi(z_r)| \ll r^{1/3} + |\varphi^{(4)}(z_r)|^{1/3} + (r^{1/9} + |\varphi(z_r)|^{1/3})(r^{2/9} + |\varphi''(z_r)|^{1/3}) + r^{1/3} + |\varphi'(z_r)|^{2/3},\]

which implies that \(\varphi(z) \equiv C \in \mathbb{C}\). Then, by \((\text{PI}_4)\),

\[0.7r \leq |z_r| \ll |w_2^{(4)}(z_r)| + |w_2(z_r)||w_2''(z_r)| + |w_2'(z_r)|^2 + |w_2(z_r)|^3 \ll r^{1-\epsilon},\]

which is a contradiction. Thus Theorem 1.1 with \(\nu = 2\) follows.

3. General case

To treat the general case, we need to know some facts related to the terms of the differential polynomial \(d_{\nu+1}[w]\). Let \([w, w', \ldots, w^{(p)}]^\iota\) denote the monomial \(w^{\iota_0}(w')^{\iota_1}\cdots(w^{(p)})^{\iota_p}\), where \(\iota = (\iota_0, \iota_1, \ldots, \iota_p) \in (\mathbb{N} \cup \{0\})^{p+1}\). For this monomial with \(\iota = (\iota_0, \iota_1, \ldots, \iota_p)\), we define the weight of it by

\[||\iota|| := \sum_{\kappa=0}^{p} (2 + \kappa)\iota_\kappa.\]

Then, \(d_{\nu+1}[w]\) is written in the form:

**Lemma 3.1.** For every \(\nu \in \mathbb{N} \cup \{0\}\),

\[d_{\nu+1}[w] = \gamma_{\nu+1}w^{\nu+1} + \sum_{||\iota|| \leq 2(\nu+1), \iota_0 \leq \nu} c_\iota[w, w', \ldots, w^{(2\nu)}]^\iota, \quad \iota = (\iota_0, \iota_1, \ldots, \iota_{2\nu}),\]

where \(c_\iota \in \mathbb{C}, \gamma_{\nu+1} \in \mathbb{C} \setminus \{0\}\).

To show Theorem 1.1 for the general case, we start from the supposition that

\[N(r, w_\nu) \ll r^{(2\nu+3)/(\nu+1)-\epsilon},\]

which implies that

\[(3.1) \quad n(r, w_\nu) \ll r^{(2\nu+3)/(\nu+1)-\epsilon}\]

for some \(\epsilon > 0\). Let \(\{a_j\}_{j=1}^\infty\) (or \(\{a_j\}_{j=1}^q\)) be a sequence of distinct poles of \(w_\nu(z)\). Around \(a_j\), we have

\[w_\nu(z) = c(a_j)(z - a_j)^{-2} + O(1),\]
where \( c(a_j) = k(a_j)(k(a_j) + 1)/2 \) for some \( k(a_j) \in \{1, ..., \nu\} \). By (3.1), \( w_\nu(z) \) is written in the form

\[
w_\nu(z) = \sum_{a_j} c(a_j)((z - a_j)^{-2} - a_j^{-2}) + \varphi(z),
\]

where \( \varphi(z) \) is an entire function. Instead of Lemmas 2.1 and 2.2, we have the following under supposition (3.1):

**Lemma 3.2.** *For every \( r > 1 \), there exists \( z_r \) satisfying*

\[
0.7r \leq |z_r| \leq r, \quad \sum_{|a_j| < 2r} |z_r - a_j|^{-2} \ll r^{1/(\nu+1)-\epsilon/2}, \quad \sum_{|a_j| < 2r} |z_r - a_j|^{-3} \ll r^{(3/2)/(\nu+1)-\epsilon}.
\]

**Lemma 3.3.** *Let \( r \) be an arbitrary number such that \( r > 1 \). Then*

\[
\sum_{|a_j| \geq 2r} |(z - a_j)^{-2} - a_j^{-2}| \ll r^{1/(\nu+1)-\epsilon}, \quad \sum_{|a_j| \geq 2r} |z - a_j|^{-3} \ll 1
\]

*for \( |z| \leq r \), and*

\[
\sum_{0 < |a_j| < 2r} |a_j^{-2}| \ll r^{1/(\nu+1)-\epsilon}.
\]

Using Lemmas 3.2 and 3.3 combined with Lemma 3.1, we prove Theorem 1.1 for the general case.

**References**


