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<td>Author(s)</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (2003), 1316: 13-18</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2003-04</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/42994">http://hdl.handle.net/2433/42994</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
On the Number of Poles of the First Painlevé Transcendents and Higher Order Analogues II

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1. Introduction

Let \( w(z) \) be an arbitrary solution of the first Painlevé equation

\[
(PI) \quad w'' = 6w^2 + z
\]

\(('=d/dz)\). Then, \( w(z) \) is a transcendental meromorphic function, and every pole is double. The counting function for poles is defined by

\[
N(r, w) = \int_0^r \left( n(\rho, w) - n(0, w) \right) \frac{d\rho}{\rho} + n(0, w) \log r,
\]

where \( n(r, w) \) denotes the number of poles inside the disk \( |z| \leq r \), each counted according to its multiplicity. By a well-known argument in the Nevanlinna theory ([4, §2.4]), we have

\[
\lim \inf \frac{m(r, w)}{T(r, w)} = 0 \quad \text{as} \quad r \to \infty,
\]

namely,

\[
\lim \sup \frac{N(r, w)}{T(r, w)} = 1,
\]

which implies \( N(r, w) \to \infty \) as \( r \to \infty \). Here, \( m(r, w) \) and \( T(r, w) \) are, respectively, the proximity and the characteristic functions defined by

\[
m(r, w) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |w(re^{i\phi})| d\phi,
\]

\[
\log^+ x = \max\{0, \log x\},
\]

\[
T(r, w) = m(r, w) + N(r, w)
\]

(for the standard notation and basic facts in the Nevanlinna theory, see [2], [4]). For the magnitude of \( N(r, w) \), the following is known ([1], [5], [6], [9]):

\[
r^{5/2} \log r \ll N(r, w) \ll r^{5/2},
\]

which implies that the growth order of \( w(z) \)

\[
\sigma(w) = \lim \sup \frac{\log T(r, w)}{\log r}
\]
is equal to $5/2$. (We write $f(r) \ll g(r)$ (or $g(r) \gg f(r)$) if $f(r) = O(g(r))$ as $r \to \infty$.)

A sequence of higher order analogues of (PI) is given by the following:

\[ (\text{PI}_{2\nu}) \quad d_{\nu+1}[w] + 4z = 0, \quad \nu \in \mathbb{N} \]

(cf. [1, §16]; [3]). Here, $d_{\nu}[w] \ (\nu = 0, 1, 2, \ldots)$ are differential polynomials in $w$ determined by

\[ (1.3) \quad d_0[w] = 1, \]
\[ (1.4) \quad Dd_{\nu+1}[w] = (D^3 - 8wD - 4w')d_{\nu}[w], \quad D = d/dz, \quad \nu \in \mathbb{N} \cup \{0\}. \]

Since

\[ d_2[w]/4 = -w'' + 6w^2 + C_1w + C_0, \]

where $C_j \in \mathbb{C}$ ($j = 0, 1$) are arbitrary, equation (PI$_2$) essentially coincides with (PI). In general, (PI$_{2\nu}$) is a $2\nu$-th order nonlinear equation; e.g. for $\nu = 2, 3$,

\[ (\text{PI}_4)_0 \quad w^{(4)} = 20ww'' + 10(w')^2 - 40w^3 + z, \]
\[ (\text{PI}_6)_0 \quad w^{(6)} = 28ww^{(4)} + 56w'w^{(3)} + 42(w')^2 - 280(w^2w'' + w(w')^2 - w^4) + z, \]

where the arbitrary constants corresponding to $C_j$ of (PI$_2$) are taken to be 0. Let $w_\nu(z)$ be an arbitrary meromorphic solution of (PI$_{2\nu}$). It is interesting to evaluate the growth order of $w_\nu(z)$. The following result gives a lower estimate of it:

**Theorem 1.1.** For every $\nu \in \mathbb{N}$,

\[ (1.5) \quad \limsup_{r \to \infty} \frac{\log N(r, w_\nu)}{\log r} \geq \frac{2\nu + 3}{\nu + 1}, \]

namely the growth order of $w_\nu(z)$ is not less than $(2\nu + 3)/(\nu + 1)$.

As an immediate consequence, we have

**Corollary 1.2.** Equation (PI$_{2\nu}$) admits no rational solutions.

Viewing Theorem 1.1 combined with (1.2), we pose the following:

**Conjecture.** The growth order of $w_\nu(z)$ is equal to $(2\nu + 3)/(\nu + 1)$.

We sketch the proof of Theorem 1.1, illustrating the particular case $\nu = 2$. The full proof is found in [8].

**2. Sketch of the proof of Theorem 1.1 for (PI$_4$)**

The basic idea is the same as in the proof for (PI) (cf. [7]). Suppose the contrary:

\[ (2.1) \quad \limsup_{r \to \infty} \frac{\log N(r, w_2)}{\log r} < \frac{7}{3}, \]
namely, for some $\varepsilon > 0$, $N(r, w_2) \ll r^{7/3-\varepsilon}$, from which it follows that
\begin{equation}
(2.2) \quad n(r) = n(r, w_2) \ll r^{7/3-\varepsilon},
\end{equation}
because
\[N(2r, w_2) \geq \int_{r}^{2r} (n(\rho, w_2) - n(0, w_2)) \frac{d\rho}{\rho} \geq (n(r, w_2) + O(1)) \log 2.
\]
Starting from (2.1), we will derive a contradiction. Let $\{a_j\}_{j=1}^{\infty}$ (or $\{a_j\}_{j=1}^{q}, q \in \mathbb{N}$) be the sequence of all distinct poles of $w_2(z)$ arranged as $|a_1| \leq \cdots \leq |a_j| \leq \cdots$. It is easy to check that, around each pole $a_j$,
\[w_2(z) = c(j)(z - a_j)^{-2} + O(1),\]
where $c(j) = 1$ or $3$. By this fact combined with (2.2), we write $w_2(z)$ in the form
\begin{equation}
(2.3) \quad w_2(z) = \Phi(z) + \varphi(z),
\end{equation}
where $\varphi(z)$ is an entire function; and in (2.4), if $a_1 = 0$ the term $(z - a_1)^{-2} - a_1^{-2}$ should be replaced by $z^{-2}$. Under (2.2), we have the following lemmas whose proofs are similar to those of [7, Lemmas 1.1 and 1.2].

**Lemma 2.1.** For every $r > 1$, there exists $z_r$ satisfying
\[0.7r \leq |z_r| \leq r, \quad \sum_{|a_j|<2r} |z_r - a_j|^{-2} \ll r^{1/3-\varepsilon/2}, \quad \sum_{|a_j|<2r} |z_r - a_j|^{-3} \ll r^{1/2-\varepsilon}.
\]

**Lemma 2.2.** Let $r$ be an arbitrary number satisfying $r > 1$. Then,
\[\sum_{|a_j| \geq 2r} |(z - a_j)^{-2} - a_j^{-2}| \ll r^{1/3-\varepsilon}, \quad \sum_{|a_j| \geq 2r} |z - a_j|^{-3} \ll 1
\]
for $|z| \leq r$, and
\[\sum_{0<|a_j|<2r} |a_j^{-2}| \ll r^{1/3-\varepsilon}.
\]

By a well-known argument of the Nevanlinna theory, it is shown that $\varphi(z)$ is a polynomial. Note that $|\Phi(z)| \leq \sum_{|a_j|<2r} + |\sum_{|a_j| \geq 2r}|$. By Lemmas 2.1 and 2.2, for every $r > 1$, there exists $z_r, 0.7r \leq |z_r| \leq r$ satisfying
\begin{equation}
(2.5) \quad |\Phi(z_r)| \ll r^{1/3-\varepsilon/2}, \quad |\Phi'(z_r)| \ll r^{1/2-\varepsilon},
\end{equation}
\[|\Phi''(z_r)| \ll r^{2/3-\varepsilon}, \quad |\Phi^{(4)}(z_r)| \ll r^{1-3\varepsilon/2}.
\]
(2.6) \( w_2(z_r) \ll (|w_2^{(4)}(z_r)| + |w_2(z_r)||w_2''(z_r)| + |w_2'(z_r)|^2 + |z_r|)^{1/3} \)
\ll |w_2^{(4)}(z_r)|^{1/3} + |w_2(z_r)||w_2''(z_r)|^{1/3} + |w_2'(z_r)|^{2/3} + |z_r|^{1/3}.

Substituting \( w_2^{(k)}(z_r) = \varphi^{(k)}(z_r) + \Phi^{(k)}(z_r) \) \((k = 0, 1, 2, 4)\) into (2.6) and using \( |\Phi^{(k)}(z_r)| \ll r^{1/3+k/6} \) (cf. (2.5)), we have

(1) \(|\varphi(z_r)| \ll r^{1/3} + |\varphi^{(4)}(z_r)|^{1/3}\)
\quad + (r^{1/9} + |\varphi(z_r)|^{1/3})(r^{2/9} + |\varphi''(z_r)|^{1/3}) + r^{1/3} + |\varphi'(z_r)|^{2/3},

which implies that \( \varphi(z) \equiv C \in \mathbb{C} \). Then, by (PI4),

\[ 0.7r \leq |z_r| \ll |w_2^{(4)}(z_r)| + |w_2(z_r)||w_2''(z_r)| + |w_2'(z_r)|^2 + |w_2(z_r)|^3 \ll r^{1-\epsilon}, \]

which is a contradiction. Thus Theorem 1.1 with \( \nu = 2 \) follows.

3. General case

To treat the general case, we need to know some facts related to the terms of the differential polynomial \( d_{\nu+1}[w] \). Let \([w, w', \ldots, w^{(p)}]^\iota\) denote the monomial \( w^{\iota_0}(w')^{\iota_1}\cdots(w^{(p)})^{\iota_p} \), where \( \iota = (\iota_0, \iota_1, \ldots, \iota_p) \in \mathbb{N} \cup \{0\} \cup \mathbb{N} \). For this monomial with \( \iota = (\iota_0, \iota_1, \ldots, \iota_p) \), we define the weight of it by

\[ ||\iota|| := \sum_{\kappa=0}^{p}(2+\kappa)\iota_\kappa. \]

Then, \( d_{\nu+1}[w] \) is written in the form:

**Lemma 3.1.** For every \( \nu \in \mathbb{N} \cup \{0\}, \)

\[ d_{\nu+1}[w] = \gamma_{\nu+1}w^{\nu+1} + \sum_{||\iota|| \leq 2(\nu+1), \iota_0 \leq \nu} c_\iota[w, w', \ldots, w^{(2\nu)}]^\iota, \quad \iota = (\iota_0, \iota_1, \ldots, \iota_{2\nu}), \]

where \( c_\iota \in \mathbb{C}, \gamma_{\nu+1} \in \mathbb{C} \setminus \{0\}. \)

To show Theorem 1.1 for the general case, we start from the supposition that

\[ N(r, w_{\nu}) \ll r^{(2\nu+3)/(\nu+1) - \epsilon}, \]

which implies that

\[ n(r, w_{\nu}) \ll r^{(2\nu+3)/(\nu+1) - \epsilon} \]

for some \( \epsilon > 0 \). Let \( \{a_j\}_{j=1}^{\infty} \) (or \( \{a_j\}_{j=1}^{q} \)) be a sequence of distinct poles of \( w_{\nu}(z) \). Around \( a_j \), we have

\[ w_{\nu}(z) = c(a_j)z^{-2} + O(1), \]
where \( c(a_j) = k(a_j)(k(a_j) + 1)/2 \) for some \( k(a_j) \in \{1, \ldots, \nu\} \). By (3.1), \( w_\nu(z) \) is written in the form

\[
w_\nu(z) = \sum_{a_j} c(a_j)((z - a_j)^{-2} - a_j^{-2}) + \varphi(z),
\]

where \( \varphi(z) \) is an entire function. Instead of Lemmas 2.1 and 2.2, we have the following under supposition (3.1):

**Lemma 3.2.** For every \( r > 1 \), there exists \( z_r \) satisfying

\[
0.7r \leq |z_r| \leq r, \quad \sum_{|a_j| < 2r} |z_r - a_j|^{-2} \ll r^{1/(\nu+1) - \varepsilon/2}, \quad \sum_{|a_j| < 2r} |z_r - a_j|^{-3} \ll r^{(3/2)/(\nu+1) - \varepsilon}.
\]

**Lemma 3.3.** Let \( r \) be an arbitrary number such that \( r > 1 \). Then

\[
\sum_{|a_j| \geq 2r} |(z - a_j)^{-2} - a_j^{-2}| \ll r^{1/(\nu+1) - \varepsilon}, \quad \sum_{|a_j| \geq 2r} |z - a_j|^{-3} \ll 1
\]

for \( |z| \leq r \), and

\[
\sum_{0 < |a_j| < 2r} |a_j^{-2}| \ll r^{1/(\nu+1) - \varepsilon}.
\]

Using Lemmas 3.2 and 3.3 combined with Lemma 3.1, we prove Theorem 1.1 for the general case.

**References**


