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WKB solutions near a hyperbolic critical point

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This report is a part of the work in progress with Jean-François Bony, Thierry Ramond and Maher Zerzeri about the semiclassical distribution of resonances created by a homoclinic orbit in the phase space.

We study the Schrödinger equation

$$P(x,hD)u = hEu, \quad P(x,hD) = h^2D^2 + V(x) \tag{0.1}$$

and we assume that the trapping set in $p^{-1}(0)$ consists of a unique homoclinic orbit. Here $p(x,\xi) = \xi^2 + V(x)$ is the symbol of the operator $P(x,hD)$, and a point (x,ξ) is said to be in a trapping set if $|\exp tH_p(x,\xi)|$ does not tend to ∞ as $t \to \pm\infty$. Homoclinic orbit is a set of (x,ξ) in $p^{-1}(0)$ such that $\exp tH_p(x,\xi)$ tends to a critical point as $t \to \pm\infty$,

Roughly speaking, the resonant state is localized near the trapping set when $h \to 0$, and so it suffices to consider solutions there for the study of the semiclassical distribution of resonances. In our case, we need to construct a WKB solution at one point on the orbit and continue it along the homoclinic orbit. Then the quantization condition of resonances is obtained by comparing the initial solution with that obtained after a round continuation.

To do this, the main problem arises at the critical point because the phase and the symbol of WKB solutions have singularities there. In this report, we restrict ourselves to the study of the connection problem at the critical point.

In the first part, we review the theory of [He-Sj] in the general setting. Thier idea is to represent the solution near the critical point as Laplace transform of a WKB solution to the time dependent Schödinger equation and to expand the phase and the symbol asymptotically as $t \to +\infty$.

In the second part, we carry out an explicit calculus for a model to demonstrate the theory of the previous section as well as to study its asymptotic behavior near the critical point.

1 General scheme

Let $\gamma(t)$ be the homoclinic orbit in the phase space. As $t \to \pm\infty$, $\gamma(t)$ converges to the critical point, which we assume to be $(x, \xi) = (0, 0)$. Let $b(x, h) \exp i\psi(x)/h$ be a microlocal WKB solution associated to a Lagrangian manifold $\Lambda_\psi = \{(x, \xi); \xi = \partial_x \psi(x)\}$, which contains the incoming part of the trajectory $\{\gamma(t)\}_{t>T}$ with large enough T. The WKB construction does not work near the critical point, that is, the symbol $b(x, h)$ is singular at $x = 0$ and its asymptotic expansion no longer has sense as $x \to 0$ rapidly with respect to h. The aim is to continue this solution to the outgoing part of the trajectory $\{\gamma(t)\}_{t<-T}$ passing through the critical point and to know the asymptotic behavior there.

The following in this section is a partial review of [He-Sj]. For the proofs and more details, see this reference.

Let $(x, \xi) = (0, 0)$ be the critical point. This means that $x = 0$ is a non-degenerate maximum of the potential $V(x)$. Then after a linear canonical transformation, we can asuume

$$V(x) = -\sum_{j=1}^{n} \frac{\lambda_j^2}{4} x_j^2 + O(|x|^3) \quad (x \to 0), \quad 0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n.$$

Let $d(0, x)$ be the geodesic distance from 0 to x with respect to the metric $ds^2 = \max(0, -V(x))dx^2$. Then

$$d(0, x) = \sum_{j=1}^{n} \frac{\lambda_j}{4} x_j^2 + O(|x|^3)$$

and if we put

$$\phi_\pm(x) = \pm d(0, x), \quad \Lambda_\pm = \{(x, \xi); \xi = \partial_x \phi_\pm\}$$

then $\Lambda_\pm$ are outgoing and incoming Lagrangian manifolds respectively.

The main idea of [He-Sj] is to represent the solution u near the origin as Laplace transform of a solution of the time dependent Schrödinger equation, that is,

$$u = I[a](x,h) = \int_{T_0}^{\infty} a(t,x,h)e^{i\phi(t,x)/h}dt, \tag{1.1}$$

$$a(t,x,h) \sim \sum_{j=0}^{\infty} a_j(t,x) \quad (h \to 0),$$

with

$$(hD_t + P - hE)(ae^{i\phi/h}) \sim 0, \tag{1.2}$$

and a large enough constant T_0. The phase ϕ and the terms a_j of the symbol a satisfy the eikonal and transport equations respectively:

$$\partial_t\phi + |\nabla\phi|^2 + V(x) = 0, \tag{1.3}$$

$$\partial_t a_0 + 2\nabla\phi \cdot \nabla a_0 + (\Delta\phi - iE)a_0 = 0, \tag{1.4}$$

$$\partial_t a_j + 2\nabla\phi \cdot \nabla a_j + (\Delta\phi - iE)a_j = i\Delta a_{j-1} \quad (j \geq 1). \tag{1.5}$$

Now we construct the phase and the symbol.

Take a point $\rho_0 = (x_0, \xi_0) \neq (0,0)$ on Λ_- and let $\gamma(t) = \exp tH_p(\rho_0)$. We assume

(H1) $\{\lambda_j\}_{j=1}^n$ are $\mathbb{Z}$-independent.

Then we have

$$\gamma(t) \sim \sum_{j=1}^{\infty} e^{-\mu_j t}\gamma_j(t) \quad (t \to +\infty),$$

where $\gamma_j(t)$ are polynomials in t and $0 < \mu_1 < \mu_2 < \cdots$ are non-zero linear combinations of $\{\lambda_j\}_{j=1}^n$ on $\mathbb{N} = \{0, 1, \ldots\}$, in particular $\mu_1 = \lambda_1$. Moreover γ_1 is independent of t and parallel to the vector $(x;\xi) = (1, 0, \ldots, 0; -\lambda_1/2, 0, \ldots, 0)$. We assume then a generic assumption

(H2) $\gamma_1 = (1, 0, \ldots, 0; -\lambda_1/2, 0, \ldots, 0)$.

This means that γ is tangent to the x_1-axis at the critical point.

Let Λ_0 be a Lagrangian manifold transverse to Λ_- at ρ_0, and put $\Lambda_t = \exp tH_p(\Lambda_0)$. Then Λ_t is also a Lagrangian manifold and there exists $\phi(t,x)$ such that

$$\Lambda_t = \{(x,\xi); \xi = \partial_x\phi(t,x)\}$$

and such that ϕ satisfies the eikonal equation (1.3). Moreover, it is *expandible*, i.e.

$$\phi(t,x) - \phi_+(x) \sim \sum_{j=1}^{\infty} e^{-\mu_j t}\phi_j(t,x) \quad t \to +\infty,$$

where $\phi_j(t,x)$ are C^∞ function in x in a neighborhood of the origin and polynomial in t, in the sense that

$$D_t^k D_x^\alpha(\phi - \phi_+ - \sum_{j=1}^{N} e^{-\mu_j t}\phi_j) = O(e^{-(\mu_{N+1}-\epsilon)t})$$

for any $N \in \mathbb{N}$, $\alpha \in \mathbb{N}^n$, $k \in \mathbb{N}$, and any $\epsilon > 0$.

With this ϕ, the solution of the transport equations (1.4), (1.5) are also *expandible* and

$$a_j(t,x) \sim e^{-St}(a_{j,0}(x) + \sum_{k=1}^{\infty} a_{j,k}(t,x)e^{-\mu_j t}),$$

where

$$S = \frac{1}{2}\sum_{j=1}^{n}\lambda_j - iE.$$

Let Ω be a sufficiently small neighborhood of 0 and $\epsilon > 0$ sufficiently small number. We define

$$W_\pm = \{x \in \Omega; \pm x_1 > |x'|^{\lambda_1/(\mu_2-\lambda_1+\epsilon)}\}.$$

Then we have the following proposition:

Proposition 1.1 *There exists a large enough T_0 such that there exists a unique critical point $t = t(x) > T_0$ for x in W_+ with $\partial^2\phi/\partial t^2(t(x),x) > 0$ and no critical point larger than T_0 for x in W_-. Moreover, the critical value $\psi(x) = \phi(t(x),x)$ satisfies*

$$\psi(x) = -\frac{\lambda_1}{4}x_1^2 + \sum_{j=2}^{n}\frac{\lambda_j}{4}x_j^2 + o(|x|^2) \quad (x \to 0 \text{ in } W_+).$$

2 A model example

In this section, we calculate $I[a]$ for a particular potential

$$V(x) = -\sum_{j=1}^{n} \frac{\lambda_j^2}{4} x_j^2$$

and for a particular choice of the Lagrangian manifold

$$\Lambda_0 = \{(x,\xi); \xi_1 = \frac{1}{2}\lambda_1(x_1 - 2), \xi_2 = \frac{1}{2}\lambda_2 x_2, \ldots, \xi_n = \frac{1}{2}\lambda_n x_n\},$$

which may be representative of the generic case, and we will clarify its asymptotic behavior of $I[a]$.

First, the phase $\phi_+(x)$, which is the Jacobi distance from 0, is given by

$$\phi_+(x) = \sum_{j=1}^{n} \frac{\lambda_j}{4} x_j^2.$$

Next, the Hamilton flow is given by

$$\exp tH_p = \otimes_{j=1}^{n} \begin{pmatrix} \cosh \lambda_j t & \frac{2}{\lambda_j} \sinh \lambda_j t \\ \frac{\lambda_j}{2} \sinh \lambda_j t & \cosh \lambda_j t \end{pmatrix}$$

and we get

$$\Lambda_t = \{(x,\xi); \xi_1 = \frac{1}{2}\lambda_1(x_1 - 2e^{-\lambda_1 t}), \xi_2 = \frac{1}{2}\lambda_2 x_2, \ldots, \xi_n = \frac{1}{2}\lambda_n x_n\}.$$

Then the phase $\phi(t,x)$ is obtained modulo a function of t, but with the eikonal equation (1.3), it is determined modulo a constant and we take

$$\phi(t,x) = \sum_{j=1}^{n} \frac{\lambda_j}{4} x_j^2 - \lambda_1 e^{-\lambda_1 t} x_1 + \frac{\lambda_1}{2} e^{-2\lambda_1 t}.$$

For this ϕ,

$$\partial_t \phi = \lambda_1^2 e^{-\lambda_1 t}(x_1 - e^{-\lambda_1 t}),$$

and if $x_1 > 0$, there is a unique critical point $t = t(x)$ in $t \in (-\infty, +\infty)$ which satisfies

$$x_1 = e^{-\lambda_1 t(x)}, \tag{2.1}$$

and the critical value is given by

$$\psi(x) = \phi(t(x), x) = -\frac{\lambda_1}{4} x_1^2 + \sum_{j=2}^{n} \frac{\lambda_j}{4} x_j^2. \tag{2.2}$$

If $x_1 < 0$, there is no critical point.

On the other hand, the transport equation (1.4) becomes

$$\left\{\partial_t + \lambda_1(x_1 - 2e^{-\lambda_1 t})\frac{\partial}{\partial x_1} + \sum_{j=2}^{n}\lambda_j x_j \frac{\partial}{\partial x_j}\right\} a_0 + S a_0 = 0.$$

The general solution is given by

$$a_0(t,x) = e^{-St} a_0^I((x_1 - e^{-\lambda_1 t})e^{-\lambda_1 t}, x_2 e^{-\lambda_2 t}, \ldots, x_n e^{-\lambda_n t}), \tag{2.3}$$

where $a_0^I(y) = a_0^I(y_1, \ldots, y_n)$ is an arbitrary function.

Let us compute the asymptotic expansion of $I[a_0]$ for $a_0^I(y) = y^\alpha$, that is, for

$$a_0(t,x) = (x_1 - e^{-\lambda_1 t})^{\alpha_1} x'^{\alpha'} e^{-\lambda_1 E(\alpha) t}, \tag{2.4}$$

where we used the notations

$$x' = (x_2, \ldots, x_n), \quad \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) = (\alpha_1, \alpha'),$$

$$x'^{\alpha'} = x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \quad \lambda \cdot \alpha = \lambda_1\alpha_1 + \cdots + \lambda_n\alpha_n,$$

$$E(\alpha) = (S + \lambda\cdot\alpha)/\lambda_1 = \sum_{j=1}^{n}\left\{(\alpha_j + \frac{1}{2})\lambda_j - iE\right\}/\lambda_1.$$

Notice that, in view of (1.5), $a_0 e^{i\phi t}$ is an exact solution to (1.2) in cases where $\alpha_1 = 0$ and 1. Substituting (2.4) in (1.1), we have

$$\begin{aligned} I[a_0] &= x'^{\alpha'} e^{i\phi_+(x)/h} \int_{T_0}^{+\infty} e^{i\lambda_1(-2x_1 e^{-\lambda_1 t} + e^{-2\lambda_1 t})/2h} e^{-\lambda_1 E(\alpha)t}(x_1 - e^{-\lambda_1 t})^{\alpha_1} dt \\ &= x'^{\alpha'} e^{i\psi(x)/h} \int_{T_0}^{+\infty} e^{i\lambda_1(x_1 - e^{-\lambda_1 t})^2/2h} e^{-\lambda_1 E(\alpha)t}(x_1 - e^{-\lambda_1 t})^{\alpha_1} dt \end{aligned}$$

By the change of variable $s = e^{-\lambda_1 t}$,

$$I[a_0] = \frac{1}{\lambda_1} x'^{\alpha'} e^{i\psi(x)/h} \int_0^{e^{-\lambda_1 T_0}} e^{i\lambda_1(x_1 - s)^2/2h} s^{E(\alpha)-1}(x_1 - s)^{\alpha_1} ds. \tag{2.5}$$

Now the problem is reduced to the study of the integral

$$J_{p,q}(z;k) = \int_0^\infty e^{-(z+s)^2/2k^2} s^{p+q-1}(z+s)^q ds.$$

We assume here that q is a non-negative integer. Then one gets a recurrence formula by an integration by parts:

$$J_{p,q} = k^2\{(p+q-1)J_{p,q-1} + (q-1)J_{p+2,q-2}\},$$

from which we obtain

$$J_{p,q} = k^{2q}(p)_q \sum_{j=0}^{[q/2]} c_j J_{p+2j,0},$$

where $(p)_q = p(p+1)\cdots(p+q-1)$ and c_j ($j = 1, \ldots, [q/2]$) are constants independent of k with $c_0 = 1$. On the other hand, one has

$$J_{p,0}(z;k) = k^p \Gamma(p) e^{-z^2/4k^2} D_{-p}(z/k),$$

where $D_\nu(z)$ is the Weber function

$$D_\nu(z) = \frac{e^{-z^2/4}}{\Gamma(-\nu)} \int_0^\infty e^{-zt-t^2/2} t^{-\nu-1} dt. \tag{2.6}$$

Hence we have

$$J_{p,q}(z;k) = k^{p+2q}(p)_q e^{-z^2/4k^2} \sum_{j=0}^{[q/2]} c_j k^{2j} \Gamma(p+2j) D_{-p-2j}(z/k)$$

Now returning back to $I[a_0]$, we write it in termes of $J_{p,0}$ with $E(\alpha') = E(\alpha)|_{\alpha_1=0} = E(\alpha) - \alpha_1$:

$$\begin{aligned}
I[a_0] &= \frac{(-1)^{\alpha_1}}{\lambda_1} e^{i\psi(x)/h} x'^{\alpha'} J_{E(\alpha'),\alpha_1}(-x_1; \sqrt{ih/\lambda_1}) \\
&= \frac{1}{\lambda_1}\left(\frac{-ih}{\lambda_1}\right)^{\alpha_1} (E(\alpha_1))_{\alpha_1} e^{i\psi(x)/h} x'^{\alpha'} \sum_{j=0}^{[\alpha_1/2]} c_j J_{E(\alpha')+2j,0}(-x_j; \sqrt{ih/\lambda_1}) \\
&= \frac{1}{\lambda_1}\left(\frac{-ih}{\lambda_1}\right)^{\alpha_1} (E(\alpha_1))_{\alpha_1} e^{i\psi(x)/h} x'^{\alpha'} \sum_{j=0}^{[\alpha_1/2]} c_j \int_0^\infty e^{i\lambda_1(x_1-s)^2/2h} s^{E(\alpha')+2j-1} ds.
\end{aligned}$$

The asymptotic behavior of the last integral is of course reduced to the well-known aymptotic formula of the Weber function, but we recall here the derivation.

If $x_1 \le 0$, the principal contribution comes from $s = 0$ and if $x_1 > 0$, the principal terms come from $s = 0$ and the critical point $s = x_1$ of the phase function $\lambda_1(x_1 - s)^2/2$. Let us study the case $x_1 > 0$.

We write

$$\int_0^{+\infty} e^{i\lambda_1(x_1-s)^2/2h} s^{E(\alpha')+2j-1} ds = \int_0^\epsilon + \int_\epsilon^\infty = I_1^j + I_2^j,$$

for sufficiently small ϵ such that $0 < \epsilon < x_1$.

The asymptotic expansion of I_2^j is obtained by the stationary phase method and it is given by

$$I_2^j \sim \sqrt{\frac{2\pi i h}{\lambda_1}} x_1^{E(\alpha')+2j-1}$$

Next, we study I_1^j. As $s \to 0$,

$$(x_1 - s)^2 \sim x_1^2 - 2x_1 s,$$

and we have

$$I_1^j \sim e^{i\lambda_1 x_1^2/2h} \int_0^\epsilon e^{-i\lambda_1 x_1 s/h} s^{E(\alpha')+2j-1} dt$$

By the change of variable $i\lambda_1 x_1 s/h = \sigma$, it is reduced to the integral of the gamma function, and one has

$$I_1^j \sim \Gamma(E(\alpha') + 2j) \left(\frac{h}{i\lambda_1 x_1}\right)^{E(\alpha')+2j} e^{i\lambda_1 x_1^2/2h}.$$

Remark that $I_2^{j+1} = x_1^{2j} I_2^j$ and $I_1^{j+1} = (h/x_1)^{2j} I_1^j$. Hence

$$\sum_{j=0}^{[\alpha_1/2]} c_j (I_1^j + I_2^j) \sim I_1^0 + I_2^0 \quad (x_1 \to 0_+, \quad h/x_1 \to 0).$$

Thus we have obtained the following proposition:

Proposition 2.1 *Let $a_0^I(y) = y^\alpha$ and a_0 be given by (2.3). Then one has as $h/x_1 \to 0$ and $x_1 \to 0+$,*

$$I[a_0](x,h) \sim \frac{1}{\lambda_1}\left(\frac{h}{i\lambda_1}\right)^{\alpha_1} (E(\alpha'))_{\alpha_1} x'^{\alpha'}$$
$$\times \left\{ \sqrt{\frac{2\pi i h}{\lambda_1}} x_1^{E(\alpha')-1} e^{i\psi(x)/h} + \left(\frac{h}{i\lambda_1}\right)^{E(\alpha')} \Gamma(E(\alpha')) x_1^{-E(\alpha')} e^{i\phi_+(x)/h} \right\}$$

参考文献

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