<table>
<thead>
<tr>
<th><strong>Title</strong></th>
<th>Effective Limit in Computable Analysis (Algebra, Logic and Geometry in Informatics)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Author(s)</strong></td>
<td>Takeuti, Izumi</td>
</tr>
<tr>
<td><strong>Citation</strong></td>
<td>数理解析研究所講究録 (2003), 1318: 3-14</td>
</tr>
<tr>
<td><strong>Issue Date</strong></td>
<td>2003-05</td>
</tr>
<tr>
<td><strong>URL</strong></td>
<td><a href="http://hdl.handle.net/2433/43024">http://hdl.handle.net/2433/43024</a></td>
</tr>
<tr>
<td><strong>Type</strong></td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td><strong>Textversion</strong></td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Effective Limit in Computable Analysis

竹内 泉
274-8510 千葉県船橋市三山2丁目2-1 東邦大学理学部情報科学科
Takeuti Izumi
Information Science, Toho Univ., Miyama Hunabasi, Tiba, 274-8510 JAPAN
takeuti@is.sci.toho-u.ac.jp

1 Introduction

Most of the studies of computable analysis have been capturing the computability of structures over real numbers. However, there are started some studies on other general mathematical structures recently.

Hertling studied computability over algebraic structures [H], however he did not discuss topology. Schröder studied weak limit spaces [S], however he did not discuss modulus of convergence. The author believe that the notions of topology and modulus of convergence are most important in the study of computable analysis. This work aims to combine their works, and to apply to the discussion on modulus of convergence.

In discussing modulus of convergence, it seems natural to do with uniform topology, which Yasugi and others studied [Y]. They discuss the computability under uniform topology with computability structure, while Hertling and Schröder did with representation.

There are several methods in studying computable analysis. One of them is representation and another is computability structure.

The notion of representation has been studied by Weihrauch [W] and others. It is a kind of a mechanical method. A representation $r$ of a mathematical structure $X$ is a surjective partial function of $\Sigma^\omega$ into $X$. A function $f : X \to X$ is computable if the is a computable partial function $\bar{f}$ of $\Sigma^*$ into $\Sigma^*$ such that $f \circ r = r \circ \bar{f}$. We will use $\mathbb{N} \to \mathbb{N}$ as $\Sigma^\omega$, the domain of representation. It seems that representations have too much information which comes from the details of implementation of computation models.

The notion of computability structure is studied by Por-El et al. [P]. It is a kind of an axiomatic method. A computability structure over a mathematical structure $X$ is a subset of $X^\omega$ which satisfies the axioms of computability structure. A function $f$ over $X$ is called computable if $f$ is locally uniformly continuous and preserves the computability structure over $X$. In order to assert that a computability structure is natural, the mathematical structure $X$ has been requested to have a distance. Yasugi et al. attempt to replace the distance with effective uniform topology in the literature [Y].

This work aims at comparing these two ways, that by computability structure and that by representation. The author would like to show that the essential notions are equivalently defined by both ways.
There are several ways to define effective uniform topology. Yasugi et al. defined it with effective uniform neighbourhood system in the literature [Y]. This work defines it with effective limit, which has not been used in any other works. An effective limit is a partial function of arity \( \omega \) over the underlying set, while an effective neighbourhood system is a family of subsets. It seems that a partial function is more familiar in computational theory than a family of subset is.

There are three similar words 'recursive', 'effective' and 'computable', all of which appear in this paper. There are only little difference in the senses of these words. In this paper, the word 'recursive' is used only for the notions concerned to recursive functions over natural numbers. The word 'computable' is used only for the notion of computability structure which defined by Pour-El et al. [P]. The word 'effective' is used for other mathematical structures, Thus, a function represented by a recursive Type-2 function is called a recursive function in this paper, although Weihrauch and others call it a computable function [W].

## 2 Uniform Topology

**Notation 2.1** We write \( \mathbb{N} \) for \( \{1, 2, 3, \ldots\} \).

**Definition 2.2 (Partial function)** A function \( f \) is a partial function of \( X \) into \( Y \) iff \( f \) is a function of \( X' \subset X \) into \( Y' \subset Y \). This \( X' \) is called the domain of \( f \) and written as \( \text{dom}(f) \). We write \( f : X \to_p Y \) when \( f \) is a partial function of \( X \) into \( Y \). For \( x \in X \), we say that \( f(x) \) is defined when \( x \in \text{dom}(f) \), and \( f(x) \) is undefined if not. When we write \( f(x) = y \), we implicitly assert \( x \in \text{dom}(f) \).

The range of \( f : X \to_p Y \) is the set \( \{f(x) \in Y \mid x \in \text{dom}(f)\} \) and written as \( \text{range}(f) \). For \( f : X \to_p Y \) and \( g : Y \to_p Z \), the concatenation \( g \circ f : X \to_p Z \) is defined as following: \( \text{dom}(g \circ f) = \{x \in X \mid x \in \text{dom}(f), f(x) \in \text{dom}(g)\} \) and \( g \circ f(x) = g(f(x)) \) for \( x \in \text{dom}(g \circ f) \).

For partial functions \( f : X \to_p Y \) and \( g : X' \to_p Y' \), the relation \( f \subseteq_p g \) holds iff \( \text{dom}(f) \subset \text{dom}(g) \) and \( f(x) = g(x) \) for all \( x \in \text{dom}(f) \). It may be the case that \( X \neq X' \) or \( Y \neq Y' \) although \( f \subseteq_p g \). For partial functions \( f \) and \( g \), it holds \( f = g \) if \( f \subseteq_p g \) and \( g \subseteq_p f \).

**Notation 2.3** We write \( X \to Y \) for the function space \( \{f \mid f : X \to Y\} \), and \( X \to_p Y \) for the set of all the partial functions \( \{f \mid f : X \to_p Y\} \). Thus, the notation \( E \subset X \to Y \) does not mean \( E : X \to_p Y \) but means \( \forall f \in E. f : X \to Y \).

The operators \( \to \) and \( \to_p \) are right associative. Thus, the notation \( X \to Y \to_p Z \) is an abbreviation of \( X \to (Y \to_p Z) \).

**Definition 2.4 (Modular Limit)** Let \( X \) be a set. An partial function \( LM \) is a modular limit over \( X \) if there is a strictly increasing function \( m : \mathbb{N} \to \mathbb{N} \) and the followings hold:

1. \( LM : X^\omega \to_p X \)
1. For each \( x \in X \), \( LM(x, x, x, \ldots) = x \)

2. Let \( f \) be a function of \( \mathbb{N} \to \mathbb{N} \) such that \( f(n) \geq n \) for any \( n \in \mathbb{N} \). If \((x_1, x_2, x_3, \ldots) \in \text{dom}(LM)\), then \((x_{f(1)}, x_{f(2)}, x_{f(3)}, \ldots) \in \text{dom}(LM)\) and \( LM(x_{f(1)}, x_{f(2)}, x_{f(3)}, \ldots) = LM(x_1, x_2, x_3, \ldots) \)

3. Let \( \{x_{i,j}\}_{i,j} \) be a double sequence in \( X \), that is, \( x_{i,j} \in X \) for any \( i, j \in \mathbb{N} \). Let \((y_1, y_2, y_3, \ldots)\) be a sequence in \( X \) and \( z \) be a point in \( X \). Suppose that \( y_i = LM(x_{i,1}, x_{i,2}, x_{i,3}, \ldots) \) for each \( i \).

   If \( LM(y_1, y_2, y_3, \ldots) = z \), then \( LM(x_{m(1),m(1)}, x_{m(2),m(2)}, x_{m(3),m(3)}, \ldots) = z \).

4. Let \( \{x_{i,j}\}_{i,j} \) be a double sequence in \( X \), that is, \( x_{i,j} \in X \) for any \( i, j \in \mathbb{N} \). Let \((y_1, y_2, y_3, \ldots)\) be a sequence in \( X \) and \( z \) be a point in \( X \). Suppose that \( y_i = LM(x_{i,1}, x_{i,2}, x_{i,3}, \ldots) \) for each \( i \).

   If \( LM(x_{1,1}, x_{2,2}, x_{3,3}, \ldots) = z \), then \( LM(y_{m(1)}, y_{m(2)}, y_{m(3)}, \ldots) = z \).

The function \( m() \), which appears in the conditions 3 and 4, is called a \textit{modulus of diagonal convergence}.

\textbf{Definition 2.5 (Effective Limit)} A modular limit is called an \textit{effective limit} when the modulus of diagonal convergence of it is a recursive function.

\textbf{Remark 2.6} The conditions 3 and 4 in the definition of modular limit means that all the converging sequence under the modular limit are uniformly converging.

\textbf{Definition 2.7 (Uniform Neighbourhood System)} Let \( X \) be a set. The series of subsets \( \{V_i(x)\}_{i \in \mathbb{N}, x \in X} \) is a \textit{uniform neighbourhood system} iff it satisfies the followings:

1. \( x \in V_i(x) \subset X \) for each \( i \in \mathbb{N} \) and each \( x \in X \).
2. \( V_i(x) \supset V_j(x) \) for \( i < j \)
3. There is a function \( m : \mathbb{N} \to \mathbb{N} \) such that if \( y \in V_{m(i)}(x) \) then \( x \in V_i(y) \)
4. There is a function \( m' : \mathbb{N} \to \mathbb{N} \) such that if \( y \in V_{m'(i)}(x) \) then \( V_{m'(i)}(y) \subset V_i(x) \)

The functions \( m \) and \( m' \) in the conditions 3 and 4 are called the \textit{moduli}.

\textbf{Definition 2.8 (Effective Uniform Neighbourhood System)} The uniform neighbourhood system is \textit{effective} if both of the moduli are recursive functions.

\textbf{Remark 2.9} When we replace the condition 4 of the definition above with:

\[ \forall x \in X. \forall i \in \mathbb{N}. \exists j \in \mathbb{N}. \forall y \in V_j(x). \exists k \in \mathbb{N}. V_k(y) \subset V_i(x) \]

then, the conditions defines the ordinary notion of countable neighbourhood systems. If we exchange the order of quantifiers \( \forall y \) and \( \exists k \), then, this condition is equivalent to the condition 4 of the definition above.
Proposition 2.10 Let $X$ be a set and $\{V_i(x)\}_{i,x}$ be a uniform neighbourhood system over $X$. Then, $X$ is a Hausdorff space iff $\bigcap_{i \in \mathbb{N}} V_i(x) = \{x\}$ for any $x \in X$.

Remark 2.11 There are several well-known definitions of uniform topology which are equivalent to each other. One of them is the definition by uniform neighbourhood systems. The following propositions show that modulus limit also gives the equivalent definition of uniform topology.

Definition 2.12 (Closeness relation) Let $LM$ be the modular limit of $X$. The relation $x \xrightarrow{LM} y$ holds iff there is a sequence $(x_1, x_2, x_3, \ldots)$ such that $x = x_i$ and $LM(x_1, x_2, x_3, \ldots) = y$. We call this relation closeness relation.

Proposition 2.13 The followings hold. The function $m(\ )$ is the modulus of diagonal convergence of the modular limit in the followings.

1. If $i \leq j$ and $x \xrightarrow{i} y$, then $x \xrightarrow{j} y$.
2. If $m(i) \leq j$ and $x \xrightarrow{j} y$, then $y \xrightarrow{i} x$.
3. If $m(i) \leq j$, $x \xrightarrow{i} y$ and $y \xrightarrow{j} z$, then $x \xrightarrow{j} z$.
4. If $m(m(i)) \leq j$ and $x \xrightarrow{j} z \xleftarrow{j} y$, then $x \xrightarrow{i} y$.
5. If it holds that $x_i \xrightarrow{m(i)} y$ for each $i$, then $LM(x_1, x_2, x_3, \ldots) = y$.

Lemma 2.14 Let $X$ be a set and $LM$ be a modular limit of modulus $m$ over $X$. For each $i \in \mathbb{N}$ and each $x \in X$, a subset $V_i(x) \subset X$ is defined as $V_i(x) = \{y \mid y \xrightarrow{i} x\}$. Then, $\{V_i(x)\}_{i,x}$ is a uniform neighbourhood system of modulus $m$ over $X$, and it makes $X$ a Hausdorff space.

Moreover, if $LM$ is an effective limit, then $\{V_i(x)\}_{i,x}$ is an effective uniform neighbourhood system.

Lemma 2.15 Let $X$ be a set and $\{V_i(x)\}_{i,x}$ be a uniform neighbourhood system over $X$ which makes $X$ a Hausdorff space. A partial function $LM : X^\omega \to_{p} X$ is defined as follows:

$$\text{dom}(LM) = \{(x_1, x_2, \ldots) \mid \exists y \in X. \forall i \in \mathbb{N}. x_i \in V_i(y)\}$$

$$LM(x_1, x_2, \ldots) = y \text{ iff } \forall i \in \mathbb{N}. x_i \in V_i(y)$$

Note that such $x$ is uniquely determined, because $X$ is a Hausdorff space by the neighbourhood system $\{V_i(x)\}_{i,x}$.

Then, $LM$ is a modular limit of modulus $m$ over $X$.

Moreover, if $\{V_i(x)\}_{i,x}$ is an effective uniform neighbourhood system, then $LM$ is an effective limit.

Remark 2.16 The previous lemmata 2.14 and 2.15 mean that modular limit is corresponds to uniform neighbourhood system, and effective limit is corresponds to effective uniform neighbourhood system.

Example 2.17 Let $\mathbb{R}$ be the set of all the real numbers. Define a partial function $\text{LimE} : \mathbb{N}^\omega \to_{p} \mathbb{N}$ as:
\[
\text{dom}(\text{LimE}) = \left\{(x_1, x_2, x_3, \ldots) \mid |x_i - x_j| < \frac{1}{i} + \frac{1}{j} \right\}
\]

\[
\text{LimE}(x_1, x_2, \ldots) = x \text{ iff } \lim_{n \to \infty} x_n = x
\]

Then, this partial function \(\text{LimE}\) is an effective limit with the modulus of diagonal convergence \(n \mapsto 2n\).

**Example 2.18** For \(i \in \mathbb{N}\) and \(x \in \mathbb{R}\), put \(V_i^R(x) \subset \mathbb{R}\) as \(y \in V_i^R(x)\) iff 
\(|y - x| < 1/i\). Then, \(V_i^R(x) = \{y \mid y \xrightarrow{\Delta} x\}\) and \(\{V_i^R(x)\}_{i,x}\) is an effective uniform neighbourhood system.

**Example 2.19** The functional space \(\mathbb{N} \rightarrow \mathbb{N}\) is regarded as a topological space. Define a partial function \(LM_{\mathbb{N},\mathbb{N}} : (\mathbb{N} \rightarrow \mathbb{N})^\omega \rightarrow_p \mathbb{N} \rightarrow \mathbb{N}\) as:

1. \((x_1, x_2, \ldots) \in \text{dom}(LM_{\mathbb{N},\mathbb{N}})\) iff for any \(i \leq j < k\), \(x_j(i) = x_k(i)\)
2. For \((x_1, x_2, \ldots) \in \text{dom}(LM_{\mathbb{N},\mathbb{N}}), LM_{\mathbb{N},\mathbb{N}}(x_1, x_2, \ldots) = (i \mapsto x_i(i))\)

The condition 1 means that \(x_j\) and \(x_k\) have the common initial segment of length \(j\) where \(j < k\). Then, this \(LM_{\mathbb{N},\mathbb{N}}\) is a modular limit with the identity function \(i \mapsto i\) as the modulus of diagonal convergence, and it induces the ordinary topology of \(\mathbb{N} \rightarrow \mathbb{N}\).

## 3 Algebraic Structure

**Definition 3.1 (Algebraic structure)** A sequence \(X = (|X|, f_1, f_2, f_3, \ldots, R_1, R_2, R_3, \ldots)\) is an algebraic structure iff it consists of the underlying set \(|X|\), partial functions \(f_i : |X|^{arity(f_i)} \rightarrow_p |X|\), and subsets \(R_i \subset |X|^{arity(R_i)}\). An algebraic structure \(X\) has finite partial functions or countably many infinite partial functions, and also \(X\) has finite relations or countably many infinite relations. For any \(f_i\) and any \(R_i\), the arities \(arity(f_i)\) and \(arity(R_i)\) belong to \(0 \cup \mathbb{N} \cup \{\omega\}\). The set of the functions and relations \(\{f_1, f_2, f_3, \ldots, R_1, R_2, R_3, \ldots\}\) are called the signature of \(X\).

We sometimes identify \(X\) to \(|X|\), and simply write \(X\) for \(|X|\).

We may define that an algebraic structure has many sorted, or it refers some other algebraic structures.

**Definition 3.2 (Algebraic Structure With Uniform Topology)** Let \(X = (|X|, f_1, f_2, f_3, \ldots, R_1, R_2, R_3, \ldots)\) be an algebraic structure. Suppose that a partial function \(f_i = LM\) belongs to the signature, and \(f_i = LM\) satisfies the definition of modular limit over the domain \(|X|\). Then, this \(X\) is an algebraic structure with uniform topology by the modular limit \(LM\).

If the modular limit is effective, then it is called an algebraic structure with effective uniform topology

**Example 3.3** we define an algebraic structure \(\mathbb{R} = (|\mathbb{R}|, 0, 1, +, -, \times, /, \text{LimE}, <)\) as follows:
The set \(|\mathbb{R}|\) is the set of all the real numbers. The constants 0 and 1 are the numbers 0 and 1 themselves. The functions +, − and × are the ordinary
summation, subtraction and multiplication. The partial function $/$ is the ordinary division. The partial function $\text{LimE} : \mathbb{R}^{\omega} \rightarrow_{p} \mathbb{R}$ is defined as in the example 2.17, which is a limit operation with modulus $1/n$. The relation $<$ is the ordinary inequality without equality.

Then, This $\mathbb{R}$ is an algebraic structure with effective uniform topology.

We sometimes identify $\mathbb{R}$ and $|\mathbb{R}|$.

4 Representation

Notation 4.1 We identify a function $f : \mathbb{N} \rightarrow \mathbb{N}$ to a infinite sequence $(f(0), f(1), f(2), \ldots) \in \mathbb{N}^{\omega}$. For a finite sequence $x \in \mathbb{N}^{*}$ and a finite or infinite sequence $y \in \mathbb{N}^{*} \cup \mathbb{N}^{\omega}$, we write $x \subseteq y$ when $x$ is an initial segment of $y$. Put $\phi_{N^{*}} : \mathbb{N} \rightarrow \mathbb{N}^{*}$ as the standard enumeration of $\mathbb{N}^{*}$ and $\psi_{N^{*}} : \mathbb{N}^{*} \rightarrow \mathbb{N}$ as its inverse.

Definition 4.2 (Recursive Type-2 Function) A partial function of $\mathbb{N}^{\omega} \rightarrow_{p} \mathbb{N}$ is called a partial Type-2 function or partial functional. A partial functional $F : \mathbb{N}^{\omega} \rightarrow_{p} \mathbb{N}$ is recursive iff there is a partial recursive function $f : \mathbb{N} \rightarrow_{p} \mathbb{N}$ which satisfies the following three conditions. We write $\hat{f}$ for $\phi_{N^{*}} \circ f \circ \psi_{N^{*}} : \mathbb{N}^{*} \rightarrow \mathbb{N}^{*}$. Then, the conditions are the following:

1. The function $\hat{f}$ is monotone with respect to $\subseteq$, that is, for any $y \subseteq z \in \mathbb{N}^{*}$, $\hat{f}(y) \subseteq \hat{f}(z)$.
2. For each $x : \mathbb{N}^{\omega}$, $x \in \text{dom}(F)$ iff there are arbitrary long $\hat{f}(y)$ such that $y \subseteq x$, that is, for every $n \in \mathbb{N}$, there exists $y \subseteq x$ such that $\hat{f}(y)$ is longer than the length $n$.
3. For $x \in \text{dom}(F)$, $F(x)$ is the infinite sequence such that $\hat{f}(y) \subseteq F(x)$ for any $y \subseteq x$.

Remark 4.3 This definition is equivalent to the notion of recursive functions relative to another functions (which is given by Odifreddi [O]). And also this is equivalent to the notion defined by Type-2 machines (by Weihrauch [W]).

Definition 4.4 (Representation) A partial function $r : \mathbb{N}^{\omega} \rightarrow_{p} X$ is a representation of $X$ iff it is a surjection, that is, $\text{range}(r) = X$.

Definition 4.5 (Pairing) A pairing function $\langle - , - \rangle : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ is defined as

$$\langle m, n \rangle = \frac{(m + n - 1)(m + n - 2)}{2} + m,$$

which is a standard bijection of $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$.

Remark 4.6 With this paring function $\langle - , - \rangle$, we can regard a unary function as a binary function, such as a unary uction $(n \mapsto f(n)) : \mathbb{N} \rightarrow \mathbb{N}$ as a binary function $(\langle m, n \rangle \mapsto f(\langle m, n \rangle)) : \mathbb{N}^{2} \rightarrow \mathbb{N}$. As inverse, we can regard a binary function as a unary function, such as a binary function $(\langle m, n \rangle \mapsto f(m, n)) : \mathbb{N}^{2} \rightarrow \mathbb{N}$ as a unary function $(\langle m, n \rangle \mapsto f(\langle m, n \rangle)) : \mathbb{N} \rightarrow \mathbb{N}$.
Definition 4.7 (Recursive function with respect to representations)
Let \( X \) and \( Y \) be sets. Let \( r \) and \( r' \) be representations of \( X \) and of \( Y \) respectively. A partial function \( f : X \rightarrow_{p} Y \) is recursive with respect to \( r \) and \( r' \) iff there is a Type-2 partial recursive functional \( F : \mathbb{N}^{\omega} \rightarrow_{p} \mathbb{N}^{\omega} \) such that \( f \circ r = r' \circ F \). If \( X = Y \) and \( r = r' \), then we say simply that the \( f \) is recursive with respect to \( r \).

Remark 4.8 We say that \( f : X^{2} \rightarrow_{p} X \) is recursive with respect to \( r \) iff it is recursive with respect to \((r', r)\) where \( r' \) is defined as:

\[
r'(n \mapsto (w_{1}(n), w_{2}(n))) = (r(w_{1}), r(w_{2})) \quad \text{for any } w_{1}, w_{2} \in \mathbb{N}^{\omega}.
\]

We say similarly for \( f : X^{3} \rightarrow_{p} X, f : X^{4} \rightarrow_{p} X \) and so forth.

We say that \( f : X^{\omega} \rightarrow_{p} X \) is recursive with respect to \( r \) iff it is recursive with respect to \((r'', r)\) where \( r'' \) is defined as:

\[
r''(n \mapsto (i \mapsto w_{i}(n, i))) = r(w) \quad \text{for any } w \in \mathbb{N}^{\omega}.
\]

Definition 4.9 (Reducibility) Let \( X \) be a set, and \( r \) be a representation of \( X \). A partial function \( f : \mathbb{N}^{\omega} \rightarrow_{p} X \) is continuously reducible to \( r \) if there is a partial continuous functional \( F : \mathbb{N}^{\omega} \rightarrow \mathbb{N}^{\omega} \) such that \( dom(f) = dom(F) \) and \( f = r \circ F \). The partial function \( f \) is reducible to \( r \) if such \( F \) is recursive.

Definition 4.10 (Admissibility) Let \( X \) be a topological space. A representation \( r : \mathbb{N}^{\omega} \rightarrow_{p} X \) is admissible if the following conditions hold:

1. (continuity) \( r \) is continuous.
2. (finality) Every partial continuous function \( q : \mathbb{N}^{\omega} \rightarrow_{p} X \) is continuously reducible to \( r \).

Remark 4.11 The notion of admissibility is just a topological notion, and does not include none of effectivity.

Lemma 4.12 Let \( X \) be an algebraic structure with uniform topology, and \( LM \) be the modular limit of \( X \), Let \( r \) be a representation of \( X \). Suppose that \( LM \) is computable with respect to \( r \). Then any continuous partial function \( f : \mathbb{N}^{\omega} \rightarrow X \) is continuously reducible to \( r \).

Theorem 4.13 If a representation \( r \) of \( X \) is continuous, and the modular limit of \( X \) is recursive with respect to \( r \), then \( r \) is admissible.

5 Computability Structure

Remark 5.1 A binary partial recursive function \( f : \mathbb{N} \times \mathbb{N} \rightarrow_{p} \mathbb{N} \) induces a total function of natural numbers into partial recursive functions \( (m \mapsto (n \mapsto f(m, n))) : \mathbb{N} \rightarrow (\mathbb{N} \rightarrow_{p} \mathbb{N}) \). When we say a recursive function \( g : \mathbb{N} \rightarrow (\mathbb{N} \rightarrow_{p} \mathbb{N}) \), we will denote that there is a binary partial recursive function \( f : \mathbb{N} \times \mathbb{N} \rightarrow_{p} \mathbb{N} \) such that \( g : m \mapsto g(m) = (n \mapsto f(m, n)) \).

Definition 5.2 (Term) Let \( \{f_{1}, f_{2}, ... \}, R_{1}, R_{2}, ... \) be a signature and \( V = \{v_{1}, v_{2}, v_{3}, ... \} \) be a set of countably infinitely many variables. Then we define the set of formal expressions \( \text{Term}_{\alpha} \) which is defined for each ordinal \( \alpha \) as
\[ Term_0 = V \]
\[ Term_{\alpha+1} = Term_{\alpha} \]
\[ \cup \{ f_i(t_{1}, t_{2}, \ldots, t_{\text{arity}(f_i)}) | \text{arity}(f_i) \in \{0\} \cup \mathbb{N}, t_j \in Term_{\alpha} \} \]
\[ \cup \{ f_i(t_{1}, t_{2}, t_{3}, \ldots) | \text{arity}(f_i) = \omega, t_j \in Term_{\alpha} \} \]
\[ Term_{\alpha} = \bigcup_{\beta < \alpha} Term_{\beta} \text{ for a limit ordinal } \alpha \]

It is easy to see that \( Term_{\alpha} \) is saturated for uncountable ordinals \( \alpha \), because the arity of every function is countable. Thus \( Term_{\alpha} \) is unique for any uncountable ordinal \( \alpha \). We define \( \text{Term} \) as:
\[ \text{Term} = Term_{\alpha} \text{ for the least uncountable ordinal } \alpha. \]

Elements of \( \text{Term} \) are called \textit{terms}.

**Definition 5.3 (Evaluation)** Let \( X \) be a structure. We define a partial function \([-]: \text{Term} \times (\mathbb{N} \rightarrow_{p} X) \rightarrow_{p} X \)
- For a variable \( v_i \in V = Term_0 \) and a partial function \( \rho : \mathbb{N} \rightarrow_{p} X \),
  \[ [v_i]_{\rho} \text{ is defined iff so is } \rho(i), \text{ and } [v_i]_{\rho} = \rho(i). \]
- For a term \( f_i(t_{1}, t_{2}, \ldots) \in Term_{\alpha+1} - Term_{\alpha} \) and a partial function \( \rho : \mathbb{N} \rightarrow_{p} X \),
  \[ [f_i(t_{1}, t_{2}, \ldots)]_{\rho} \text{ is defined iff } f_i([t_{1}]_{\rho}, [t_{2}]_{\rho}, \ldots) \text{ is defined} \]
  and \[ [f_i(t_{1}, t_{2}, \ldots)]_{\rho} = f_i([t_{1}]_{\rho}, [t_{2}]_{\rho}, \ldots) \]

The function \( r_0 \) of \( V = Term_0 \) is defined as follows.
- For \( x : \mathbb{N}^{\omega}, x \in \text{dom}(r_0) \) iff \( x(1) = 1 \)
- For \( x \in \text{dom}(r_0), r_0(x) = v_0(x) \)

The representation \( r_{\alpha+1} \) of \( Term_{\alpha+1} \) is defined as follows.
- For \( x : \mathbb{N}^{\omega}, x \in \text{dom}(r_0) \) iff either
  \[ x \in \text{dom}(r_{\alpha}) \]
  or,
  \[ x(1) \neq 1, \]
  the signature has the function \( f_{x(1)-1} \)
  and \( x \circ j_i \in \text{dom}(r_{\alpha}) \) for each \( i < \text{arity}(f_{x(1)-1}) \)
- For \( x \in \text{dom}(r_{\alpha+1}), \)
  if \( x \in \text{dom}(r_{\alpha}) \) then \( r_{\alpha+1}(x) = r_{\alpha}(x) \),
  otherwise \( r_{\alpha+1}(x) = f_{x(1)-1}(r_{\alpha}(x \circ j_1), r_{\alpha}(x \circ j_2), r_{\alpha}(x \circ j_3), \ldots) \)

The representation \( r_{\alpha} \) of \( Term_{\alpha} \) for a limit ordinal \( \alpha \) is defined as follows.
\[ \text{dom}(r_{\alpha}) = \bigcup_{\beta < \alpha} \text{dom}(r_{\beta}) \]

For \( a \in \text{dom}(r_{\alpha}) \),
- if \( x \in \text{dom}(r_0) \) then \( r_{\alpha}(x) = r_0(x) = v_{x(1)} \),
- and if \( x \in \text{dom}(r_{\beta+1}) - \text{dom}(r_{\beta}) \) then \( r_{\alpha}(x) = r_{\beta+1}(x) \)

This \( r_{\alpha} \) is saturate as \( \alpha \) is uncountable. We define \( r_T \) as
\[ r_T(x) = r_{\alpha}(x) \text{ for the least uncountable ordinal } \alpha. \]
Remark 5.5 It is obvious that this $r_\alpha : N^\omega \rightarrow_p \mathrm{Term}_\alpha$ is surjective. Thus, $r_T$ is a surjection into $\mathrm{Term}$.

Definition 5.6 (Computability Structure) Let $X$ be an algebraic structure. A subset $S \subset N \rightarrow X$ is a computability structure over $X$ if $S$ satisfies the followings:

1. (Permutation) For each $s \in S$ and each total recursive function $f : N \rightarrow N$, it holds that $s \circ f \in S$.
2. (Merging) For any $s, s' \in S$, there is $s'' \in S$ such that $s''(2n) = s(n)$ and $s''(2n + 1) = s'(n)$.
3. (Effective sequence of terms) For each $s \in S$ and each recursive function $f : N \rightarrow (N \rightarrow_p N)$, if $f(m) \in \mathrm{dom}(r_T)$ and $[r_T(f(m))]|_{\tau}(i \rightarrow s((n,i)))$ is defined for each $m, n \in N$, then there is $s' \in S$ such that $s'((m, n)) = [r_T(f(m))]|_{\tau}(i \rightarrow s((n,i)))$

Example 5.7 Regard $N^\omega$ as an algebraic structure with uniform topology $(N^\omega, LM_{N,N,N})$, as in Example 2.19. Put $S_{N^\omega}$ as

$$\{ f : N \rightarrow N^\omega \mid f(i)(j) = g(i, j), g \text{ is a binary total recursive function} \}.$$

Then, this $S_{N^\omega}$ is a computability structure over $N^\omega$. This $S_{N^\omega}$ is the standard computability structure of it.

Definition 5.8 (Computable function) Let $X$ and $Y$ be algebraic structure with effective uniform topology, and $LM^X$ and $LM^Y$ be the effective limits of them, respectively. Let $S_X$ and $S_Y$ be computability structures of $X$ and $Y$, respectively.

Then, a partial function $f : X \rightarrow Y$ is computable if there exists a total recursive function $m : N^2 \rightarrow N$ and the following five conditions hold:

1. There is $a \in S_X$ such that $s(N)$ is dense in $\mathrm{dom}(f)$.
2. For each $s \in S_X$, there is a partial recursive function $i : N^2 \rightarrow_p N$ such that for each $n \in N$, if $s(n) \in \mathrm{dom}(f)$, then $LM^X(a(i(n, 1)), a(i(n, 2)), a(i(n, 3)), \ldots) = s(n)$.
3. $f \circ a \in S_Y$
4. For every $(i_1, i_2, i_3, \ldots) \in N^\omega$, if $(a(i_1), a(i_2), a(i_3), \ldots) = \mathrm{dom}(LM^X)$, then there is a sequence $(j_1, j_2, j_3, \ldots) \in N^\omega$ such that $m(i_{j_n}, n) \leq j_n$ for any $n \in N$.
5. if $LM^X(a(i_1), a(i_2), a(i_3), \ldots) = x$, and $m(i_{j_n}, n) \leq j_n$ for any $n \in N$, then $LM^Y(f(a(i_{j_1})), f(a(i_{j_2})), f(a(i_{j_3})), \ldots) = f(x)$.

Proposition 5.9 Let $f : X \rightarrow_p Y$ and $g : Y \rightarrow_p Z$ be computable functions with respect to $(S_X, S_Y)$ and $(S_Y, S_Z)$, respectively. If $\mathrm{range}(f) \subset \mathrm{dom}(g)$, then $g \circ f$ is computable with respect to $(S_X, S_Z)$. 
Notation 5.10 We abbreviate the following condition over a quadruple \((X, LM, r, S)\) as the condition \((*)\):

- The first component \(X\) is an algebraic structures with effective uniform topology.
- The second \(LM\) is the effective limits of \(X\).
- The third \(r\) is a representation of \(X\) such that all the partial functions of the signature of \(X\) are recursive with respect to \(r\). Thus, \(S_r\) is a computable structures over \(X\).
- As for the forth component, \(S = S_r\).

Lemma 5.11 Let \((X, LM, r, S_r)\) be a quadruple which satisfies the condition \((*)\). Then, for each function \(f : N^\omega \to X\), if \(f\) is computable with respect to \((S_{N^\omega}, S_r)\), then \(f\) is reducible to \(r\).

Remark 5.12 This proof follows the steps similar to those in the proof of Lemma 4.12.

Corollary 5.13 A partial function \(f : N^\omega \to_p N^\omega\) is recursive if it is com-
putable with respect to \(S_{N^\omega}\).

Lemma 5.14 Let \(f\) be a partial function of \(N^\omega \to_p N^\omega\). Suppose that \(f\) is recursive and there is \(a \in S_{N^\omega}\) such that \(a(N)\) is dense in \(\text{dom}(f)\). Then, \(f\) is computable with respect to \(S_{N^\omega}\).

Corollary 5.15 Let \(f\) be a partial function of \(N^\omega \to_p N^\omega\). Suppose that there is \(a \in S_{N^\omega}\) such that \(a(N)\) is dense in \(\text{dom}(f)\). Then, \(f\) is recursive iff it is computable with respect to \(S_{N^\omega}\).

Lemma 5.16 Let \((X, LM^X, r, S_X)\) be a quadruple which satisfies the condition \((*)\). Let \(Y\) be an algebraic structure with effective uniform topology, and \(LM^Y\) be the effective limit of it. Let \(S_Y\) be a computability structure over \(Y\). Suppose that \(r\) is computable with respect to \((S_{N^\omega}, S_X)\).

Then, for any \(f : X \to_p Y\), if \(f \circ r\) is computable with respect to \((S_{N^\omega}, S_Y)\), then \(f\) is computable with respect to \((S_X, S_Y)\).

Theorem 5.17 (Main Theorem) Let \((X, LM^X, r_X, S_X)\) and \((Y, LM^Y, r_Y, S_Y)\) be quadruples which satisfy the condition \((*)\). Suppose that \(r_X\) and \(r_Y\) are computable with respect to \((S_{N^\omega}, S_X)\) and \((S_{N^\omega}, S_Y)\), respectively.

Let \(f : X \to_p Y\) and \(F : N^\omega \to_p N^\omega\) be partial functions which satisfy the
following:
1. \(f \circ r_X = r_Y \circ F\)
2. \(\text{range}(F) \subset \text{dom}(R_Y)\)
3. There is \(a \in S_{N^\omega}\) such that \(a(N)\) is dense in \(\text{dom}(F)\).

Then, \(F\) is recursive iff \(f\) is computable with respect to \((S_X, S_Y)\).
6 Computability in Yasugi's sense

Remark 6.1 There is another definition of computable functions, which is given by Yasugi et al. [Y]. Their definition is defined by using uniform neighbourhood system. We call the computability defined by their definition the computability in Yasugi's sense, or Y-computability.

Definition 6.2 (Computable function in Yasugi's sense) Let $X = (|X|, \{V^X_i(x)\}_{i,x}, S_X)$ be a triples which consists of an underlying space $X$, an effective uniform neighbourhood system $\{V^X_i(x)\}$, and a computability structure $S_X$. As is usual, we identify $|X|$ to $X$.

A partial function $f : X \to \mathbb{R}$ is computable in Yasugi's sense, or Y-computable, iff the following hold:
1. For $s \in S_X$, if $s(N) \subseteq dom(f)$, then $f \circ s \in S_R$.
2. For any $s \in S_X$, there is a total recursive function $j_s : \mathbb{N} \to \mathbb{N}^\omega$ such that $f(V^X_{j_s(i,n)}(x_i)) \subseteq V^R_{\circ s}(f(x_i))$ for any $i$ and $n$.
3. There are $e \in S_X$ and a total recursive function $j_e : \mathbb{N} \to \mathbb{N}^\omega$ such that
   (3.1) The range $e(N)$ is dense in $X$.
   (3.2) $f(V^X_{j_e(i,n)}(x_i)) \subseteq V^R_{\circ s}(f(x_i))$ for any $i$ and $n$.
   (3.3) $\bigcup_{i \in \mathbb{N}} V^X_{j_e(i,n)}(x_i) = X$.

Lemma 6.3 Let $X$ be an algebraic structure with effective uniform topology, and $LM$ be the effective limit of it. Put $\{V_i(x)\}$ be the effective uniform neighbourhood system defined as Lemma 2.14. Let $S_X$ be a computable structure over $X$.

If a function $f : X \to \mathbb{R}$ is computable, then it is Y-computable.

Lemma 6.4 Let $X$ be an algebraic structure with effective uniform topology, and $LM$ be the effective limit of it. Put $\{V_i(x)\}$ be the effective uniform neighbourhood system defined as Lemma 2.14. Let $S_X$ be a computable structure over $X$. Suppose the following set is recursively enumerable for each $s \in S_X$:
$$\{(i,j,n,n') \in \mathbb{N}^4 \mid V^R_j(s(n')) \subseteq V^X_i(s(n))\}$$

If a function $f : X \to \mathbb{R}$ is Y-computable, then it is computable.

Corollary 6.5 Under the same assumption as the previous lemma (6.4), a function $f : X \to \mathbb{R}$ is computable iff it is Y-computable.

Remark 6.6 The algebraic structure $\mathbb{R}$ satisfies the assumption of the previous lemma (6.4). Therefore, for each function of $\mathbb{R} \to \mathbb{R}$, it is computable iff it is Y-computable.

Remark 6.7 This condition appears in the definition of computability by Yasugi et al.:
$$\bigcup_{i \in \mathbb{N}} V^X_{j_e(i,n)}(x_i) = X.$$ The function $j_e$ corresponds to the modulus $m$ in our definition.
In their definition, the condition asserts only the existence of \( i \) such that \( x \in V^X j_e(i, n)(x_i) \) for \( x \), but not the effectivity of such \( i \). Our definition asserts the effectivity of such \( i \), because our definition asserts the numerical inequality, which is recursively justified.

7 Conclusional remark

The essential definition of this work is Definition 5.8, which defines computable functions over algebraic structures with effective uniform topology. The main theorem (Theorem 5.17) says that the computability in our definition is equivalent to that in the definition by Weihrauch and others, with some suitable assumption. And also, Corollary 6.5 says that our computability is equivalent to that by Yasugi and others, with some suitable assumption.

The author think that this definition 5.8 has some conditions on the domain of the function, which seems inessential. The author would like to make this condition more natural as a future work.

Acknowledgements
The author is grateful to Brattka, Kamo, Mori, Tsuiki and Yasugi for discussions and comments.

References


