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HEIGHT FUNCTIONS OVER FUNCTION FIELDS

DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY
ATSUSHI MORIWAKI

For details of this talk, see [1], [2], [3] and [4].

1. Function fields

First of all, we fix two kinds of functions fields, namely, an arithmetic function field and a geometric function field.

- An arithmetic function field is a finitely generated extension field of $\mathbb{Q}$.
- A geometric function field is a finitely generated extension field of an algebraically closed field.

2. Height function on $\mathbb{P}^1(\mathbb{Q})$

First, let us review a height of a rational number. Roughly speaking, it measures the complexity of rational numbers, and you may agree with the following:

$$\boxed{\text{The complexity of rational numbers}} \fallingdotseq \boxed{\text{The magnitude of numerators and denominators}}$$

Hence, for $a/b \in \mathbb{Q}$ ($a, b \in \mathbb{Z}$ and $a\mathbb{Z} + b\mathbb{Z} = \mathbb{Z}$), the complexity h of a/b should be

$$h = \log \max\{|a|, |b|\}.$$

This gives rise to a height function h^{arith} on

$$\mathbb{P}^1(\mathbb{Q}) = \{(a : b) \mid a, b \in \mathbb{Q}, (a, b) \neq (0, 0)\},$$

namely, for $x = (a : b)$ with $a, b \in \mathbb{Z}$ and $a\mathbb{Z} + b\mathbb{Z} = \mathbb{Z}$,

$$\boxed{h^{\text{arith}}(x) = \log \max\{|a|, |b|\}.}$$

3. Height function on $\mathbb{P}^1(\mathbb{Q}(t))$

For this purpose, we need to ask again what

$$\boxed{\text{the complexity of polynomials}}$$

3.1. **Geometric case.** (Complexity = Degree)
For $x = (f(t) : g(t))$ with $f(t), g(t) \in \mathbb{Z}[t]$ and $f(t), g(t)$ relatively prime,

$$\boxed{h^{\mathrm{geom}}(x) = \max\{\deg(f(t)), \deg(g(t))\}.}$$

h^{geom} is NOT an extension of h^{arith} when we view $\mathbb{Q}$ as a subfield of $\mathbb{Q}(t)$.

3.2. **Arithmetic case.** (Complexity = Degree + Largeness of coefficients)
For $f = \sum_i a_i t^i \in \mathbb{Q}[t]$, we set

$$|f|_\infty = \max_i\{|a_i|\}.$$

Then, as before, we may consider

$$\max\{\deg(f(t)), \deg(g(t))\} \\ + \log\max\{|f|_\infty, |g|_\infty\},$$

which is NOT good from the geometric view point. Thus, we need a more sophisticated invariant to measure the largeness of coefficients. For this purpose, let us fix a positive (1,1)-form Ω on $\mathbb{P}^1(\mathbb{C})$ with $\int_{\mathbb{P}^1(\mathbb{C})} \Omega = 1$. Then, we set

$$v(f) = \exp\left(\int_{\mathbb{P}^1(\mathbb{C})} \log|f|\Omega\right).$$

We can see $\|f\|_\infty \fallingdotseq v(f)$. Hence, we may define

$$\boxed{\begin{aligned} h^{\mathrm{arith}}(x) = {} & \max\{\deg(f(t)), \deg(g(t))\} \\ & + \int_{\mathbb{P}^1(\mathbb{C})} \log\max\{|f(t)|, |g(t)|\}\Omega. \end{aligned}}$$

4. A quick review of Arakelov Geometry

4.1. **Arithmetic curve.** Let K be a number field and O_K the ring of integers in K. Let $K(\mathbb{C})$ be the set of all embeddings $K \hookrightarrow \mathbb{C}$. Let L be a flat and finitely generated O_K-module of rank 1. For an embedding $\sigma \in K(\mathbb{C})$, the tensor product $L \otimes_K \mathbb{C}$ in terms of the embedding σ is denoted by $L \otimes_\sigma \mathbb{C}$. Let $\|\cdot\|_\sigma$ be a hermitian metric of $L \otimes_\sigma \mathbb{C}$. The collection $\left(L, \{\|\cdot\|_\sigma\}_{\sigma \in K(\mathbb{C})}\right)$ is called a hermitian line bundle on $C = \mathrm{Spec}(O_K)$. For simplicity, it is denoted by $\overline{L}$.

Let s be a non-zero element of L. Then, let us consider:

$$\log\#(L/sL) - \sum_\sigma \log(\|s \otimes_\sigma 1\|_\sigma).$$

Then, by the product formula, it does not depend on the choice of s, so that it is denoted by $\widehat{\deg}(\overline{L})$.

4.2. **General case.**

X : a projective and flat integral scheme over $\mathbb{Z}$ such that $X \to \mathrm{Spec}(\mathbb{Z})$ is smooth over $\mathbb{Q}$.

(Z,T) : for a non-negative integer p, a pair (Z,T) is called an arithmetic cycle codimension p if Z is a cycle of codimension p and T is a current of type $(p-1,p-1)$ on $X(\mathbb{C})$.

$\widehat{Z}^p(X)$: the set of all arithmetic cycles of codimension p.

$\widehat{R}^p(X)$: the subgroup of $\widehat{Z}^p(X)$ generated by the following elements:

(1) $((f), -[\log |f|^2])$, where f is a non-zero rational function on an integral closed subscheme Y of codimension $p-1$ and $[\log |f|^2]$ is the current defined by

$$[\log |f|^2](\gamma) = \int_{Y(\mathbb{C})} (\log |f|^2)\gamma.$$

(2) $(0, \partial(\alpha)+\bar{\partial}(\beta))$, where α and β are currents of type $(p-2,p-1)$ and $(p-1,p-2)$ respectively.

Note that $\widehat{Z}^0(X) = \mathbb{Z}(X,0)$ and $\widehat{R}^0(X) = 0$.

Here we define

$$\widehat{\mathrm{CH}}^p(X) := \widehat{Z}^p(X)/\widehat{R}^p(X).$$

Let $\overline{L} = (L, \|\cdot\|)$ be a C^∞-hermitian line bundle on X, that is, L is a line bundle on X and $\|\cdot\|$ is a C^∞-hermitian metric of $L_{\mathbb{C}}$ on $X(\mathbb{C})$. We define a homomorphism

$$\hat{c}_1(\overline{L})\cdot : \widehat{\mathrm{CH}}^p(X) \to \widehat{\mathrm{CH}}^{p+1}(X)$$

in the following way: Let (Z,T) be an element of $\widehat{Z}^p(X)$. For simplicity, we assume that Z is integral. Then, taking a non-zero rational section s of $L|_Z$, we consider an arithmetic cycle of codimension $p+1$:

$$(\mathrm{div}(s) \text{ on } Z, -[\log(\|s\|_Z^2)] + c_1(\overline{L}) \wedge T),$$

where $[\log(\|s\|_Z^2)]$ is a current given by $\phi \mapsto \int_{Z(\mathbb{C})} \log(\|s\|_Z^2)\phi$.

Let $\overline{L}_1, \ldots, \overline{L}_{\dim X}$ be C^∞-hermitian line bundles on X. Then,

$$\hat{c}_1(\overline{L}_1)\cdots\hat{c}_1(\overline{L}_{\dim X}) \in \widehat{\mathrm{CH}}^{\dim X}(X).$$

Moreover, we have a homomorphism

$$\widehat{\deg} : \widehat{\mathrm{CH}}^{\dim X}(X) \to \mathbb{R}$$

given by

$$\widehat{\deg}\left(\sum_P n_P P, T\right) = \sum_P n_P \log \#(\kappa(P)) + \frac{1}{2}\int_{X(\mathbb{C})} T.$$

Thus, we have the number

$$\widehat{\deg}(\hat{c}_1(\overline{L}_1)\cdots\hat{c}_1(\overline{L}_{\dim X})),$$

which is called the intersection number of $\overline{L}_1, \ldots, \overline{L}_{\dim X}$. Note that the intersection number

$$\widehat{\deg}(\hat{c}_1(\overline{L}_1)\cdots\hat{c}_1(\overline{L}_{\dim X}))$$

can be defined even if $X \to \mathrm{Spec}(\mathbb{Z})$ is not smooth over $\mathbb{Q}$.

5. Polarization and height function

K : an arithmetic function field, i.e., a field finitely generated over $\mathbb{Q}$.

d : the transcendental degree of K over $\mathbb{Q}$.

B : a projective and flat integral scheme over $\mathbb{Z}$ whose function field is K.

$\overline{H}$: a nef hermitian line bundle on B, i.e. the Chern form $c_1(\overline{H})$ on $B(\mathbb{C})$ is semi-positive and $\widehat{\deg}\left(\hat{c}_1(\overline{H})\cdot(Z,0)\right) \geq 0$ for every integral 1-dimensional subscheme Z on B.

$(B,\overline{H})$: A pair $(B,\overline{H})$ is called a polarization of K, denoted by $\overline{B}$.

For $(\phi_0,\ldots,\phi_n) \in K^{n+1}\setminus\{0\}$, we define

$$h^{\overline{B}}(\phi_0,\ldots,\phi_n) := \sum_{\Gamma} \max_i\{-\mathrm{ord}_\Gamma(\phi_i)\}\widehat{\deg}\left(\hat{c}_1\left(\overline{H}|_\Gamma\right)^d\right) + \int_{B(\mathbb{C})} \log\left(\max_i\{|\phi_i|\}\right) c_1(\overline{H})^{\wedge d}.$$

(Γ's run over all prime divisors on B)

It is easy to see

$$h^{\overline{B}}(x\phi_0,\ldots,x\phi_n) = h^{\overline{B}}(\phi_0,\ldots,\phi_n).$$

Thus we get

$$h^{\overline{B}} : \mathbb{P}^n(K) \to \mathbb{R}.$$

⋆ In the case where K is a number field, $h^{\overline{B}}$ is the arithmetic height function.

⋆ In the case where B is an arithmetic surface and $\overline{H} = (\mathcal{O}_B, c|\cdot|_{\mathrm{can}})$ $(0 < c < 1)$, $h^{\overline{B}}$ is a constant multiple of the geometric height function as

6. Another description

$*$ Fix a polarization:

$$\begin{cases} K : \text{ an arithmetic function field} \\ d := \text{tr.deg}_{\mathbb{Q}}(K). \\ B : \text{ a projective and flat integral scheme} \\ \quad \text{over } \mathbb{Z} \text{ whose function field is } K. \\ \overline{H} : \text{ a nef hermitian line bundle on } B. \\ \overline{B} = (B, \overline{H}) : \text{ a polarization of } K. \end{cases}$$

$*$ Variety and line bundle over K

$$\begin{cases} X : \text{ a projective variety over } K. \\ L : \text{ a line bundle on } X. \end{cases}$$

$*$ Model of (X, L)

$$\begin{cases} \mathcal{X} : \text{ an integral projective scheme over } B \\ \quad \text{whose generic fiber of } \mathcal{X} \to B \text{ is } X. \\ \overline{\mathcal{L}} : \text{ a hermitian line bundle on } \mathcal{X} \text{ which gives} \\ \quad \text{rise to } L \text{ on the generic fiber of } \mathcal{X} \to B. \end{cases}$$

A pair $(\mathcal{X}, \overline{\mathcal{L}})$ is called a model of (X, L).

$*$ Δ_P for $P \in X(\overline{K})$

For $P \in X(\overline{K})$, the Zariski closure of the image

$$\text{Spec}(\overline{K}) \xrightarrow{P} X \hookrightarrow \mathcal{X}$$

is denoted by Δ_P.

Then we define $h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})} : X(\overline{K}) \to \mathbb{R}$ to be

$$h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(P) := \frac{\widehat{\deg}\left(\widehat{c}_1(\overline{\mathcal{L}}|_{\Delta_P}) \cdot \widehat{c}_1(f^*(\overline{H})|_{\Delta_P})^d\right)}{[K(P) : K]},$$

where f is the canonical morphism $\mathcal{X} \to B$. Note that if $(\mathcal{X}', \overline{\mathcal{L}}')$ is another model of (X, L), then there is a constant C with

$$\left| h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}(P) - h^{\overline{B}}_{(\mathcal{X}',\overline{\mathcal{L}}')}(P) \right| \leq C \quad (\forall P \in X(\overline{K}))$$

This means that $h^{\overline{B}}_{(\mathcal{X},\overline{\mathcal{L}})}$ is uniquely determined modulo bounded functions on $X(\overline{K})$, so that we may write it as $h^{\overline{B}}_{L}$.

7. Northcott's theorem

Theorem 1 (Northcott's theorem). *We assume that $\overline{H}$ is big, i.e., $\mathrm{rk}_{\mathbb{Z}} H^0(B,$ $O(m^d)$ and for a sufficient large n, there is a non-zero $s \in H^0(B, H^{\otimes n})$ with $\|s\|_{\sup} < 1$. Then, for any M and e, the set*

$$\{P \in X(\overline{K}) \mid h^{\overline{B}}_{L}(P) \leq M, \quad [K(P) : K] \leq e\}$$

is finite.

Theorem 2 (Refinement). *We assume that $\overline{H}$ is big. Then, for a fixed e,*

$$\frac{\log \#\{P \in X(\overline{K}) \mid h^{\overline{B}}_{L}(P) \leq h, [K(P) : K] \leq e\}}{h^{d+1}}$$

is bounded above as h goes to the infinity.

8. The number of algebraic cycles

In the similar techniques, we have the following:

Theorem 3 (Geometric version). *Let X be a projective scheme over a finite field $\mathbb{F}_q$ and H a very ample line bundle on X. For a non-negative integer k, we denote by $n_k(X, H, l)$ the number of effective l-dimensional cycles with*

$$\deg(H^{\cdot l} \cdot V) = k.$$

Then, there is a constant C depending only on l and $\dim_{\mathbb{F}_q} H^0(X, H)$ such that

$$\log_q(n_k(X, H, l)) \leq C k^{l+1}$$

for all $k \geq 1$.

Theorem 4 (Arithmetic version). *Let X be a projective and flat integral scheme over $\mathbb{Z}$ and $\overline{H}$ an ample C^∞-hermitian line bundle X. For a real number h, we denote by $n_{\leq h}(X, \overline{H}, l)$ the number of effective l-dimensional cycles with*

$$\widehat{\deg}(\widehat{c}_1(H)^{\cdot l} \cdot V) \leq h.$$

Then, there is a constant C such that

$$\log(n_{\leq h}(X, \overline{H}, l)) \leq C h^{l+1}$$

for all $h \geq 1$.

Remark 5. The above two theorems might give rise to new zeta functions. For example, in Theorem 3, if we set

$$Z(X,H,l)(T) = \sum_{k=0}^{\infty} n_k(X,H,l)T^{k^{l+1}},$$

then $Z(X,H,l)$ is a convergent power series at 0. Moreover, in Theorem 4, if we set

$$\zeta(X,\overline{H},l)(s) = \sum_{V} \exp\left(-s\cdot \widehat{\deg}(\widehat{c}_1(H)^{\cdot l}\cdot V)^{l+1}\right)$$

is a convergent Dirichlet series on $\mathrm{Re}(s) \gg 0$, where V runs over all effective l-dimensional cycles.

9. Height function on an abelian variety

We assume that X is an abelian variety A. Let L be a symmetric ample line bundle on A. Then, as in the usual theory of height functions, we have the canonical quadratic function

$$\hat{h}_L^{\overline{B}} : A(\overline{K}) \to \mathbb{R}.$$

Actually, it is defined by

$$\hat{h}_L^{\overline{B}}(P) := \lim_{n\to\infty} \frac{h_L^{\overline{B}}(nP)}{n^2}.$$

By Northcott's theorem, if $\overline{H}$ is big, then

$$\hat{h}_L^{\overline{B}}(P) = 0 \quad \Longleftrightarrow \quad P \in A(\overline{K})_{\mathrm{tor}}.$$

From now on, we assume that $\overline{H}$ is big. Here we set

$$\langle x,y\rangle_L^{\overline{B}} = \frac{1}{2}\left(\hat{h}_L^{\overline{B}}(x+y) - \hat{h}_L^{\overline{B}}(x) - \hat{h}_L^{\overline{B}}(y)\right)$$

Then, $\langle\ ,\ \rangle_L^{\overline{B}}$ gives rise to an inner product $A(\overline{K})\otimes\mathbb{R}$. For $x_1,\ldots,x_l \in A(\overline{K})$, we set

$$\delta_L^{\overline{B}}(x_1,\ldots,x_l) := \det\left(\langle x_i,x_j\rangle_L^{\overline{B}}\right).$$

10. Bogomolov + Mordell

Theorem 6. *Let Γ be a subgroup of finite rank in $A(\overline{K})$, and Y a subvariety of $A_{\overline{K}}$. Let us fix a basis $\{\gamma_1,\ldots,\gamma_n\}$ of $\Gamma\otimes\mathbb{Q}$. If the set*

$$\{x\in Y(\overline{K}) \mid \delta_L^{\overline{B}}(\gamma_1,\ldots,\gamma_n,x) \le \epsilon\}$$

is Zariski dense in Y for every positive number ϵ, then Y is a translation of an abelian subvariety of $A_{\overline{K}}$ by an element of Γ_{div}, where

$$\Gamma_{\mathrm{div}} = \{x\in A(\overline{K}) \mid \exists n\in\mathbb{Z}_{>0}\ nx\in\Gamma\}.$$

Corollary 7 (Bogomolov's conjecture). *Let Y be a subvariety of $A_{\overline{K}}$. If the set*

$$\{x \in Y(\overline{K}) \mid \hat{h}_L^{\overline{B}}(x) \leq \epsilon\}$$

is Zariski dense in Y for every positive number ϵ, then Y is a translation of an abelian subvariety of $A_{\overline{K}}$ by a torsion point.

Corollary 8 (Mordell-Lang conjecture). *Let A be a complex abelian variety, Γ a subgroup of finite rank in $A(\mathbb{C})$, and Y a subvariety of A. Then, there are abelian subvarieties $C_1, \ldots, C_n$ of A, and $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that*

$$\overline{Y(\mathbb{C}) \cap \Gamma} = \bigcup_{i=1}^{n} (C_i + \gamma_i)$$

and

$$Y(\mathbb{C}) \cap \Gamma = \bigcup_{i=1}^{n} (C_i(\mathbb{C}) + \gamma_i) \cap \Gamma.$$

11. Outline of the proof

Step 1: Prove Bogomolov's conjecture, i.e. the case where $\Gamma = 0$.
Step 2: Verify the special case of Mordell-Lang conjecture:

If $Y(K)$ is dense in Y, then Y is a translation of an abelian subvariety.

Step 3: Poonen's idea + Step 1 + Step 2

12. Poonen's idea

K : a field finitely generated over $\mathbb{Q}$.

$\overline{B} = (B, \overline{H})$: a big polarization of K ($\overline{H}$: big).

A : an abelian variety over K.

L : a symmetric ample line bundle on A.

Γ : a subgroup of finite rank in $A(\overline{K})$ such that there is a finitely generated subgroup Γ_0 of $A(K)$ with $\Gamma_0 \otimes \mathbb{Q} = \Gamma \otimes \mathbb{Q}$.

Let S be an infinite subset of $A(\overline{K})$. We say S is small with respect to Γ if there is a decomposition $s = \gamma(s) + z(s)$ for each $s \in S$ with the following properties:

(1) $\gamma(s) \in \Gamma$ for all $s \in S$;
(2) for any $\epsilon > 0$, there is a finite proper subset S' of S such that $\hat{h}_L^{\overline{B}}(z(s)) \leq \epsilon$ for all $s \in S \setminus S'$.

Let F be a finite extension of K. For $x \in A(\overline{K})$, we set

$$O_F(x) := \{\sigma(x) \mid \sigma \in \mathrm{Gal}(\overline{K}/F)\}.$$

For an integer $n \geq 2$, let $\beta_n : A^n \to A^{n-1}$ be a homomorphism given by

$$\beta_n(x_1, \ldots, x_n) = (x_2 - x_1, x_3 - x_1, \ldots, x_n - x_1).$$

For a subset T of S and a finite extension F of K, we set

$$\mathcal{D}_n(T, F) = \bigcup_{s \in T} \beta_n(O_F(s)^n).$$

Moreover, we denote by $\overline{\mathcal{D}}_n(T, F)$ the Zariski closure of $\mathcal{D}_n(T, F)$.

A pair (S, K) is said to be minimized if

(1) for any infinite subset T of S and any finite extension F of K, $\overline{\mathcal{D}}_2(T, F) = \overline{\mathcal{D}}_2(S, K)$;

(2) $\overline{\mathcal{D}}_2([N](S), K) = \overline{\mathcal{D}}_2(S, K)$ for all integers $N \geq 1$.

Note that if an infinite subset S of $A(\overline{K})$ is small with respect to Γ, then there are an infinite subset T of S, a finite extension F of K, and a positive integer N such that $([N](T), F)$ is minimized.

Theorem 9 (Poonen-Moriwaki). *Let S be an infinite subset of $A(\overline{K})$ such that S is small with respect to Γ. If (S, K) is minimized, then there is an abelian subvariety C of $A_{\overline{K}}$ such that $\overline{\mathcal{D}}_n(S, K) = C^{n-1}$ for all $n \geq 2$.*

The above theorem is a consequence of Bogomolov's conjecture.

Three ingredients:

1 the above theorem

2 the special case of Mordell-Lang conjecture

3 a geometric trick to remove a measure-theoretic argument in Poonen's paper

imply the main theorem.

More precisely, we can prove it in the following way:

Replacing K by a finite extension of K, we may assume that there is a finitely generated subgroup Γ_0 of $\Gamma \cap A(K)$ with $\Gamma_0 \otimes \mathbb{Q} = \Gamma \otimes \mathbb{Q}$. We set

$$\mathrm{Stab}(Y) = \{a \in A \mid Y + a = Y\}.$$

Considering $A/\mathrm{Stab}(Y)$, it is sufficient to show the following claim.

Claim: If $\mathrm{Stab}(Y) = \{0\}$, then Y is a point.

We assume that $\dim Y > 0$. Then, replacing K by a finite extension of K, we can find an infinite subset S of $Y(\overline{K})$ with the following properties:

(1) S is small with respect to Γ_{div}.

(2) S is Zariski dense in Y.

(3) (S, K) is minimized.

Then, there is an abelian subvariety C of $A_{\overline{K}}$ with $\overline{\mathcal{D}}_n(S, K) = C^{n-1}$ for all $n \geq 2$. If $\dim C = 0$, then $S \subseteq A(K)$. Thus, by the special case of Mordell-Lang conjecture, Y is a translation of an abelian subvariety B of $A_{\overline{K}}$. Then, $\mathrm{Stab}(Y) = B$. Thus, $\dim B = 0$, which implies $\dim Y = 0$, so that we have a contradiction.

Next we assume that $\dim C > 0$. Let us fix a positive integer n with $n > 2\dim(A)$. Let $\pi : A \to A/C$ be the natural homomorphism and $T = \pi(Y)$. Let Y_T^n be the fiber product over T in Y^n. Then, we have a morphism $\beta_n : Y_T^n \to A^{n-1}$ given by

$$\beta_n(x_1, \ldots, x_n) = (x_2 - x_1, \ldots, x_n - x_1).$$

Since $O_K(s)^n \subseteq X_T^n$, let Y be the Zariski closure of $\bigcup_{s \in S} O_K(s)^n$. Then, $\beta_n(Y) \supseteq C^{n-1}$. Thus, we get

$$\dim(X_T^n) \geq \dim(C^{n-1}).$$

On the other hand, since $\mathrm{Stab}(Y) = \{0\}$,

$$\dim(X/T) \leq \dim(C) - 1.$$

Thus,

$$\begin{aligned} \dim(X_T^n) - \dim(C^{n-1}) &= (n\dim(X/T) + \dim(T)) \\ &\quad -(n-1)\dim(C) \\ &\leq \dim(C) + \dim(T) - n \\ &\leq 2\dim(A) - n < 0. \end{aligned}$$

This is a contradiction.

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