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Constructing topological groups through unit equations

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Abstract

We treat problems concerning duality properties of topological groups. To solve them, we make the additive group of the integers into topological groups. The onstruction depends on a family of exponential Diophantine equations.

1 Introduction

We exhibit an application of exponential Diophantine equations to some problems on characters of topological groups. In Section 2, we introduce two duality properties we consider. Section 3 is for the explanation of the metrics on the integers due to J. W. Nienuys [4]. In Section 4, we find particular metrics answering the questions. The construction is closely tied with a family of S-unit equations. As an appendix, we mention the ineffectiveness of the method.

Most of the contents of this article overlap those of [5] or [6], which is mainly intended for the audience with a topological background. Here we proceed more number-theoretically.

2 Problems

All topological groups we treat are Hausdorff and Abelian, and a character is a continuous homomorphism into the torus $\mathbf{T} = \mathbf{R}/\mathbf{Z}$. A subgroup H of a topological group G is *dually closed* if for each $g \in G$ on the outside of H, there exists a character χ of G separateing g from H; i.e., χ vanishes on H but does not at g. We say that H is *dually embedded* if every character of H is obtained as the restriction of one of G.

Our concern is for the following two properties: "every closed subgroup is dually closed" and "every closed subgroup is dually embedded." We denote the former by X(1) and the latter by X(2) after [1].

The problem is whether these are preserved under direct products. Constructing a counterexample, We show that so is neither against misunderstanding in the literature ([8]).

3 Metrics on the Integers

We begin with some metric group topologies on the integers as in [4]. Suppose that $\delta : \{p^n : n \in \mathbf{N}\} \to \mathbf{R}_{>0}$ is a non-increasing function defined on the powers of a prime p with $\delta(p^n) \to 0$ as $n \to \infty$. We define a function $||\cdot||_\delta : \mathbf{Z} \to \mathbf{R}$ by

$$||u||_\delta = \inf\left\{\sum_i \delta(p^{n_i}) : u = \sum_i e_i p^{n_i}, e_i \in \{1,-1\}, n_i \in \mathbf{N}\right\}.$$

We denote by $\mathbf{Z}_\delta$ the topological group $\mathbf{Z}$ with the metric induced by $||\cdot||_\delta$. This topology is finer than or equal to the p-adic topology.

Our counterexample consists of $\mathbf{Z}_\delta$ and $\mathbf{Z}_\varepsilon$ for some δ defined on the powers of p and ε on those of another prime q. Here we must choose 'nice' δ and ε with a certain number-theoretic property, which is made precise in the next section.

We have rather straightforward observations unconditionally:

1. Both groups have X(1) and X(2);
2. The diagonal $\Delta = \{(u,u) : u \in \mathbf{Z}\} \subset \mathbf{Z}_\delta \times \mathbf{Z}_\varepsilon$ is dually-closed.
3. There exists a homomorphism $\Delta \to \mathbf{T}$ that is not obtained as the restriction of a character of the whole product.

Accordingly if Δ is discrete (and closed in the product), then the product has neither X(1) nor X(2).

4 Number-theoretic Requirements

For the diagonal Δ to be discrete, we find 'nice' δ and ε such that

$$\inf\{||u||_\delta + ||u||_\varepsilon : u \in \mathbf{Z}, u \neq 0\} > 0.$$

Here we invoke a finiteness theorem for S-unit equations, which is similar to [3, Theorem 8].

Theorem 4.1 *Suppose that G and H are finitely generated subgroups of $\mathbf{C}^*$. For any positive integers k and l, there are finite sets $A \subseteq G$ and $B \subseteq H$ such that for every solution of the equation*

$$x_1 + \cdots + x_k = y_1 + \cdots + y_l$$

with $x_1, ..., x_k \in G$, $y_1, ..., y_l \in H$ and no vanishing subsums, one has $x_1, ..., x_k \in A$ and $y_1, ..., y_l \in B$. □

Now we construct a pair of metrics as desired. Let p and q be distinct primes and k, l and s positive integers. We apply the theorem above to the groups $G = \langle p, -1\rangle$ and $H = \langle q, -1\rangle$, and set

$$F(p,q,k,l) = \{a \in A : a \geq 1\}$$

with respect to the purported set A and

$$F(p,q,s) = \bigcup_{k+l\leq s} F(p,q,k,l).$$

Then the final definition follows:

$$\delta(p^n) = 1/\min\{s : p^n \leq \max F(p,q,s)\},$$

$$\varepsilon(q^n) = 1/\min\{s : q^n \leq \max F(q,p,s)\}.$$

Note that if

$$e_1p^{m_1} + \cdots + e_kp^{m_k} = f_1q^{n_1} + \cdots + f_lq^{n_l}$$

has no vanishing subsums with non-negative integers $m_1, ..., m_k, n_1, ...n_l$ and $e_1, ..., e_k, f_1, ..., f_l \in \{\pm 1\}$, then we have $p^{m_i} \in F(p,q,k+l), q^{n_j} \in F(q,p,k+l)$, and hence

$$\delta(p^{m_i}), \varepsilon(q^{n_j}) \geq \frac{1}{k+l}$$

for each $1 \leq i \leq k$ and $1 \leq j \leq l$. Accordingly for a non-zero integer u with

$$u = e_1p^{m_1} + \cdots + e_kp^{m_k} = f_1q^{n_1} + \cdots + f_lq^{n_l},$$

it holds that

$$||u||_\delta + ||u||_\varepsilon \geq \delta(p^{m_1}) + \cdots + \delta(p^{m_k}) + \varepsilon(q^{n_1}) + \cdots \varepsilon(q^{n_l}) \geq 1.$$

Thus we are done.

Theorem 4.2 *Neither* X(1) *nor* X(2) *is preserved under the product* $\mathbf{Z}_\delta \times \mathbf{Z}_\varepsilon$ *for* δ *and* ε *decreasing slowly enough.* □

A Appendix

Since Theorem 4.1 is ineffective, we do not have explicit functions in Theorem 4.2 or even the estimation of their order. Here we exhibit a now unsuccessful attempt at effectivization.

We recall an analogue due to C.L. Stewart [11, Theorem 1]. Suppose that a and b are integers greater than 1 with $\log a/\log b$ irrational. Then, from some estimations for linear forms in logarithms, effective lower bound is obtained for the sum of the numbers of non-zero digits of a positive integer n in base a and in base b.

We would like to find a similar bound in case 'negative digits' are allowed. That is, for an integer n with a representation, which may not be unique,

$$\begin{aligned} n &= a_1 a^{m_1} + a_2 a^{m_2} + \cdots + a_r a^{m_r} \\ &= b_1 b^{l_1} + b_2 b^{l_2} + \cdots + b_t b^{l_t}, \end{aligned} \tag{1}$$

where the integers satisfy following conditions:

$$0 < |a_i| < a,$$

$$0 < |b_j| < b,$$

for $i = 1, 2, ..., r$ and $j = 1, 2, ..., t$, and

$$m_1 > m_2 > ... > m_r \geq 0,$$

$$l_1 > l_2 > ... > l_t \geq 0,$$

we want an effective lower bound for $r + t$ in term of n.

We assume that n is positive and sufficiently large and try to proceed as in [11]. For appropriate $1 \leq p \leq r$ and $1 \leq q \leq t$, set

$$\begin{aligned} A_1 a^{m_p} &= a_1 a^{m_1} + \cdots + a_p a^{m_p}, \\ A_2 &= a_{p+1} a^{m_{p+1}} + \cdots + a_r a^{m_r}, \\ B_1 b^{l_q} &= b_1 b^{l_1} + \cdots + b_q b^{l_q}, \\ B_2 &= b_{q+1} b^{l_{q+1}} + \cdots + b_t b^{l_t}, \end{aligned} \tag{2}$$

$$R = \frac{A_1 a^{m_p}}{B_1 b^{l_q}}.$$

A parallel argument breaks down at the upper estimation for $\max\{R, R^{-1}\}$, since we have no efficient lower bound for $A_1 a^{m_p}$.

We may save part of the proof as follows: if there exists a positive integer n with (1) and (2) such that

$$\begin{aligned} &4 \max \left\{ \frac{|A_2|}{A_1 a^{m_p}}, \frac{|B_2|}{B_1 b^{l_q}} \right\} \\ \leq\ &\exp\left(-C(3,1) \log(\max\{e, A_1, B_1\}) \log(\max\{e, a\}) \log(\max\{e, b\}) \log(\max\{e, m_p, l_q\})\right), \end{aligned} \tag{3}$$

$$\max\{m_p, l_q\} > C_1(3,1) \log a \log b \log(\max\{A_1, B_1\}), \tag{4}$$

where the constants C and C_1 are from [2] and from [9], respectively,

$$C(n,d) = 18(n+1)! n^{n+1} (32d)^{n+2} \log(2nd),$$

$$C_1(n,d) = \left(\frac{3}{2}nd\right)^{n-1} (21d \log(6d))^{\min\{n, d+1\}},$$

then it follows that $\log a / \log b$ is rational. More precisely, (3) implies that $R = 1$, which, in turn combined with (4), yields the rationality results. So it suffices to get a lower bound for $r + s$ assuming that for every representation (1) and partition (2) at least one of (3) and (4) fails. We, however, have no idea about

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