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On sums of cubes of almost primes.

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1. Introduction—The Waring-Goldbach problem for cubes.

Being based on Vinogradov's method in his proof of the three prime theorem, Hua started discussing representations of natural numbers as sums of powers of prime numbers. This problem is often refered as the Waring-Goldbach problem. In 1938, Hua [4] proved, amongst others, that every sufficiently large odd integer can be written as the sum of nine cubes of primes. In fact, he established an asymptotic formula for the number of representations of large natural numbers as the sum of s cubes of primes, for each s exceeding 8.

The above result of Hua implies that for each s exceeding 9, every large integer can be written as the sum of s cubes of primes, because for any integer n, either $n-(s-9)2^{3}$ or $n-(s-10)2^{3}-3^{3}$ is odd. In this direction, therefore, the next target is the sum of eight cubes of primes. It is conjectured that every large even number can be written in the latter manner. Although no one could succeed in proving it by now, some results were shown concerning this problem. First, Roth [6] proved that every large integer can be written in the form

$$n=p_{1}^{3}+\cdots+p_{7}^{3}+x^{3},$$

where p_{i}'s are primes and x is a natural number. Brüdern [2] improved this result, by showing that when n is even and large, one can restrict x to P_{4} in the above representation. (P_{r} denotes a natural number having at most r prime divisors, counted according to multiplicity.) By changing the sieve procedure in Brüdern's proof, the author [5] substituted P_{3} for P_{4} in Brüdern's theorem. It is of course desired to replace P_{3} further with

P_1, that is, a prime, but it seems to me that even the improvement to P_2 is way beyond the current state of technique.

We then proceed to the sum of seven cubes. Needless to say, the situation becomes harder than the case of eight cubes, but it is still possible to show that every large integer can be written as the sum of seven cubes of almost primes. Indeed, Brüdern [3] established that every large integer n can be written in the form

$$n = p^3 + x_1^3 + \cdots + x_5^3 + y^3,$$

where p is a prime, x_i's are P_5, and y is P_{69}. The purpose of this short account is to report several refinements on the latter result of Brüdern. This work was essentially done while I visited the University of Michigan at Ann Arbor through courtesy of Professor Trevor D. Wooley, and enjoyed the benefits of a Fellowship from the David and Lucile Packard Foundation, from April to June 1997. I am disappointed and ashamed that even now (as of March 2003) I could not complete the paper of this work yet. Also, I would like to apologize to the organizer, Professor Noriko Hirata-Kohno, for violating the deadline for this report (...as usual).

2. Problems on sums of seven cubes.

We are concerned with the conclusions of the following form:

"Every large integer can be written as $x_1^3 + \cdots + x_7^3$, where x_i is P_{r_i} for each i."

As Brüdern [3] mentioned, there are various combinations of r_i's for which one can prove the latter statement. As regards this problem, personally I am interested in three questions. Here they are:

(A) What is the possible least value of $\max\{r_1, \dots, r_7\}$?

(B) What is the possible least value of $r_1 + \cdots + r_7$?

(C) How many variables can we force to be primes? (In other words, what is the possible largest number of i's with $r_i = 1$?)

Probably one can expect that every sufficiently large natural number can be written as the sum of seven cubes of primes, so we may conjecture that the answers for these questions are 1 for (A), and 7 for both (B) and (C), in truth. Before writing down the answers that I actually proved, we should mention a congruence condition relating the sum of seven cubes of primes.

When we consider representaions of n as the sum of seven cubes of primes, it is natural to impose the following condition on n: For every natural number q, the congruence

$$n \equiv x_1^3 + \cdots + x_7^3 \pmod q$$

has a solution such that all the x_i's are coprime to q. By elementary argument, one may easily confirm that the latter condition is eventually equivalent to $2 \nmid n$ and $9 \nmid n$. What happens if n violates it? For instance, suppose that $9|n$. Then for $q = 9^*$, the above congruence has no solution with $(x_1 x_2 \ldots x_7, 9) = 1$. Therefore, if such an n is written as the sum of seven cubes of primes, then at least one of the primes must be not coprime to 9, namely, it must be 3. Thus n is a sum of seven cubes of primes, if and only if $n - 3^3$ is a sum of six cubes of primes. Still n may be written as the sum of seven cubes of primes, but the representation problem no longer involves seven variables in practice. In this sense the above congruence condition arises naturally, when one investigates the sum of seven cubes of primes. And as a modest form of the above conjecture, one may say that every large n satisfying $2 \nmid n$ and $9 \nmid n$ can be written as the sum of seven cubes of primes[†].

*It may be worth pointing out not only that 9 looks similar to q in shape, but also that in Japanese, 9 is pronounced exactly the same way as the letter "q". How curious!

†An elementary deliberation on the above congruences may convince us that most likely this statement is true even in the cases where $2|n$ or $9|n$. The hardest case will be the integers n satisfying $14|n$ and $n \equiv 0$ or $-2 \pmod 9$. If such an n is written as the sum of seven cubes of primes, then three of the seven primes must be 2, 3 and 7, which means that $n - 2^3 - 3^3 - 7^3$ must be the sum of four cubes of primes. Although one may expect so if n is large, one must face a problem on four cubes which seems very hard to solve.

In this context, moreover, it may be worth recording here that 7 is presumably the least value of s for which every large integer can be written as the sum of s cubes of primes. To see this, suppose that an odd natural number n satisfies the congruences $n + 1 \equiv 0$ or $\pm 1 \pmod 9$ and $n \equiv \pm 1 \pmod 7$, and that n is written as the sum of six cubes of primes. Then elementary argument on congruences reveals that there must exist three primes p_1, p_2, p_3 such that $n - 2^3 - 3^3 - 7^3 = p_1^3 + p_2^3 + p_3^3$. But if $n \le X$, then $p_i \le X^{1/3}$, so simply there are at most $O(X(\log X)^{-3})$ such numbers n. Hence almost all numbers satisfying the congruence conditions we are now assuming cannot be written as the sum of six cubes of primes.

3. Results.

In connection with the congruence condition observed in the previous section, we introduce the numbers a_n and b_n as follows;

$$a_n = \begin{cases} 1, & \text{when } n \text{ is odd,} \\ 2, & \text{when } n \text{ is even,} \end{cases} \qquad b_n = \begin{cases} 1, & \text{when } 9 \nmid n, \\ 3, & \text{when } 9 | n. \end{cases}$$

Then, as for the questions (A) and (B) above, the following conclusions may be obtained.

Theorem 1 *Every sufficiently large integer n can be written as each of the following forms;*

(i) $n = x_1^3 + \cdots + x_5^3 + (a_n y_1)^3 + (b_n y_2)^3$, *where x_i's are P_4, and y_1, y_2 are P_3.*

(ii) $n = p^3 + x_1^3 + \cdots + x_4^3 + (a_n b_n x_5)^3 + y^3$, *where p is a prime, x_i's are P_3, and y is P_5.*

The part (i) says that every large integer can be written as the sum of seven cubes of P_4, and we may consequently answer 4 to the question (A). By (ii), as regards odd integers n with $9 \nmid n$, we may answer 21 to the question (B).

Theorem 1 may be proved by adding two ingredients to the method of Brüdern [3]. One of them is concerned with the technique that is called, for example, as "Vaughan's iterative method restricted to minor arcs". In [3], a simpler version of the latter method was adopted, but we may do better at this point by appealing to the original argument of Vaughan [8]. Another point is about sieve methods. Brüdern [3] used a weighted sieve, but we apply the switching principle (or, the reversal rôle technique) with the ordinary linear sieve. It seems that the switching principle may give stronger conclusions than weighted sieves, in most of the situations where it can be applied.

We then proceed to the problem (C). I confess that all my efforts to answer 4 to this problem have ended in failure by now, while one can easily answer 3 to (C) by a couple of known results. In fact, it follows immediately from the work of Vaughan [7] that when n is a large integer, the number of positive integers of the form $n - p_1^3 - p_2^3 - p_3^3$ with primes p_i

is $\gg n^{8/9+\varepsilon}$, for any fixed $\varepsilon > 0$. On the other hand, Brüdern [1] showed that the number of positive integers less than n that is not the sum of four cubes is $\ll n^{37/42+\varepsilon}$, for any given $\varepsilon > 0$. Since $8/9 > 37/42$, these results together indicate the existence of an integer of the form $n - p_1^3 - p_2^3 - p_3^3$ with primes p_i that is written as the sum of four cubes, whenever n is large. This tells that 3 is a possible answer for (C).

Moreover, it is possible to refine the last conclusion by saying that every large n can be written as $n = p_1^3 + p_2^3 + p_3^3 + x_1^3 + \cdots + x_4^3$ with primes p_i and almost primes x_j. In this respect, I suppose that the best result may be given by the vector sieve of Brüdern and Fouvry. Our method of the proof of Theorem 1 can also say something on this, due to Wooley's breaking classical convexity device. Via the latter way and some numerical computation, I confirmed that in the last representation of n, one can restrict x_1, x_2 and x_3 to P_{10}, and x_4 to P_{24}, for example. In any case, rather than refining the quality of almost primes here, I would like to make serious effort towards establishing 4 as a possible answer for (C).

4. Sums of eight cubes of almost primes.

Finally I would like to discuss the questions on the sum of eight cubes corresponding to (A), (B) and (C) above. As is written in the introduction, Roth [6] answered 7 for the question corresponding to (C), and the improvement on this seems far beyond our grasp at present. The result of the author [5] shows that 10 is a possible answer for the question corresponding to (B). To beat it, one must work with the sum of seven cubes of primes and a cube of P_2, and it is again quite hard to do for now, I think. As regards (A), the best known answer is 3 in view of [5], and there is room for improvement on this. Actually, I discussed this problem with Brüdern, and obtained the following result recently. We now recall the definition of a_n in the previous section.

Theorem 2 *Every sufficiently large integer n can be written as*

$$n = p_1^3 + \cdots + p_5^3 + (a_{n-1}p_6)^3 + x^3 + y^3,$$

where p_i denotes primes, x and y are P_2.

I confess that I spent quite a lot of time on this problem, but at last it could be proved within the techniques of Brüdern [2] and Kawada [5].

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