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Arithmetical properties of solutions of certain q-difference equations

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In the present note we report certain results obtained by the author with Keijo Väänänen on the linear independence of the values of theta series and Tschakaloff functions.

1

Let $\Theta(\boldsymbol{q},z)$ be the theta series defined by

$$\Theta(\boldsymbol{q},z)=\sum_{n=-\infty}^{\infty}\boldsymbol{q}^{n^2}z^n,\quad |\boldsymbol{q}|<1, \tag{1}$$

and let $T_q(z)$ be the Tschakaloff function defined by

$$T_q(z)=\sum_{n=0}^{\infty}\frac{z^n}{q^{\binom{n}{2}}},\quad |q|>1. \tag{2}$$

If $\boldsymbol{q}q=1$, then these functions are connected by the equation

$$\Theta(\boldsymbol{q},z)=T_{q^2}(q^{-1}z)+T_{q^2}(q^{-1}z^{-1})-1. \tag{3}$$

It was shown in [3] and [5] that Nesterenko's result [6] on the values of Ramanujan functions implies the transcendence of $\Theta(\boldsymbol{q},1)$ for all algebraic $\boldsymbol{q}$, $0<|\boldsymbol{q}|<1$ and that of $T_q(1)$ for all algebraic q, $|q|>1$. However there are no such results for $\Theta(\boldsymbol{q},z)$ with general algebraic α, even if we take some special value $\boldsymbol{q}=1/q$, $q\in\mathbf{Z}\backslash\{0,\pm1\}$ for $\boldsymbol{q}$. On the other hand, since the paper [8] of Tschakaloff there are a lot of works containing linear independence results on the values of $T_q(z)$ and its derivatives, a good overview is given in [4]. In particular, it was proved in [8] if q is an integer in an imaginary quadratic field $\mathcal{I}$, $|q|>1$, and $\alpha_1,\ldots,\alpha_\ell$ are nonzero elements of $\mathcal{I}$ satisfying $\alpha_i/\alpha_j\notin q^{\mathbf{Z}}$ for all $i\neq j$, then the numbers $1,T_q(\alpha_1),\ldots,T_q(\alpha_\ell)$ are linearly independent over $\mathcal{I}$, see also [9] for some results on more general algebraic number fields. By (3), one can obtain linear independence of $1, \Theta(\boldsymbol{q}, \alpha_1), ..., \Theta(\boldsymbol{q}, \alpha_\ell)$ under some extra conditins on α_i, see [7], Korollar 2.

In the present note we are interested in the linear independence of the numbers

$$1, \Theta(\boldsymbol{q}_1, \alpha_1), ..., \Theta(\boldsymbol{q}_\ell, \alpha_\ell)$$

with different $\boldsymbol{q}_i$. As far as we know there are no earlier results of this type.

2

In the following we assume that $q \in \mathbf{Z} \backslash \{0, \pm 1\}$.

Theorem 1. *Let ℓ and L be positive integers with $L \geq 2(\ell - 1)$. Then there exists an effectively computable positive constant $\gamma_1(\ell, L)$ such that, for any positive integers $s_1, ..., s_\ell$ satisfying*

$$\gamma_1(\ell, L) \leq s_1 < s_2 < \cdots < s_\ell \leq s_1 + L/2$$

and for any nonzero rational numbers $\alpha_1, ..., \alpha_\ell$ satisfying

$$\alpha_i \notin -q^{s_i(1+2\mathbf{Z})} \quad (i = 1, ..., \ell),$$

the numbers

$$1, \Theta(q^{-s_1}, \alpha_1), ..., \Theta(q^{-s_\ell}, \alpha_\ell)$$

are linearly independent over the rationals.

This result is a consequence of the following result on the values of Tschakaloff functions.

Theorem 2. *Let ℓ and L be positive integers with $L \geq \ell - 1$, and let $m_1, ..., m_\ell$ be positive integers. Then there exists an effectively computable positive constant $\gamma_2(m, L)$ with $m = m_1 + \cdots + m_\ell$ such that, for any positive integers $s_1, ..., s_\ell$ satisfying*

$$\gamma_2(m, L) \leq s_1 < s_2 < \cdots < s_\ell \leq s_1 + L$$

and for any nonzero rational numbers α_{ij} $(i = 1, ..., \ell;\ j = 1, ..., m_i)$ satisfying

$$\alpha_{ij_1}/\alpha_{ij_2} \notin q^{s_i \mathbf{Z}} \quad (i = 1, ..., \ell;\ j_1 \neq j_2),$$

the numbers

$$1,\ T_{q^{s_i}}(\alpha_{ij}) \quad (i=1,...,\ell;\ j=1,...,m_i)$$

are linearly independent over the rationals.

We here deduce Theorem 1 from Theorem 2. Denoting $q_i = q^{s_i}$, we have by (3)

$$\Theta(q_i^{-1},\alpha_i) = T_{q_i^2}(q_i^{-1}\alpha_i) + T_{q_i^2}(q_i^{-1}\alpha_i^{-1}) - 1 \quad (i=1,...,\ell).$$

Since

$$\frac{q_i^{-1}\alpha_i}{q_i^{-1}\alpha_i^{-1}} = \alpha_i^2 \notin (q_i^2)^{\mathbf{Z}} \quad \Longleftrightarrow \quad \alpha_i \notin \pm q_i^{\mathbf{Z}},$$

the statement of Theorem 1 follows from Theorem 2 under the right-hand conditions inthe above. Therefore, our task is to relax these conditions into the conditions given in the theorem. To this aim let us fix i and assume $\alpha_i \in \pm q_i^{\mathbf{Z}}$. Then we have $\alpha_i = \epsilon q_i^{-n}$ for some nonnegative integer n assuming $|\alpha_i| \le 1$ without loss of generality, where ϵ is 1 or -1. (We may replace the role of α_i by that of α_i^{-1} if necessary.) By denoting $\beta_i = q_i^{-1}\alpha_i$ this gives $q_i^{-1}\alpha_i^{-1} = \beta_i q_i^{2n}$. Since the repeated application of the functional equation

$$T_{q_i^2}(q_i^2 z) = q_i^2 z T_{q_i^2}(z) + 1$$

for $T_{q_i^2}(z)$ implies

$$T_{q_i^2}(q_i^{2n} z) = q_i^{2\binom{n+1}{2}} z^n T_{q_i^2}(z) + P_{in}(z), \quad P_{in}(z) \in \mathbf{Z}[z],$$

on noting $\beta_i = \epsilon q_i^{-(n+1)}$, we have

$$T_{q_i^2}(q_i^{-1}\alpha_i^{-1}) = q_i^{2\binom{n+1}{2}} \beta_i^n T_{q_i^2}(\beta_i) + P_{in}(\beta_i) = \epsilon^n T_{q_i^2}(\beta_i) + P_{in}(\beta_i).$$

Hence,

$$\Theta(q_i^{-1},\alpha_i) = \begin{cases} P_{in}(\beta_i) - 1, & \epsilon = -1 \text{ and } n \text{ is odd}, \\ 2T_{q_i^2}(\beta_i) + P_{in}(\beta_i) - 1, & \text{otherwise}, \end{cases}$$

which completes the deduction of Theorem 1 from Theorem 2.

For the proof of Theorem 2 we use the method given in [2] and [10]. More precisely, we define

$$f_{ij}(z)=\sum_{n=0}^{\infty}\frac{z^{s_i n}}{q^{s_i\binom{n+1}{2}}}\alpha_{ij}^n \quad (i=1,...,\ell;\ j=1,...,m_i),$$

which satisfy the q-difference equations

$$\alpha_{ij}z^{s_i}f_{ij}(z)=f_{ij}(qz)-1.$$

Since $f_{ij}(q)=T_{q^{s_i}}(\alpha_{ij})$, the statement of Theorem 2 follows from the linear independence over the rationals of the numbers

$$1,\ f_{ij}(q) \quad (i=1,...,\ell;\ j=1,...,m_i).$$

To prove this statement we construct Padé-type approximations of the second kind for $f_{ij}(z)$ taking the conditions $s_1<s_2<\cdots<s_\ell$ into account (see [1] for details).

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