**Arithmetical properties of solutions of certain $q$-difference equations**

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In the present note we report certain results obtained by the author with Keijo Väänänen on the linear independence of the values of theta series and Tschakaloff functions.

1

Let $\Theta(q, z)$ be the theta series defined by

$$\Theta(q, z) = \sum_{n=-\infty}^{\infty} q^{n^2} z^n, \quad |q| < 1,$$

and let $T_q(z)$ be the Tschakaloff function defined by

$$T_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{q^{(n^2/2)}}, \quad |q| > 1.$$

If $qq = 1$, then these functions are connected by the equation

$$\Theta(q, z) = T_{q^2}(q^{-1}z) + T_{q^2}(q^{-1}z^{-1}) - 1.$$

It was shown in [3] and [5] that Nesterenko's result [6] on the values of Ramanujan functions implies the transcendence of $\Theta(q, 1)$ for all algebraic $q$, $0 < |q| < 1$ and that of $T_q(1)$ for all algebraic $q$, $|q| > 1$. However there are no such results for $\Theta(q, z)$ with general algebraic $\alpha$, even if we take some special value $q = 1/q$, $q \in \mathbb{Z}\setminus\{0, \pm 1\}$ for $q$. On the other hand, since the paper [8] of Tschakaloff there are a lot of works containing linear independence results on the values of $T_q(z)$ and its derivatives, a good overview is given in [4]. In particular, it was proved in [8] if $q$ is an integer in an imaginary quadratic field $\mathcal{I}$, $|q| > 1$, and $\alpha_1, ..., \alpha_k$ are nonzero elements of $\mathcal{I}$ satisfying $\alpha_i/\alpha_j \not\in q^\mathbb{Z}$ for all $i \neq j$, then the numbers $1, T_q(\alpha_1), ..., T_q(\alpha_k)$ are linearly independent over $\mathcal{I}$, see also [9] for some results on more general algebraic number...
fields. By (3), one can obtain linear independence of $1, \Theta(q, \alpha_1), ..., \Theta(q, \alpha_{\ell})$ under some extra conditions on $\alpha_i$, see [7], Korollar 2.

In the present note we are interested in the linear independence of the numbers

$$1, \Theta(q_1, \alpha_1), ..., \Theta(q_\ell, \alpha_\ell)$$

with different $q_i$. As far as we know there are no earlier results of this type.

2

In the following we assume that $q \in \mathbb{Z}\setminus\{0, \pm 1\}$.

**Theorem 1.** Let $\ell$ and $L$ be positive integers with $L \geq 2(\ell - 1)$. Then there exists an effectively computable positive constant $\gamma_1(\ell, L)$ such that, for any positive integers $s_1, ..., s_\ell$ satisfying

$$\gamma_1(\ell, L) \leq s_1 < s_2 < \cdots < s_\ell \leq s_1 + L/2$$

and for any nonzero rational numbers $\alpha_1, ..., \alpha_\ell$ satisfying

$$\alpha_i \notin -q^{s_i(1+2\mathbb{Z})} \quad (i = 1, ..., \ell),$$

the numbers

$$1, \Theta(q^{-s_1}, \alpha_1), ..., \Theta(q^{-s_\ell}, \alpha_\ell)$$

are linearly independent over the rationals.

This result is a consequence of the following result on the values of Tschakaloff functions.

**Theorem 2.** Let $\ell$ and $L$ be positive integers with $L \geq \ell - 1$, and let $m_1, ..., m_\ell$ be positive integers. Then there exists an effectively computable positive constant $\gamma_2(m, L)$ with $m = m_1 + \cdots + m_\ell$ such that, for any positive integers $s_1, ..., s_\ell$ satisfying

$$\gamma_2(m, L) \leq s_1 < s_2 < \cdots < s_\ell \leq s_1 + L$$

and for any nonzero rational numbers $\alpha_{ij}$ ($i = 1, ..., \ell; j = 1, ..., m_i$) satisfying

$$\alpha_{ij_1}/\alpha_{ij_2} \notin q^{s_i\mathbb{Z}} \quad (i = 1, ..., \ell; j_1 \neq j_2),$$
the numbers
\[ 1, T_{q^{s_{i}}}^{n}(\alpha_{ij}) \quad (i=1,\ldots,\ell; \ j=1,\ldots,m_{i}) \]
are linearly independent over the rationals.

We here deduce Theorem 1 from Theorem 2. Denoting \( q_{i} = q^{s_{i}} \), we have by (3)
\[
\Theta(q_{i}^{-1}, \alpha_{i}) = T_{q_{i}^{-1}}^{n}(q_{i}^{-1}\alpha_{i}) + T_{q_{i}^{-1}}^{n}(q_{i}^{-1}\alpha_{i}^{-1}) - 1 \quad (i=1,\ldots,\ell).
\]
Since
\[
\frac{q_{i}^{-1}\alpha_{i}}{q_{i}^{-1}\alpha_{i}^{-1}} = \alpha_{i}^{2} \notin (q_{i}^{2})^{\mathbb{Z}} \iff \alpha_{i} \notin \pm q_{i}^{\mathbb{Z}},
\]
the statement of Theorem 1 follows from Theorem 2 under the right-hand conditions in the above. Therefore, our task is to relax these conditions into the conditions given in the theorem. To this aim let us fix \( i \) and assume \( \alpha_{i} \in \pm q_{i}^{\mathbb{Z}} \). Then we have \( \alpha_{i} = \epsilon q_{i}^{-n} \) for some nonnegative integer \( n \) assuming \( |\alpha_{i}| \leq 1 \) without loss of generality, where \( \epsilon \) is 1 or \(-1\). (We may replace the role of \( \alpha_{i} \) by that of \( \alpha_{i}^{-1} \) if necessary.) By denoting \( \beta_{i} = q_{i}^{-1}\alpha_{i} \) this gives \( q_{i}^{-1}\alpha_{i}^{-1} = \beta_{i}q_{i}^{2n} \). Since the repeated application of the functional equation
\[
T_{q_{i}^{2}}(q_{i}^{2}z) = q_{i}^{2}zT_{q_{i}^{2}}(z) + 1
\]
for \( T_{q_{i}^{2}}(z) \) implies
\[
T_{q_{i}^{2}}(q_{i}^{2n}z) = q_{i}^{2(n+1)}z^{n}T_{q_{i}^{2}}(z) + P_{n}(z), \quad P_{n}(z) \in \mathbb{Z}[z],
\]
on noting \( \beta_{i} = \epsilon q_{i}^{-(n+1)} \), we have
\[
T_{q_{i}^{2}}(q_{i}^{-1}\alpha_{i}^{-1}) = q_{i}^{2(n+1)}\beta_{i}^{n}T_{q_{i}^{2}}(\beta_{i}) + P_{n}(\beta_{i}) = \epsilon^{n}T_{q_{i}^{2}}(\beta_{i}) + P_{n}(\beta_{i}).
\]
Hence,
\[
\Theta(q_{i}^{-1}, \alpha_{i}) = \begin{cases} 
P_{n}(\beta_{i}) - 1, & \epsilon = -1 \text{ and } n \text{ is odd,} \\
2T_{q_{i}^{2}}(\beta_{i}) + P_{n}(\beta_{i}) - 1, & \text{otherwise,}
\end{cases}
\]
which completes the deduction of Theorem 1 from Theorem 2.
For the proof of Theorem 2 we use the method given in [2] and [10]. More precisely, we define

$$f_{ij}(z) = \sum_{n=0}^{\infty} \frac{z^{s_{i}n}}{q^{s_{i}(\binom{n+1}{2})}} \alpha_{ij}^{n} \quad (i = 1, \ldots, \ell; \ j = 1, \ldots, m_{i}),$$

which satisfy the $q$-difference equations

$$\alpha_{ij} z^{s_{i}} f_{ij}(z) = f_{ij}(qz) - 1.$$

Since $f_{ij}(q) = T_{q^{s_{i}}}(\alpha_{ij})$, the statement of Theorem 2 follows from the linear independence over the rationals of the numbers

$$1, \ f_{ij}(q) \quad (i = 1, \ldots, \ell; \ j = 1, \ldots, m_{i}).$$

To prove this statement we construct Padé-type approximations of the second kind for $f_{ij}(z)$ taking the conditions $s_{1} < s_{2} < \cdots < s_{\ell}$ into account (see [1] for details).

References


[8] L. Tschakaloff, Arithmetische Eigenschaften der unendlichen Reihe $\sum_{\nu=0}^{\infty} \pi^{\nu} a^{-\frac{1}{2}\nu(\nu+1)} I$, Math. Ann. 80 (1921), 62–74; II, ibid. 84 (1921), 100–114.
