<table>
<thead>
<tr>
<th>Title</th>
<th>THE DISTRIBUTION OF CLASS NUMBERS OF PURE NUMBER FIELDS (Diophantine Problems and Analytic Number Theory)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Peter, Manfred</td>
</tr>
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</tbody>
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THE DISTRIBUTION OF CLASS NUMBERS OF PURE NUMBER FIELDS

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Much is known about the statistical distribution of class numbers of binary quadratic forms and quadratic fields. Let \( d \equiv 0, 1 \mod 4 \) and \( d \) not a perfect square. Define \( h(d) \) as the number of equivalence classes of primitive binary quadratic forms with discriminant \( d \) (and positive definite in case \( d < 0 \)). For \( d > 0 \), let \( \epsilon_d := (u_d + v_d \sqrt{d})/2 \), where \((u_d, v_d)\) is the fundamental solution of Pell's equation \( u^2 - dv^2 = 4 \). If \( d \) is a fundamental discriminant then \( h(d) \) is also the class number of \( \mathbb{Q}(\sqrt{d}) \) in the narrow sense.


\[
\sum_{0 < d \leq x} h(d) \log \epsilon_d \sim \frac{\pi^2}{18 \zeta(3)} x^{3/2}, \quad \sum_{0 > d \geq -x} h(d) \sim \frac{\pi}{18 \zeta(3)} x^{3/2}.
\]

Chowla and Erdös [3] proved that there is a continuous distribution function \( F \) such that for all \( z \in \mathbb{R} \),

\[
\lim_{x \to \infty} \frac{1}{x/2} \# \left\{ 0 < d \leq x \left| \frac{h(d) \log \epsilon_d}{d^{1/2}} \leq e^z \right. \right\} = F(z),
\]

\[
\lim_{x \to \infty} \frac{1}{x/2} \# \left\{ 0 > d \geq -x \left| \frac{h(d) \pi}{|d|^{1/2}} \leq e^z \right. \right\} = F(z).
\]

Elliott [4] showed that \( F \in C^\infty(\mathbb{R}) \) and it has the characteristic function

\[
\Psi(t) = \prod_p \left( \frac{1}{p} + \frac{1}{2} \left( 1 - \frac{1}{p} \right) \left( 1 - \frac{1}{p} \right)^{-it} + \frac{1}{2} \left( 1 - \frac{1}{p} \right) \left( 1 + \frac{1}{p} \right)^{-it} \right), \quad t \in \mathbb{R}.
\]

Barban [1] proved that for \( q \in \mathbb{N} \), the \( q \)-th moment \( \beta_q \) of \( F(\log z) \) exists and that

\[
\lim_{x \to \infty} \frac{1}{x/2} \sum_{0 < d \leq x} \left( \frac{h(d) \log \epsilon_d}{d^{1/2}} \right)^q = \beta_q = \sum_{n \geq 1} \frac{\varphi(n) d_q(n^2)}{2n^3},
\]

\[
\lim_{x \to \infty} \frac{1}{x/2} \sum_{0 > d \geq -x} \left( \frac{h(d) \pi}{|d|^{1/2}} \right)^q = \beta_q,
\]

where \( \varphi \) is Euler's totient function and \( d_q(n) \) is the number of ways one can write \( n \) as a product of \( q \) positive integers. For all these results, error term estimates can be given (see [2], [6], [10], [12], [13]).

It seems that for number fields of higher degree, no analogous results are known. The Brauer-Siegel Theorem (see, e.g., [8], Chapter XVI) gives a rough idea of the size of the class number times the regulator: Let \( k \) range over a sequence of number fields which are galois over \( \mathbb{Q} \) such that \( n/ \log d \to 0 \), where \( n := [k : \mathbb{Q}] \) is the degree and \( d = d_k/q \) is the
absolute discriminant of $k$. Let $h_k$ be the class number of $k$ and $R_k$ its regulator. Then
\[
\frac{\log(h_kR_k)}{\log d^{1/2}} \to 1.
\]
When looking for more precise information on the value distribution of
\[
\frac{h_kR_k}{d^{1/2}},
\]
we run into the problem of how to effectively parametrize number fields. This problem is avoided in the present paper by choosing a special class of number fields: Let $l$ be a fixed rational prime and
\[
S_l := \{m \in \mathbb{N} \setminus \{1\} \mid m \text{ is } l\text{-power-free}\}.
\]
For $m \in S_l$, define the pure number field $k_m := \mathbb{Q}(\sqrt{m})$ where the radical is chosen in $\mathbb{R}^+$. Let $r(m) := \text{res}_{s=1} \zeta_{k_m}(s)$ where $\zeta_{k_m}$ is the Dedekind zeta function of $k_m$. Then
\[
r(m) = \frac{h_{k_m}R_{k_m}}{d_{k_m}^{1/2}} c(l), \quad c(l) = \begin{cases} 2, & l = 2, \\ \frac{(2\pi)^{(l-1)/2}}{l}, & l \geq 3, \end{cases}
\]
and $d_{k_m} \asymp K(m)^{l-1}$, where $K(m)$ is the squarefree kernel of $m$. For $m \in \mathbb{N} \setminus S_l$, define $r(m) := 0$.

**Theorem.** There is a distribution function $F \in C^\infty(\mathbb{R})$ such that for all $z \in \mathbb{R}$,
\[
\lim_{x \to \infty} \frac{\# \{m \in S_l \mid m \leq x, r(m) \leq e^z\}}{\# \{m \in S_l \mid m \leq x\}} = F(z).
\]
Furthermore,
\[
\lim_{x \to \infty} \frac{1}{\# \{m \in S_l \mid m \leq x\}} \sum_{m \in S_l : m \leq x} r(m)^q = \int_{\mathbb{R}^+} z^q dF(\log z)
\]
for all $q \in \mathbb{N}$. The characteristic function $\Psi(t)$ of $F$ is an Euler product whose factors depend on $t \in \mathbb{R}$.

In order to give an idea of the proof let us first review the method for the well-known case $l = 2$. For $m > 1$ squarefree, Dirichlet’s class number formula gives
\[
\zeta_{\mathbb{Q}(\sqrt{m})}(s) = \zeta(s) L(s, \chi_d),
\]
where
\[
d = \begin{cases} m, & m \equiv 1 \text{ mod } 4, \\ 4m, & m \equiv 2, 3 \text{ mod } 4, \end{cases}
\]
is the discriminant of $\mathbb{Q}(\sqrt{m})$ and $\chi_d$ is the Jacobi character for the modulus $|d|$. Therefore
\[
r(m) = L(1, \chi_d) = \sum_{n \geq 1} \frac{\chi_d(n)}{n} = \prod_p \left(1 - \frac{\chi_d(p)}{p^s}\right)^{-1} \Bigg|_{s=1}.
\]
The idea of proof is as follows: For $q \geq 1$, the function $r$ is approximated in the $q$-th mean by functions $R_P$, $P \in \mathbb{N}$, such that
\[
\|r - R_P\|_q \to 0 \text{ as } P \to \infty. \quad (1)
\]
Here
\[ \|f\|_q := \left( \limsup_{x \to \infty} \frac{1}{x} \sum_{m \leq x} |f(m)|^q \right)^{1/q} \in [0, \infty] \]
for \( f : \mathbb{N} \to \mathbb{C} \). The functions \( R_P \) are partial products of the Euler product above, i.e.
\[ R_P(m) := \prod_{p \leq P} \left( 1 - \frac{\chi_{d}(p)}{p} \right)^{-1}. \]
They are periodic in \( m \) since for \( p > 2 \), we have
\[ \chi_{d}(p) = \left( \frac{d}{p} \right) = \begin{cases} 1 & x^2 \equiv d \mod p \text{ solvable, } p \not| d, \\ -1 & x^2 \equiv d \mod p \text{ unsolvable,} \\ 0 & p|d. \end{cases} \]
Since periodic functions have limit distributions a standard procedure shows the same for \( r \). In fact the procedure in this last step is somewhat different since we also want to show the smoothness of \( F \).

The approximation (1) could be done with character sum estimates. More suitable for generalizations is the following method which uses contour integration and zero density estimates. Let \( \mathcal{K} \) be the rectangle with vertices \( 2 + iT, \gamma + iT, \gamma - iT \) and \( 2 - iT \), and \( N, T \geq 1 \) and \( 1/2 < \gamma < 1 \) free parameters. The Residue Theorem gives
\[ \frac{1}{2\pi i} \int_{\mathcal{K}} L(s, \chi_{d}) \Gamma(s-1) N^{s-1} ds = L(1, \chi_{d}). \]
Since the \( \Gamma \)-function decays exponentially in vertical strips of finite width the limit \( T \to \infty \) together with Mellin’s inversion formula gives
\[ L(1, \chi_{d}) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} = \sum_{n \geq 1} \frac{\chi_{d}(n)}{n} e^{-n/N} - I(m, N). \]
If we assume the Generalized Lindelöf Hypothesis
\[ L(s, \chi_{d}) \ll_{\epsilon} (d(1 + |\Im s|))^\epsilon \]
for \( \gamma \leq \Re s \leq 1 \) and \( m > 1 \) squarefree, we easily get the estimate
\[ I(m, N) \ll_{\epsilon} d^\epsilon N^{\gamma - 1}. \]
Here it is important that the exponent of \( d \) can be made arbitrarily small and the exponent of \( N \) is negative.

Without any assumption this procedure can be immitated as follows: If \( L(s, \chi_{d}) \) has no zeros in the rectangle
\[ \{ s \in \mathbb{C} \mid \Re s \geq \gamma - \epsilon, |\Im s| \leq (\log x)^2 \}, \]
then the usual combination of the Borel-Caratheodory Theorem and Hadamard’s Three Circles Theorem gives (2) for \( \gamma \leq \Re s \leq 2 \) and \( |\Im s| \leq (\log x)^2/2 \). Using the exponential decay of the \( \Gamma \)-function on \( |\Im s| \geq (\log x)^2/2 \) we again get (3). If there is a zero of \( L(s, \chi_{d}) \) in the rectangle (4) all we can say is that
\[ I(m, N) \ll d^\epsilon + N^\epsilon. \]
Now zero density estimates can be used to show that the second case does not happen too often, i.e.

$$\# \{1 < m \leq x \mid m \text{ squarefree}, L(s, \chi_d) \text{ has a zero in the rectangle (4)} \} \ll e^{x^{1-c(\gamma) + \epsilon}}$$

with some constant $c(\gamma) > 0$.

In the $q$-th mean we have the approximation

$$\sum_{n \geq 1} \frac{\chi_d(n)}{n} e^{-n/N} \approx \sum_{n \leq N} \frac{\chi_d(n)}{n}.$$

Choosing $N$ as a small power of $x$ proves the statement (1).

In the general case $l \geq 2$ we have, for $Rs > 1$,

$$\zeta_{k_m}(s) = \prod_p \prod_{\mathfrak{p} | p: f(\mathfrak{p}/p) \geq 2} \left(1 - \frac{1}{p^{f(\mathfrak{p}/p)s}}\right)^{-1} \prod_{p \neq l} \left(1 - \frac{1}{p^s}\right)^{-\chi(m,p)} \zeta(s),$$

where $f(p/p) := [O_{k_m}/p : \mathbb{Z}/p\mathbb{Z}]$ is the residue class degree of $p$ and

$$\chi(m, p) := \#\{p | f(p/p) = 1\} - 1.$$

Thus

$$r(m) = \prod_p \prod_{p | p: f(p/p) \geq 2} \left(1 - \frac{1}{p^{f(p/p)}}\right)^{-1} \left(1 - \frac{1}{l}\right)^{-\chi(m,l)} \prod_{p \neq l} \left(1 - \frac{1}{p^s}\right)^{-\chi(m,p)} |_{s=1}.$$

In order to get the almost periodicity of the partial products of this Euler product we exploit the relation between the splitting of rational primes $p$ in $k_m$ and the splitting of $X^l - m$ in $\mathbb{F}_p[X]$ and $\mathbb{Q}_p^{\text{unram}}[X]$. Here $\mathbb{Q}_p^{\text{unram}}$ is the maximal unramified extension of $\mathbb{Q}_p$.

The following lemmas give the necessary information.

**Lemma.** For $p \neq l$, we have

$$\chi(m, p) = \#\{x \mod p | x^l \equiv m \mod p\} - 1.$$

In particular, the function $\chi(\cdot, p)$ is $p$-periodic and

$$\sum_{m \mod p} \chi(m, p) = 0,$$

which serves as a substitute for the orthogonality relation for characters.

**Lemma.** Let $p$ be a prime, $m \in S_l$ and $b \in \mathbb{N}_0$ such that $p^b \parallel m$. Then the factor

$$\prod_{p | p: f(p/p) \geq 2} \left(1 - \frac{1}{p^{f(p/p)}}\right)^{-1}$$

is constant on the residue class $m \mod p^{b(1-1)+\text{ord}_p l+1}$.

Both lemmas are used to show the almost periodicity of $R_F$ in the general case. In order to prove the approximation (1) we use the following zero density estimate of Kawada [7].
Theorem. For sufficiently small $\eta > 0$, we have
\[ \sum_{m \in S; \ x < m \leq 2x} N(m; 1 - \eta, T) \ll (xT)^{1-\eta}, \quad x \geq T \geq 1, \]
where $N(\ldots)$ is the number of zeros of $\zeta_{km}(s)\zeta(s)^{-1}$ in the rectangle $[1 - \eta, 1] \times [-T, T]$.

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