数理解析研究所講究録 1319 巻 2003 年 220-229

The Representation of Unity by Quartic Forms

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October 25, 2002

1 Theorem

Situation: Let

$$f(X,Y) \in \mathbf{Z}[X,Y]$$

be given. Assume that $f(X,Y)$ is homogeneous, irreducible and quartic. Assume also $f(X,Y)$ splits completely in a totally real field. Denote by $\mathcal{R}(f)$ the number of integer points on the curve

$$\mathcal{T} : f(x,y) = \pm 1.$$

(Count $\pm(x,y)$ as one point.) Denote by $D(f)$ the discriminant of $f(X,Y)$.
Assertion: If $D(f) \gg 0$, we have

$$\mathcal{R}(f) \leq 12.$$

2 Thue Curve and its Parameterization

Let

$$\mathcal{A} = \{\alpha_1 < \alpha_2 < \ldots < \alpha_4\}$$

be a given configuration of 4 distinct points. Let

$$f(X,Y) = f(\mathcal{A},X,Y) = \prod_{i=1}^{4}(X - Y\alpha_i)$$

and consider the *Thue curve*

$$\mathcal{T} : |f(x,y)| = 1.$$

Take the *projective point*

$$t = \frac{x}{y} \in \mathbf{P}^1(\mathbf{R})$$

and parameterize $\mathcal{T}/\{\pm 1\}$ by

$$\begin{cases} y(t) &= y(\mathcal{A},t) &= |f(t)|^{-1/4}, \\ x(t) &= x(\mathcal{A},t) &= t\,y(t), \end{cases}$$

where

$$f(t) = f(\mathcal{A};t) = f(t,1).$$

3 Projective Transformation and Change of Variables

A *projective transformation* of $t \in \mathbf{P}^1(\mathbf{R})$ is given by

$$G = (g_{ij}) \in GL_2(\mathbf{R}) : t \longmapsto G\langle t\rangle = \frac{g_{11}t + g_{12}}{g_{21}t + g_{22}}.$$

We adopt the convention

$$\tilde{t} = G\langle t\rangle, \quad \tilde{\alpha}_i = G\langle \alpha_i\rangle, \quad \tilde{\mathcal{A}} = \{\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n\}, \quad \tilde{x} = x(\tilde{\mathcal{A}}, \tilde{t}), \quad \tilde{y} = y(\tilde{\mathcal{A}}, \tilde{t}).$$

We have the following *transformation law of difference*: For

$$u, u' \in \mathbf{R} \subset \mathbf{P}^1(\mathbf{R}),$$

we have

$$\tilde{u} - \tilde{u}' = \frac{(u-u')\det G}{\chi(G,u)\chi(G,u')}, \quad \text{where} \quad \chi(G,t) = g_{21}t + g_{22}.$$

Consider $f(x,y)$ as

$$f(x,y) = \prod_{i=1}^{4} \det \begin{pmatrix} x & \alpha_i \\ y & 1 \end{pmatrix}.$$

When

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = G \begin{pmatrix} x \\ y \end{pmatrix},$$

we have

$$\prod_{i=1}^{4} \det \begin{pmatrix} x_1 & \tilde{\alpha}_i \\ y_1 & 1 \end{pmatrix} = \frac{\det G^4}{\prod_{i=1}^{4} \chi(G,\alpha_i)} \prod_{i=1}^{4} \det \begin{pmatrix} x & \alpha_i \\ y & 1 \end{pmatrix}.$$

Thus,

$$G\begin{pmatrix} x \\ y \end{pmatrix} = \pm \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \Longleftrightarrow |\det G^4| = \left| \prod_{i=1}^{4} \chi(G,\alpha_i) \right|.$$

This *condition of compatibility* is suitable for real algebraic geometry.

4 Invariant Coordinate and Transcendental Curve

Define the coordinates $\phi_m(t)$, $(m = 1, 2, \ldots, 4)$ by

$$\phi_m(t) = \phi_m(\mathcal{A}, t) = \log \left| \frac{D^{1/8}(x - y\alpha_m)}{|f'(\alpha_m)|^{1/2}} \right|$$

with $D = D(\mathcal{A}) = \prod_{1 \le i < j \le 4} |\alpha_i - \alpha_j|^2$. Then, define

$$\phi(t) = \phi(\mathcal{A}, t) = (\phi_1(t), \phi_2(t), \ldots, \phi_4(t)).$$

Since each coordinate $\phi_m(t)$ is invariant under the action of $GL_2(\mathbf{R})$ (on t and α_i's), the point $\phi(t)$ is invariant upto permutation of coordinates under the action of $GL_2(\mathbf{R})$. Deep consequences come from geometry of *the transcendental curve*

$$\mathcal{C} = \phi(\mathbf{P}^1(\mathbf{R}) \setminus \mathcal{A}).$$

5 Asymptotic Line of $\mathcal{C}$

The curve $\mathcal{C}$ has four *asymptoic lines.* We choose one of them and discuss what happens along it. The situation around the other three asymptotic lines are the same. Let

$$\boldsymbol{b}_1 = -\frac{1}{4}(-3,1,1,1),\quad \boldsymbol{b}_2 = -\frac{1}{4}(1,-3,1,1),\quad \boldsymbol{b}_3 = -\frac{1}{4}(1,1,-3,1),\quad \boldsymbol{b}_4 = -\frac{1}{4}(1,1,1,-3)$$

and

$$\boldsymbol{c}_i = \boldsymbol{b}_i + \frac{1}{3}\boldsymbol{b}_4,\quad (i<4)\quad (\boldsymbol{c}_i \perp \boldsymbol{b}_4).$$

Then,

$$\phi(t) = \sum_{i=1}^{4} \log\frac{|t-\alpha_i|}{|f'(\alpha_i)|^{1/2}} \cdot \boldsymbol{b}_i = \sum_{i=1}^{3} \log\frac{|t-\alpha_i|}{|f'(\alpha_i)|^{1/2}} \cdot \boldsymbol{c}_i + \frac{2\ell_4}{\sqrt{3}}\boldsymbol{b}_4,$$

where

$$\ell_4 = \sqrt{\frac{3}{4}}\left(\log\frac{|t-\alpha_4|}{|f'(\alpha_4)|^{1/2}} - \frac{1}{3}\sum_{i=1}^{3}\log\frac{|t-\alpha_i|}{|f'(\alpha_i)|^{1/2}}\right).$$

Let

$$\mathcal{L}_4 = \sum_{i=1}^{3}\log\frac{|\alpha_4-\alpha_i|}{|f'(\alpha_i)|^{1/2}} \cdot \boldsymbol{c}_i + \mathbf{R}\boldsymbol{b}_4.$$

Then, $\phi(t)$ approaches $\mathcal{L}_4$ as t approaches α_4. If $t = \alpha_4 + u$ with $|u|/(\alpha_4-\alpha_3) \ll 1$, we have

$$\operatorname{dist}(\phi(t),\mathcal{L}_4) = \left\|\sum_{i=1}^{3}\log\frac{|t-\alpha_i|}{|\alpha_4-\alpha_i|}\cdot \boldsymbol{c}_i\right\| \ll \frac{3|u|}{\alpha_4-\alpha_3};\qquad \ell_4 = \frac{\log|u|}{\sqrt{4/3}} + O_{\mathcal{A}}(1).$$

Thus, we have $r = \|\phi(t)\| = -\ell_4 + O_{\mathcal{A}}(1)$. Therefore,

$$\operatorname{dist}(\phi(t),\mathcal{L}_4) \ll_{\mathcal{A}} \exp\left(-\sqrt{4/3}\,r\right).$$

6 Convexity of $\mathcal{C}$ and Intersection with Line

The transcendental curve $\mathcal{C}$ has *convexity in a certain sense.* For observing it, we calculate

$$\phi(t) - \boldsymbol{v} = \sum_{i\neq 2}\log|t-\alpha_i|\cdot \boldsymbol{c}_i + \frac{2\ell_2}{\sqrt{3}}\boldsymbol{b}_2 = \sum_{i\neq 2,4}\log\frac{|t-\alpha_i|}{|t-\alpha_4|}\cdot \boldsymbol{c}_i + \frac{2\ell_2}{\sqrt{3}}\boldsymbol{b}_2,$$

where $\boldsymbol{v}$ is a certain vector independent of t. Since $\boldsymbol{c}_1, \boldsymbol{b}_2, \boldsymbol{c}_3$ form a basis of the orthogonal space $\Pi_{\log}$ of $(1,1,\ldots,1)$,

$$(u(t), w(t)) = \left(\log\frac{|t-\alpha_1|}{|t-\alpha_4|}, \log\frac{|t-\alpha_3|}{|t-\alpha_4|}\right)$$

is a linear projection of $\phi(t)$.

The curve $(u(t), w(t))$ with $t \in]\alpha_1, \alpha_3[$ is a convex curve as verified below: Observe

$$\frac{du}{dt} = \frac{\alpha_1 - \alpha_4}{(t-\alpha_1)(t-\alpha_4)} > 0 > \frac{\alpha_3 - \alpha_4}{(t-\alpha_3)(t-\alpha_4)} = \frac{dw}{dt}$$

and calculate

$$\frac{d^2 w}{du^2} = \frac{\frac{d}{dt}\frac{dw/dt}{du/dt}}{du/dt} = \frac{(t-\alpha_1)(t-\alpha_4)(\alpha_3-\alpha_4)}{(\alpha_1-\alpha_4)^2}\frac{d}{dt}\frac{t-\alpha_1}{t-\alpha_3}$$
$$= \frac{(t-\alpha_1)(t-\alpha_4)(\alpha_3-\alpha_4)(\alpha_1-\alpha_3)}{(\alpha_1-\alpha_4)^2(t-\alpha_3)^2} < 0.$$

The convexity implies that an intersection of the part $\phi(]\alpha_1, \alpha_3[)$ with any given line always consists of at most two points.

Since we can projectively transform

$$\begin{array}{cccc} & \alpha_{m-2}, & \alpha_{m-1} & \text{and} \quad \alpha_m \\ \text{to} & +1, & -1 & \text{and} \quad 0 \end{array}$$

without altering the point $\phi(t)$, the same property is enjoyed by every intervals $]\alpha_{m-1}, \alpha_{m+1}[$. Here we read the subscript modulo 4 and also read $]\alpha_4, \alpha_2[=]\alpha_4, \infty] \cap [-\infty, \alpha_2[$ and $]\alpha_3, \alpha_1[=]\alpha_3, \infty] \cap [-\infty, \alpha_1[$.

Therefore, *the intersection of the part*

$$\phi(]\alpha_{m-1}, \alpha_{m+1}[)$$

with any given line always consists of at most two points, regardless of the value of $m = 1, 2, \ldots, 4$.

7 Intersection of C with Plane

An intersection of a plane of $\Pi_{\log}$ *with* C *always consists of at most* 6 *points.* To see this, we denote the normal vector of $\Pi_{\log}$ by $(w_1, w_2, \ldots, w_n) \in \Pi_{\log}$ and count solutions to

$$c = \sum_{i=1}^{4} w_i \phi_i(t).$$

We have

$$c = \sum_{i=1}^{4} w_i \log\left|\frac{D^{1/8}(x - y\alpha_m)}{|f'(\alpha_m)|^{1/2}}\right| = \sum_{i=1}^{4} w_i \log|t - \alpha_i|\,;$$

$$\frac{d}{dt}\sum_{i=1}^{4} w_i \log|t-\alpha_i| = \frac{\sum_{i=1}^{n} w_i f_i(t)}{f(t)},$$

where $f_i(t) = f(t)/(t-\alpha_i)$ is a monic polynomial of degree 3 ($i = 1, 2, \ldots, 4$). Since the leading terms of the numerator of the right hand side cancel out, the right hand side has at most two roots. Thus, the function $\sum_{i=1}^{4} w_i \phi_i(t)$ has at most 2 critical points. On the other hand it has exacltly 4 singular points. Therefore, its mapping degree is at most 6.

8 Admissible Transformation and Discreteness

Let $G \in GL_2(\mathbf{R})$. We consider G preserves discreteness if it preserves

$$\left|\det\begin{pmatrix} x & x' \\ y & y' \end{pmatrix}\right|$$

and is compatible with change of variables:

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \pm G \begin{pmatrix} x \\ y \end{pmatrix}.$$

As we have seen in §3, the latter is characterized by

$$\left|\det G^4\right| = \left|\prod_{i=1}^{4} \chi(G, \alpha_i)\right|.$$

We say G is *admissible* for $\mathcal{A}$ if these conditions hold, i.e.,

$$\left|\det G\right| = \left|\prod_{i=1}^{4} \chi(G, \alpha_i)\right| = 1.$$

An admissible transformation always preserves the discriminant:

$$D(\tilde{\mathcal{A}}) = D(\mathcal{A})$$

since

$$\prod_{1\le i<j\le 4} |\tilde{\alpha}_i - \tilde{\alpha}_j| = \prod_{1\le i<j\le 4} \left|\frac{(\alpha_i - \alpha_j)\det G}{\chi(G,\alpha_i)\chi(G,\alpha_j)}\right|.$$

Admissible transformation has freedom of degree 2, i.e., it can transform given two points, say $\mu, \nu \notin \mathcal{A}$, to $0, \infty$: Choose v and w suitably to make

$$G = \begin{pmatrix} v & -v\mu \\ w & -w\nu \end{pmatrix}$$

admissible for $\mathcal{A}$.

9 Normalization of "Roots" and Symmetry of the Curve $\mathcal{C}$

We write $\alpha = \alpha_1, \beta = \alpha_2, \gamma = \alpha_3$ and $\delta = \alpha_4$. Set

$$\boldsymbol{e}_i = \boldsymbol{b}_i + \boldsymbol{b}_4, \quad (i = 1, 2, 3).$$

Then, $\boldsymbol{e}_1, \boldsymbol{e}_2$ and $\boldsymbol{e}_3$ constitute a basis of the space $\Pi_{\log}$. We get

$$\begin{aligned} 2\phi(t) &= \log\left|\frac{(t-\alpha)(t-\delta)(\gamma-\beta)}{(t-\beta)(t-\gamma)(\delta-\alpha)}\right| \cdot \boldsymbol{e}_1 \\ &\quad + \log\left|\frac{(t-\beta)(t-\delta)(\gamma-\alpha)}{(t-\alpha)(t-\gamma)(\delta-\beta)}\right| \cdot \boldsymbol{e}_2 \\ &\quad + \log\left|\frac{(t-\gamma)(t-\delta)(\beta-\alpha)}{(t-\alpha)(t-\beta)(\delta-\gamma)}\right| \cdot \boldsymbol{e}_3 \\ &=: 2z_1(t)\boldsymbol{e}_1 + 2z_2(t)\boldsymbol{e}_2 + 2z_3(t)\boldsymbol{e}_3. \end{aligned}$$

The argument for intersection with subspace implies $z_i(t)$ has at most 2 critical points. Therefore, $z_1(t)$ has one critical point in each of $]\beta, \gamma[$ and $]\delta, \alpha[$. We call them $\mu(\beta, \gamma)$ and $\mu(\delta, \alpha)$. Similarly, $\mu(\alpha, \beta)$ and $\mu(\gamma, \delta)$ are defined by z_3.

We can transform $\mu(\beta, \gamma)$ and $\mu(\delta, \alpha)$ respectively to 0 and ∞ by an admissible transformation. Therefore, we assume $\alpha = -\delta, \quad \beta = -\gamma$ without altering the geometry of the curve $\mathcal{C}$. The cross ratio

$$\lambda = -\frac{(\gamma-\beta)(\alpha-\delta)}{(\delta-\gamma)(\beta-\alpha)}$$

of $\mathcal{A}$ is a projective invariant (upto permutation of "roots").

Admissible transformation determined by $\mu(\alpha, \beta) \longmapsto 0$ and $\mu(\gamma, \delta) \longmapsto \infty$ inverts λ.

We say $\mathcal{A}$ is *normalized* if $\alpha = -\delta, \quad \beta = -\gamma$ and $4\gamma\delta/(\delta-\gamma)^2 = \lambda \geq 1$. We can assume that $\mathcal{A}$ is normalized without altering the geometry of the curve $\mathcal{C}$.

We now have $\gamma \geq \delta/(3 + 2\sqrt{2})$.

Set $L = \gamma + \delta$. Then, $\sqrt{D} = 4\gamma\delta L^2(\delta-\gamma)^2 \leq L^6/\lambda$. We now have

$$\begin{aligned} 2\phi(t) &= z_1(t)\boldsymbol{e}_1 + z_2(t)\boldsymbol{e}_2 + z_3(t)\boldsymbol{e}_3 \\ &= \log\left|\frac{2\gamma(t-\alpha)(t-\delta)}{2\delta(t-\beta)(t-\gamma)}\right| \cdot \boldsymbol{e}_1 \\ &\quad + \log\left|\frac{(t-\beta)(t-\delta)}{(t-\alpha)(t-\gamma)}\right| \cdot \boldsymbol{e}_2 \\ &\quad + \log\left|\frac{(t-\gamma)(t-\delta)}{(t-\alpha)(t-\beta)}\right| \cdot \boldsymbol{e}_3. \end{aligned}$$

We set $\mu = -\mu(\alpha, \beta) = \mu(\gamma, \delta) = \sqrt{\gamma\delta}$.

Then, the curve C is preserved by the projective transformations $t \longmapsto -t$, $t \longmapsto -\mu^2/t$ and $t \longmapsto \mu^2/t$. Note: transformations

$$\begin{pmatrix} -1 & \\ & 1 \end{pmatrix}, \quad \begin{pmatrix} & -\mu \\ \mu^{-1} & \end{pmatrix}, \quad \begin{pmatrix} & \mu \\ \mu^{-1} & \end{pmatrix}$$

are admissible for $\mathcal{A}$.

The three transformations have the same effect on the curve C as the rotations around $\mathbf{R}e_1$, $\mathbf{R}e_2$ and $\mathbf{R}e_3$ of angle π in the space $\Pi_{\log}$.

10 Four Asymptotic Parts and One Bridge of C

Hereafter, we assume $D > 10^{20}$.

We wrap C by five"cylinders" (four "asymptotic cylinders" and "the bridge"). The part of C corresponding to $\phi(\delta + u)$ with

$$s^{-1} := \frac{|u|}{\delta - \gamma + u} \leq \frac{4}{L^2}$$

will be called the asymptotic part of C at δ. Asymptotic part of C at other "roots" are defined by symmetry.

The rest of the part of C will be called the bridge.

In the asymptotic part, we have (e.g.)

$$\mathrm{dist}(\phi(\delta + u), \mathcal{L}_4) \ll s^{-1}$$

and

$$\begin{aligned} (2r)^2 &> 2(\log s)^2 + (\log s + \log\lambda - 0.2)^2, \\ (2r)^2 &< 2(\log s + 2)^2 + (\log s + \log\lambda + 2)^2, \end{aligned}$$

where $r = \|\phi(t)\|$. The first inequality and $D \leq L^{12}$ imply

$$\log D \ll r. \tag{1}$$

Since $1 \leq \lambda \leq L^6/\sqrt{D}$, we have $\lambda < s^3$. Thus, the second inequality implies

$$\log s > \sqrt{2}r/3$$

and

$$\mathrm{dist}(\phi(\delta + u), \mathcal{L}_4) \ll e^{-\sqrt{2}r/3}.$$

We have the Gap Principle

$$r'' \gg \triangle\left(\phi(t), \phi(t'), \phi(t'')\right)\exp\left(\sqrt{2}r/3\right) > 0 \tag{2}$$

when $\phi(t)$, $\phi(t')$ *and* $\phi''(t)$ *belongs to the same asymptotic part of* C *and* $\|\phi(t)\| \leq \|\phi(t')\| \leq r'' := \|\phi(t'')\|$. This follows from the previous estimate and the simple estimate

$$\triangle\left(\phi(t), \phi(t'), \phi(t'')\right) \ll r'' \cdot \mathrm{dist}(\phi(t), \mathcal{L}_4)$$

and the result of §6.

11 Original Arithmetic Situation

We say $f(X,Y)$ is *arithmetic* (or $\mathcal{A}$ is arithmetic) if $f(X,Y) \in \mathbf{Z}[X,Y]$ is irreducible. We say t is arithmetic if $f(X,Y)$ is arithmetic and $x(t), y(t) \in \mathbf{Z}$. (Later, we shall extend its use.)

When t and t' are arithmetic, $\phi(t) - \phi(t')$ belongs to the image $\mathfrak{E}$ of the regulator map of the unit group of the field defined by $f(X,1)$:

$$\phi(t) - \phi(t') = \log \vec{\varepsilon} = \left(\log \left|\varepsilon^{(i)}\right|\right)_{1 \le i \le 4} \in \mathfrak{E}.$$

(Just recall $\phi_m(t) = \log \left|D^{1/8}(x - y\alpha_m)/|f'(\alpha_m)|^{1/2}\right|$ and $|f(x(t), y(t))| = 1$.)

By tuning the Gap-Principle of Bombieri-Schmidt in our setting, we see there are at most 4 arithmetic points t such that $\phi(t)$ is on the bridge.

We have seen, under the normality of the roots,

$$\operatorname{dist}(\phi(\delta + u), \mathcal{L}_4) \ll e^{-\sqrt{2}r/3}.$$

The left hand side has an invariant representation:

$$9 \cdot \operatorname{dist}(\phi(\delta+u), \mathcal{L}_4)^2 = \log^2 \left| \frac{(t-\alpha)(\delta-\beta)}{(\delta-\alpha)(t-\beta)} \right|$$
$$+ \log^2 \left| \frac{(t-\beta)(\delta-\gamma)}{(\delta-\beta)(t-\gamma)} \right| + \log^2 \left| \frac{(t-\gamma)(\delta-\alpha)}{(\delta-\gamma)(t-\alpha)} \right|. \tag{3}$$

Thus, we get the inequality

$$\Lambda := \log \left| \frac{(t-\alpha)(\delta-\beta)}{(\delta-\alpha)(t-\beta)} \right| \ll e^{-\sqrt{2}r/3},$$

of the invariant quantity Λ under $GL_2(\mathbf{R})$.

Switching back to the original configuration and assume $\mathcal{A}$ is an arithmetic configuration and t, t_0 are arithmetic points. Let $\mathfrak{K} = \mathbf{Q}(\alpha)$. Let $\log\zeta$, $\log\eta$, $\log\xi$ be successive minima of $\log \mathfrak{O}(\mathfrak{K})^\times$. ($\|\log\zeta\| \le \|\log\eta\| \le \|\log\xi\|$.) Then, Λ is a linear combination with rational integral coefficients in $\log((t_0-\alpha)(\delta-\beta)/(\delta-\alpha)(t_0-\beta))$, $\log(\zeta_1/\zeta_2)$,$\log(\eta_1/\eta_2)$ and $\log(\xi_1/\xi_2)$. Here, the subscript of ζ_i, η_i and ξ_i denotes the embedding of $\mathfrak{K}$ induced by $\alpha \longmapsto \alpha_i$.

By using Matveev's lower bound (E. M. Matveev, "An explicit lower bound for a homogeneous rational linear form in the logarithms of algebraic numbers. II", Izvestiya Mathematics 64 (2000) 1217–1269.), we get

$$r \ll -\log|\Lambda| \ll \log\left(\frac{r + \log D}{A_4}\right) \cdot \prod_{k=1}^{4} A_k, \tag{4}$$

where we set

$$A_1 = h\left(\frac{(t_0-\alpha)(\delta-\beta)}{(\delta-\alpha)(t_0-\beta)}\right), \quad (t_0 : \text{ arithmetic point});$$
$$A_2 = \|\log\zeta\|, \quad A_3 = \|\log\eta\|, \quad A_4 = \|\log\xi\|.$$

12 Controlling the Parameter A_1

We want to control the size of

$$\log\left|\frac{(t_0-\alpha_j)(\alpha_i-\alpha_k)}{(\alpha_i-\alpha_j)(t_0-\alpha_k)}\right|.$$

The identity (3) implies

$$\log\left|\frac{(t_0-\alpha_j)(\delta-\alpha_k)}{(\delta-\alpha_j)(t_0-\alpha_k)}\right| < \mathrm{dist}(\phi(t_0),\mathcal{L}_4) \ll 1.$$

For other α_i, we have

$$\log\left|\frac{(t_0-\alpha_j)(\alpha_i-\alpha_k)}{(\alpha_i-\alpha_j)(t_0-\alpha_k)}\right| < \mathrm{dist}(\phi(t_0),\mathcal{L}_i).$$

By symmetry of the curve, there is a point z such that

$$\mathrm{dist}(z,\mathcal{L}_4)=\mathrm{dist}(\phi(t_0),\mathcal{L}_i),\quad \|z\|=\|\phi(t_0)\|.$$

Thus,

$$\mathrm{dist}(\phi(t_0),\mathcal{L}_i) < 2\|\phi(t_0)\|+o(1).$$

We now see

$$\log\left|\frac{(t_0-\alpha_j)(\alpha_i-\alpha_k)}{(\alpha_i-\alpha_j)(t_0-\alpha_k)}\right| \ll \|\phi(t_0)\|.$$

Hence, $A_1 \ll \|\phi(t_0)\| + \log D$.

13 Counting All Arithmetic Points

Suppose 13 arithmetic points exist. Remove 4 arithmetic points of minimal "radii" $\|\phi(t)\|$. The arithmetic points on the bridge are removed. (See §11.) For the rest of the arithmetic points t, we have $\log D \ll \|\phi(t)\|$. (See (1) of §10.)

At least 3 arithmetic points t, t' and t'' concentrate on an asymptotic part. Write $r''=\|\phi(t'')\|, r'=\|\phi(t')\|, r=\|\phi(t)\|$. WLOG, $r''\geq r'\geq r$. We get

$$\frac{r''/A_4}{\log(r''/A_4)} \ll \prod_{k=1}^{3} A_k$$

from $\log D \ll r$ and (4). Thus, we get

$$r'' \ll \prod_{k=1}^{4} A_k\cdot\log\left(\prod_{k=1}^{3} A_k\right)$$

We set $t_0=t$. Then, we get

$$r'' \ll \prod_{k=2}^{4} A_k\cdot r\log r$$

since $A_1 \ll \|\phi(t_0)\| + \log D \ll r$ by the result of §12 and the analytic class number formula implies $\log A_2A_3A_4 \ll \log D(\mathfrak{K}) \ll \log D \ll r$. For showing $r \ll 1$, we would like to combine this inequality with the Gap Principle (2):

$$r'' \gg \triangle\left(\phi\left(t\right), \phi\left(t'\right), \phi\left(t''\right)\right) \exp\left(\sqrt{2}\, r/3\right).$$

It will establish the theorem since $\log D \ll r$.

The result of §6 implies linear independence of $\phi(t') - \phi(t)$ and $\phi(t'') - \phi(t)$ over $\mathbf{R}$. Let $\log\tilde{\zeta}$ and $\log\tilde{\xi}$ be a reduced basis of the plane lattice

$$\mathbf{Z}(\phi(t') - \phi(t)) + \mathbf{Z}(\phi(t'') - \phi(t)).$$

Then, the theory of basis reduction of plane lattice implies

$$\triangle\left(\phi\left(t\right), \phi\left(t'\right), \phi\left(t''\right)\right) \gg \|\log\tilde{\zeta}\| \cdot \|\log\tilde{\xi}\| \gg A_2A_3.$$

Easier Case: If $A_4 \leq 2r$, we easily argue as follows:

$$A_2A_3e^{\sqrt{2}\,r/3} \ll r'' \ll A_2A_3r^2 \log r;$$

$$r \ll 1.$$

Harder Case: We now treat the harder case of $A_4 > 2r$. The lattice generated by vectors

$$\phi(T) - \phi(t), \quad (T : \text{arithmetic point}, \|\phi(T)\| \leq \|\phi(t)\|)$$

is a sublattice of finite index of the lattice $\mathbf{Z}\log\zeta + \mathbf{Z}\log\eta$. (Here, we use the result of §6 noting that there are at least five points of the form $\phi(T)$.) Therefore, $A_2, A_3 \leq 2r$. Those T's and t', t'' form a set of 7 or more points. Hence, $\log\zeta, \log\eta, \log\tilde{\zeta}$ and $\log\tilde{\xi}$ generate a space lattice by the result of §7. Therefore, $\|\log\tilde{\xi}\| \geq A_4$. (Obviously, $\|\log\tilde{\zeta}\| \geq A_2$.) Now, we can argue as follows:

$$A_2A_4e^{\sqrt{2}\,r/3} \ll r'' \ll A_2A_4r^2 \log r;$$

$$r \ll 1.$$