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The Representation of Unity by Quartic Forms

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1 Theorem

Situation: Let

\[ f(X, Y) \in \mathbb{Z}[X, Y] \]

be given. Assume that \( f(X, Y) \) is homogeneous, irreducible and quartic. Assume also \( f(X, Y) \) splits completely in a totally real field. Denote by \( \mathcal{R}(f) \) the number of integer points on the curve

\[ T: f(x, y) = \pm 1. \]

(Count \( \pm (x, y) \) as one point.) Denote by \( D(f) \) the discriminant of \( f(X, Y) \).

Assertion: If \( D(f) \gg 0 \), we have

\[ \mathcal{R}(f) \leq 12. \]

2 Thue Curve and its Parameterization

Let

\[ A = \{\alpha_1 < \alpha_2 < \ldots < \alpha_4\} \]

be a given configuration of 4 distinct points. Let

\[ f(X, Y) = f(A, X, Y) = \prod_{i=1}^{4} (X - Y\alpha_i) \]

and consider the Thue curve

\[ T: |f(x, y)| = 1. \]

Take the projective point

\[ t = \frac{x}{y} \in \mathbb{P}^1(\mathbb{R}) \]

and parameterize \( T/\{\pm 1\} \) by

\[
\begin{align*}
    y(t) &= y(A, t) = |f(t)|^{-1/4}, \\
    x(t) &= x(A, t) = ty(t),
\end{align*}
\]
where

\[ f(t) = f(A; t) = f(t, 1). \]

### 3 Projective Transformation and Change of Variables

A projective transformation of \( t \in \mathbb{P}^1(\mathbb{R}) \) is given by

\[ G = (g_{ij}) \in GL_2(\mathbb{R}) : t \mapsto G(t) = \frac{g_{11}t + g_{12}}{g_{21}t + g_{22}}. \]

We adopt the convention

\[ \tilde{t} = G(t), \quad \tilde{\alpha}_i = G(\alpha_i), \quad \tilde{A} = \{\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_n\}, \quad \tilde{x} = x(\tilde{A}, \tilde{t}), \quad \tilde{y} = y(\tilde{A}, \tilde{t}). \]

We have the following transformation law of difference: For

\[ u, u' \in \mathbb{R} \subset \mathbb{P}^1(\mathbb{R}), \]

we have

\[ \tilde{u} - \tilde{u}' = \frac{(u - u') \det G}{\chi(G, u)\chi(G, u')}, \]

where \( \chi(G, t) = g_{21}t + g_{22} \).

Consider \( f(x, y) \) as

\[ f(x, y) = \prod_{i=1}^{4} \det \begin{pmatrix} x & \alpha_i \\ y & 1 \end{pmatrix}. \]

When

\[ \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = G \begin{pmatrix} x \\ y \end{pmatrix}, \]

we have

\[ \prod_{i=1}^{4} \det \begin{pmatrix} x_1 & \tilde{\alpha}_i \\ y_1 & 1 \end{pmatrix} = \frac{\det G^4}{\prod_{i=1}^{4} \chi(G, \alpha_i)} \prod_{i=1}^{4} \det \begin{pmatrix} x & \alpha_i \\ y & 1 \end{pmatrix}. \]

Thus,

\[ G \begin{pmatrix} x \\ y \end{pmatrix} = \pm \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} \iff |\det G^4| = \left| \prod_{i=1}^{4} \chi(G, \alpha_i) \right|. \]

This condition of compatibility is suitable for real algebraic geometry.

### 4 Invariant Coordinate and Transcendental Curve

Define the coordinates \( \phi_m(t), (m = 1, 2, \ldots, 4) \) by

\[ \phi_m(t) = \phi_m(A, t) = \log \left| \frac{D^{1/8}(x - y\alpha_m)}{|f'(\alpha_m)|^{1/2}} \right| \]

with \( D = D(A) = \prod_{1 \leq i < j \leq 4} |\alpha_i - \alpha_j|^2 \). Then, define

\[ \phi(t) = \phi(A, t) = (\phi_1(t), \phi_2(t), \ldots, \phi_4(t)). \]
Since each coordinate $\phi_m(t)$ is invariant under the action of $GL_2(\mathbb{R})$ (on $t$ and $\alpha_i$'s), the point $\phi(t)$ is invariant up to permutation of coordinates under the action of $GL_2(\mathbb{R})$. Deep consequences come from geometry of the transcendental curve $C = \phi(\mathbb{P}^1(\mathbb{R}) \setminus A)$.

5 Asymptotic Line of $C$

The curve $C$ has four asymptotic lines. We choose one of them and discuss what happens along it. The situation around the other three asymptotic lines are the same. Let

$$b_1 = -\frac{1}{4}(-3, 1, 1, 1), \quad b_2 = -\frac{1}{4}(1, -3, 1, 1), \quad b_3 = -\frac{1}{4}(1, 1, -3, 1), \quad b_4 = -\frac{1}{4}(1, 1, 1, -3)$$

and

$$c_i = b_i + \frac{1}{3}b_4, \quad (i < 4) \quad (c_i \perp b_4).$$

Then,

$$\phi(t) = \sum_{\iota=1}^{4} \log^i \cdot b_i|f(\alpha_{\iota})|^{1/2}|t\alpha| = \sum_{i=1}^{3}\log\frac{|t-\alpha_i|}{|f'\alpha_i|^{1/2}}\cdot q + \frac{2\ell_4}{\sqrt{3}}b_4,$$

where

$$\ell_4 = \sqrt{3/4} \left( \log \frac{|t-\alpha_4|}{|f'(\alpha_4)|^{1/2}} - \frac{1}{3} \sum_{i=1}^{3} \log \frac{|t-\alpha_i|}{|f'(\alpha_i)|^{1/2}} \right).$$

Let

$$L_4 = \sum_{i=1}^{3}\log\frac{|\alpha_4 - \alpha_i|}{|f'(\alpha_i)|^{1/2}}\cdot c_i + Rb_4.$$

Then, $\phi(t)$ approaches $L_4$ as $t$ approaches $\alpha_4$. If $t = \alpha_4 + u$ with $|u|/(\alpha_4 - \alpha_3) \ll 1$, we have

$$\text{dist}(\phi(t), L_4) = \left\| \sum_{i=1}^{3}\log\frac{|t-\alpha_i|}{|\alpha_4 - \alpha_i|}\cdot c_i \right\| \ll \frac{3|u|}{\alpha_4 - \alpha_3}; \quad \ell_4 = \log \frac{|u|}{\sqrt{4/3}} + O_A(1).$$

Thus, we have $r = \|\phi(t)\| = -\ell_4 + O_A(1)$. Therefore,

$$\text{dist}(\phi(t), L_4) \ll_A \exp \left(-\sqrt{4/3} r \right).$$

6 Convexity of $C$ and Intersection with Line

The transcendental curve $C$ has convexity in a certain sense. For observing it, we calculate

$$\phi(t) - v = \sum_{i \neq 2} \log|t-\alpha_i| \cdot c_i + \frac{2\ell_2}{\sqrt{3}}b_2 = \sum_{i \neq 2,4} \log\frac{|t-\alpha_i|}{|t-\alpha_4|} \cdot c_i + \frac{2\ell_2}{\sqrt{3}}b_2,$$
where \( v \) is a certain vector independent of \( t \). Since \( c_1, b_2, c_3 \) form a basis of the orthogonal space \( \Pi_{\log} \) of \((1, 1, \ldots, 1)\),

\[
(u(t), w(t)) = \left( \log \frac{|t - \alpha_1|}{|t - \alpha_4|}, \log \frac{|t - \alpha_3|}{|t - \alpha_4|} \right)
\]
is a linear projection of \( \phi(t) \).

The curve \((u(t), w(t))\) with \( t \in ]\alpha_1, \alpha_3[ \) is a convex curve as verified below: Observe

\[
\frac{du}{dt} = \frac{\alpha_1 - \alpha_4}{(t - \alpha_1)(t - \alpha_4)} > 0 > \frac{\alpha_3 - \alpha_4}{(t - \alpha_3)(t - \alpha_4)} = \frac{dw}{dt}
\]
and calculate

\[
\frac{d^2 w}{du^2} = \frac{\frac{d}{dt} \frac{dw}{dt}}{du} = \frac{(t - \alpha_1)(t - \alpha_4)(\alpha_3 - \alpha_4)}{(\alpha_1 - \alpha_4)^2} \frac{d}{dt} \frac{t - \alpha_1}{t - \alpha_3}
\]
\[
= \frac{(t - \alpha_1)(t - \alpha_4)(\alpha_3 - \alpha_4)(\alpha_1 - \alpha_3)}{(\alpha_1 - \alpha_4)^2(t - \alpha_3)^2} < 0.
\]
The convexity implies that an intersection of the part \( \phi(]\alpha_1, \alpha_3[) \) with any given line always consists of at most two points.

Since we can projectively transform

\[
\alpha_{m-2}, \alpha_{m-1} \text{ and } \alpha_m
\]
to

\[
+1, -1 \text{ and } 0
\]
without altering the point \( \phi(t) \), the same property is enjoyed by every intervals \( ]\alpha_{m-1}, \alpha_{m+1}[ \).

Here we read the subscript modulo 4 and also read \( ]\alpha_4, \alpha_2[ = ]\alpha_4, \infty[ \) \( \cap [-\infty, \alpha_2[ \) and \( ]\alpha_3, \alpha_1[ = ]\alpha_3, \infty[ \) \( \cap [-\infty, \alpha_1[ \).

Therefore, \textit{the intersection of the part}

\[
\phi(]\alpha_{m-1}, \alpha_{m+1}[)
\]
\textit{with any given line always consists of at most two points, regardless of the value of } \( m = 1, 2, \ldots, 4 \).

7 Intersection of \( C \) with Plane

\textit{An intersection of a plane of } \( \Pi_{\log} \) \textit{with } \( C \) \textit{always consists of at most 6 points.} To see this, we denote the normal vector of \( \Pi_{\log} \) by \((w_1, w_2, \ldots, w_n) \in \Pi_{\log} \) and count solutions to

\[
c = \sum_{i=1}^{4} w_i \phi_i(t).
\]
We have

\[
c = \sum_{i=1}^{4} w_i \log \left| \frac{D^{1/8}(x-y\alpha_m)}{|f'(\alpha_m)|^{1/2}} \right| = \sum_{i=1}^{4} w_i \log |t - \alpha_i|;
\]
\[
\frac{d}{dt} \sum_{i=1}^{4} w_i \log |t - \alpha_i| = \frac{\sum_{i=1}^{n} w_i f_i(t)}{f(t)},
\]
where \( f_i(t) = f(t)/(t - \alpha_i) \) is a monic polynomial of degree 3 \((i = 1, 2, \ldots, 4)\). Since the leading terms of the numerator of the right hand side cancel out, the right hand side has at most two roots. Thus, the function \( \sum_{i=1}^{4} w_i \phi_i(t) \) has at most 2 critical points. On the other hand it has exactly 4 singular points. Therefore, its mapping degree is at most 6.

8 Admissible Transformation and Discreteness

Let \( G \in GL_2(\mathbb{R}) \). We consider \( G \) preserves discreteness if it preserves
\[
\left| \det \begin{pmatrix} x & x' \\ y & y' \end{pmatrix} \right|
\]
and is compatible with change of variables:
\[
\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \pm G \begin{pmatrix} x \\ y \end{pmatrix}.
\]
As we have seen in §3, the latter is characterized by
\[
|\det G^4| = \left| \prod_{i=1}^{4} \chi(G, \alpha_i) \right|.
\]
We say \( G \) is admissible for \( A \) if these conditions hold, i.e.,
\[
|\det G| = \left| \prod_{i=1}^{4} \chi(G, \alpha_i) \right| = 1.
\]
An admissible transformation always preserves the discriminant:
\[
D(\tilde{A}) = D(A)
\]
since
\[
\prod_{1 \leq i < j \leq 4} |\tilde{\alpha}_i - \tilde{\alpha}_j| = \prod_{1 \leq i < j \leq 4} \left| \frac{(\alpha_i - \alpha_j) \det G}{\chi(G, \alpha_i) \chi(G, \alpha_j)} \right|.
\]
Admissible transformation has freedom of degree 2, i.e., it can transform given two points, say \( \mu, \nu \notin A \), to 0, \( \infty \): Choose \( v \) and \( w \) suitably to make
\[
G = \begin{pmatrix} v & -v \mu \\ w & -w \nu \end{pmatrix}
\]
admissible for \( A \).
9 Normalization of "Roots" and Symmetry of the Curve $\mathcal{C}$

We write $\alpha = \alpha_1, \beta = \alpha_2, \gamma = \alpha_3$ and $\delta = \alpha_4$. Set

$$e_i = b_i + b_4, \quad (i = 1, 2, 3).$$

Then, $e_1, e_2$ and $e_3$ constitute a basis of the space $\Pi_{\log}$. We get

$$2\phi(t) = \log \left| \frac{(t-\alpha)(t-\delta)(\gamma-\beta)}{(t-\beta)(t-\gamma)(\delta-\alpha)} \right| \cdot e_1$$

$$+ \log \left| \frac{(t-\beta)(t-\delta)(\gamma-\alpha)}{(t-\alpha)(t-\gamma)(\delta-\beta)} \right| \cdot e_2$$

$$+ \log \left| \frac{(t-\gamma)(t-\delta)(\beta-\alpha)}{(t-\alpha)(t-\gamma)(\delta-\gamma)} \right| \cdot e_3$$

$$= : 2z_1(t)e_1 + 2z_2(t)e_2 + 2z_3(t)e_3.$$

The argument for intersection with subspace implies $z_i(t)$ has at most 2 critical points. Therefore, $z_1(t)$ has one critical point in each of $|\beta, \gamma|$ and $|\delta, \alpha|$. We call them $\mu(\beta, \gamma)$ and $\mu(\delta, \alpha)$. Similarly, $\mu(\alpha, \beta)$ and $\mu(\gamma, \delta)$ are defined by $z_3$.

We can transform $\mu(\beta, \gamma)$ and $\mu(\delta, \alpha)$ respectively to $0$ and $\infty$ by an admissible transformation. Therefore, we assume $\alpha = -\delta, \beta = -\gamma$ without altering the geometry of the curve $\mathcal{C}$. The cross ratio

$$\lambda = -\frac{(\gamma-\beta)(\alpha-\delta)}{(\delta-\gamma)(\beta-\alpha)}$$

of $\mathcal{A}$ is a projective invariant (upto permutation of "roots").

Admissible transformation determined by $\mu(\alpha, \beta) \rightarrow 0$ and $\mu(\gamma, \delta) \rightarrow \infty$ inverts $\lambda$.

We say $\mathcal{A}$ is normalized if $\alpha = -\delta, \beta = -\gamma$ and $4\gamma\delta/(\delta-\gamma)^2 = \lambda \geq 1$. We can assume that $\mathcal{A}$ is normalized without altering the geometry of the curve $\mathcal{C}$.

We now have $\gamma \geq \delta/(3 + 2\sqrt{2})$.

Set $L = \gamma + \delta$. Then, $\sqrt{D} = 4\gamma\delta L^2(\delta-\gamma)^2 \leq L^6/\lambda$. We now have

$$2\phi(t) = z_1(t)e_1 + z_2(t)e_2 + z_3(t)e_3$$

$$= \log \left| \frac{2\gamma(t-\alpha)(t-\delta)}{2\delta(t-\beta)(t-\gamma)} \right| \cdot e_1$$

$$+ \log \left| \frac{(t-\beta)(t-\delta)}{(t-\alpha)(t-\gamma)} \right| \cdot e_2$$

$$+ \log \left| \frac{(t-\gamma)(t-\delta)}{(t-\alpha)(t-\beta)} \right| \cdot e_3.$$

We set $\mu = -\mu(\alpha, \beta) = \mu(\gamma, \delta) = \sqrt{\gamma\delta}$.
Then, the curve $C$ is preserved by the projective transformations $t \mapsto -t$, $t \mapsto -\mu^2/t$ and $t \mapsto \mu^2/t$. Note: transformations
\[
\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \mu^{-1} & 0 \\ 0 & -\mu \end{pmatrix}, \quad \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu \end{pmatrix}
\]
are admissible for $\mathcal{A}$.

The three transformations have the same effect on the curve $C$ as the rotations around $\Re e_1$, $\Re e_2$ and $\Re e_3$ of angle $\pi$ in the space $\Pi_{\log}$.

10 Four Asymptotic Parts and One Bridge of $C$

Hereafter, we assume $D > 10^{20}$.

We wrap $C$ by five "cylinders" (four "asymptotic cylinders" and "the bridge"). The part of $C$ corresponding to $\phi(\delta + u)$ with
\[
s^{-1} := \frac{|u|}{\delta - \gamma + u} \leq \frac{4}{L^2}
\]
will be called the asymptotic part of $C$ at $\delta$. Asymptotic part of $C$ at other "roots" are defined by symmetry.

The rest of the part of $C$ will be called the bridge.

In the asymptotic part, we have (e.g.)
\[
\text{dist}(\phi(\delta + u), \mathcal{L}_4) \ll s^{-1}
\]
and
\[
(2r)^2 > 2(\log s)^2 + (\log s + \log \lambda - 0.2)^2,
\]
\[
(2r)^2 < 2(\log s + 2)^2 + (\log s + \log \lambda + 2)^2,
\]
where $r = ||\phi(t)||$. The first inequality and $D \leq L^{12}$ imply
\[
\log D \ll r.
\]
(1)

Since $1 \leq \lambda \leq L^6/\sqrt{D}$, we have $\lambda < s^3$. Thus, the second inequality implies
\[
\log s > \sqrt{2}r/3
\]
and
\[
\text{dist}(\phi(\delta + u), \mathcal{L}_4) \ll e^{-\sqrt{2}r/3}.
\]

We have the Gap Principle
\[
r'' \gg \Delta (\phi(t), \phi(t'), \phi(t'')) \exp(\sqrt{2}r/3) > 0
\]
(2)

when $\phi(t)$, $\phi(t')$ and $\phi''(t)$ belongs to the same asymptotic part of $C$ and $||\phi(t)|| \leq ||\phi(t')|| \leq r'' := ||\phi(t'')||$. This follows from the previous estimate and the simple estimate
\[
\Delta (\phi(t), \phi(t'), \phi(t'')) \ll r'' \cdot \text{dist}(\phi(t), \mathcal{L}_4)
\]
and the result of §6.
11 Original Arithmetic Situation

We say $f(X, Y)$ is arithmetic (or $A$ is arithmetic) if $f(X, Y) \in \mathbb{Z}[X, Y]$ is irreducible. We say $t$ is arithmetic if $f(X, Y)$ is arithmetic and $x(t), y(t) \in \mathbb{Z}$. (Later, we shall extend its use.)

When $t$ and $t'$ are arithmetic, $\phi(t) - \phi(t')$ belongs to the image $\mathcal{E}$ of the regulator map of the unit group of the field defined by $f(X, 1)$:

$$\phi(t) - \phi(t') = \log \epsilon = (\log |\epsilon^{(i)}|)_{1 \leq i \leq 4} \in \mathcal{E}.$$ 

(Just recall $\phi_{m}(t) = \log|D^{1/8}(x - y\alpha_{m})/|f'\alpha_{m}|^{1/2}|$ and $|f(x(t), y(t))| = 1$.)

By tuning the Gap-Principle of Bombieri-Schmidt in our setting, we see there are at most 4 arithmetic points $t$ such that $\phi(t)$ is on the bridge.

We have seen, under the normality of the roots,

$$\text{dist}(\phi(\delta + u), \mathcal{L}_{4}) \ll e^{-\sqrt{2}r/3}.$$ 

The left hand side has an invariant representation:

$$9 \cdot \text{dist}(\phi(\delta + u), \mathcal{L}_{4})^2 = \log^2 \left| \frac{(t - \alpha)(\delta - \beta)}{(\delta - \alpha)(t - \beta)} \right| + \log^2 \left| \frac{(t - \beta)(\delta - \gamma)}{(\delta - \beta)(t - \gamma)} \right| + \log^2 \left| \frac{(t - \gamma)(\delta - \alpha)}{(\delta - \gamma)(t - \alpha)} \right|. \quad (3)$$

Thus, we get the inequality

$$\Lambda := \log \left| \frac{(t - \alpha)(\delta - \beta)}{(\delta - \alpha)(t - \beta)} \right| \ll e^{-\sqrt{2}r/3},$$

of the invariant quantity $\Lambda$ under $GL_{2}(\mathbb{R})$.

Switching back to the original configuration and assume $A$ is an arithmetic configuration and $t, t_0$ are arithmetic points. Let $\mathcal{R} = \mathbb{Q}(\alpha)$. Let $\log \zeta, \log \eta, \log \xi$ be successive minima of $\log D(\mathcal{R})^x$. ($\| \log \zeta \| \leq \| \log \eta \| \leq \| \log \xi \|$.) Then, $\Lambda$ is a linear combination with rational integral coefficients in $\log((t_0 - \alpha)/(\delta - \alpha)(t_0 - \beta))$, $\log(\zeta_{1}/\zeta_{2}), \log(\eta_{1}/\eta_{2})$ and $\log(\xi_{1}/\xi_{2})$. Here, the subscript of $\zeta_{i}, \eta_{i}$ and $\xi_{i}$ denotes the embedding of $\mathcal{R}$ induced by $\alpha \rightarrow \alpha_{i}$.


$$r \ll -\log |\Lambda| \ll \log \left( \frac{r + \log D}{A_4} \right) \prod_{k=1}^{4} A_k, \quad (4)$$

where we set

$$A_1 = h \left( \frac{(t_0 - \alpha)(\delta - \beta)}{(\delta - \alpha)(t_0 - \beta)} \right), \quad (t_0 : \text{arithmetic point});$$

$$A_2 = \| \log \zeta \|, \quad A_3 = \| \log \eta \|, \quad A_4 = \| \log \xi \|.$$
12 Controlling the Parameter $A_1$

We want to control the size of

$$\log \left| \frac{(t_0 - \alpha_j)(\alpha_i - \alpha_k)}{(\alpha_i - \alpha_j)(t_0 - \alpha_k)} \right|.$$ 

The identity (3) implies

$$\log \left| \frac{(t_0 - \alpha_j)(\delta - \alpha_k)}{(\delta - \alpha_j)(t_0 - \alpha_k)} \right| < \text{dist}(\phi(t_0), \mathcal{L}_4) \ll 1.$$ 

For other $\alpha_i$, we have

$$\log \left| \frac{(t_0 - \alpha_j)(\alpha_i - \alpha_k)}{(\alpha_i - \alpha_j)(t_0 - \alpha_k)} \right| < \text{dist}(\phi(t_0), \mathcal{L}_i).$$

By symmetry of the curve, there is a point $z$ such that

$$\text{dist}(z, \mathcal{L}_4) = \text{dist}(\phi(t_0), \mathcal{L}_i), \quad ||z|| = ||\phi(t_0)||.$$ 

Thus,

$$\text{dist}(\phi(t_0), \mathcal{L}_i) < 2||\phi(t_0)|| + o(1).$$

We now see

$$\log \left| \frac{(t_0 - \alpha_j)(\alpha_i - \alpha_k)}{(\alpha_i - \alpha_j)(t_0 - \alpha_k)} \right| \ll ||\phi(t_0)||.$$ 

Hence, $A_1 \ll ||\phi(t_0)|| + \log D.$

13 Counting All Arithmetic Points

Suppose 13 arithmetic points exist. Remove 4 arithmetic points of minimal "radii" $||\phi(t)||$. The arithmetic points on the bridge are removed. (See §11.) For the rest of the arithmetic points $t$, we have $\log D \ll ||\phi(t)||$. (See (1) of §10.)

At least 3 arithmetic points $t, t'$ and $t''$ concentrate on an asymptotic part. Write $r'' = ||\phi(t'')||, r' = ||\phi(t')||, r = ||\phi(t)||$. WLOG, $r'' \geq r' \geq r$. We get

$$\frac{r''/A_4}{\log (r''/A_4)} \ll \prod_{k=1}^{3} A_k$$

from $\log D \ll r$ and (4). Thus, we get

$$r'' \ll \prod_{k=1}^{4} A_k \cdot \log \left( \prod_{k=1}^{3} A_k \right)$$

We set $t_0 = t$. Then, we get

$$r'' \ll \prod_{k=2}^{4} A_k \cdot r \log r$$
since $A_1 \ll \|\phi(t_0)\| + \log D \ll r$ by the result of §12 and the analytic class number formula implies $\log A_2A_3A_4 \ll \log D(R) \ll \log D \ll r$. For showing $r \ll 1$, we would like to combine this inequality with the Gap Principle (2):

$$r'' \gg \triangle (\phi(t), \phi(t'), \phi(t'')) \exp \left( \sqrt{2} \frac{r}{3} \right).$$

It will establish the theorem since $\log D \ll r$.

The result of §6 implies linear independence of $\phi(t') - \phi(t)$ and $\phi(t'') - \phi(t)$ over $\mathbb{R}$. Let $\log \tilde{\zeta}$ and $\log \tilde{\xi}$ be a reduced basis of the plane lattice

$$\mathbb{Z}(\phi(t') - \phi(t)) + \mathbb{Z}(\phi(t'') - \phi(t)).$$

Then, the theory of basis reduction of plane lattice implies

$$\triangle (\phi(t), \phi(t'), \phi(t'')) \gg \|\log \tilde{\zeta}\| \cdot \|\log \tilde{\xi}\| \gg A_2A_3.$$

**Easier Case:** If $A_4 \leq 2r$, we easily argue as follows:

$$A_2A_3 e^{\sqrt{2}r/3} \ll r'' \ll A_2A_3r^2 \log r;$$

$$r \ll 1.$$  

**Harder Case:** We now treat the harder case of $A_4 > 2r$. The lattice generated by vectors

$$\phi(T) - \phi(t), \quad (T : \text{arithmetic point, } \|\phi(T)\| \leq \|\phi(t)\|)$$

is a sublattice of finite index of the lattice $\mathbb{Z}\log \zeta + \mathbb{Z}\log \eta$. (Here, we use the result of §6 noting that there are at least five points of the form $\phi(T)$.) Therefore, $A_2, A_3 \leq 2r$. Those $T$'s and $t', t''$ form a set of 7 or more points. Hence, $\log \zeta, \log \eta, \log \tilde{\zeta}$ and $\log \tilde{\xi}$ generate a space lattice by the result of §7. Therefore, $\|\log \tilde{\xi}\| \geq A_4$. (Obviously, $\|\log \tilde{\zeta}\| \geq A_2$.)

Now, we can argue as follows:

$$A_2A_4 e^{\sqrt{2}r/3} \ll r'' \ll A_2A_4r^2 \log r;$$

$$r \ll 1.$$