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Partitioned Runge-Kutta methods for partial differential equations

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Abstract

We consider implicit-explicit Runge-Kutta (IMEX RK) schemes for time-dependent semi-linear partial differential equations. We show that the error of a scheme is of $O(\Delta t^2)$ in time under some conditions, where $\Delta t$ is the stepsize. This result is, in a sense, optimal. The so-called order reduction phenomena occur, i.e., the error of a scheme based on a partitioned RK method whose order $\geq 3$ behaves as $O(\Delta t^2)$, which is shown numerically.

1. Introduction

We consider initial-boundary value problems of the form

\[
\begin{align*}
\frac{\partial u}{\partial t} &= Lu + g(t, x, u), \quad 0 \leq t \leq T, \quad x \in \Omega, \\
\Phi_b u(t, x) &= \varphi(t, x), \quad 0 \leq t \leq T, \quad x \in \partial \Omega, \\
u(0, x) &= u^0(x), \quad x \in \Omega.
\end{align*}
\]

Here, $u = u(t, x)$ is an $\mathbb{R}^m$-valued unknown function, $\Omega$ is a bounded domain in $\mathbb{R}^d$ with the boundary $\partial \Omega$, $L$ is a linear partial differential operator with constant coefficients with respect to $x$, $\Phi_b$ is a boundary operator. Various reaction-diffusion equations and nonlinear Schrödinger equations such as the Gross-Pitaevskii equation (see, e.g., [4]) are typical examples of (1.1).

Many numerical schemes for evolitional problems in partial differential equations (PDEs) are derived and implemented along the idea of the method of lines (MOL). In this approach a PDE is first discretized in space by finite difference or finite element techniques to be converted into a system of ordinary differential equations (ODEs). We consider the grid $\Omega_h$ defined by $\Omega_h = \Omega \cap h\mathbb{Z}^d$ for $h > 0$, and MOL approximations of (1.1)–(1.2) in the form

\[
\frac{dU_h}{dt} = L_h U_h + \varphi_h(t) + g_h(t, U_h).
\]

Here, $U_h$ is an approximate function of $u$ on $\Omega_h$, $L_h$ is a difference approximation of $L$, $g_h$ is the restriction of $g$ onto $\Omega_h$, and $\varphi_h(t)$ is a function determined from the boundary condition (1.2).
The ODE (1.4) is usually a stiff equation, not easily treated with the standard explicit methods. In some cases (e.g., [11]), the equation (1.4) is solved by a scheme of the form

$$U_{h}^{n+1} = U_{h}^{n} + \Delta t \left( L_{h} U_{h}^{n+1} + \varphi_{h}(t_{n+1}) \right) + \Delta t g_{h}(t_{n}, U_{h}^{n}),$$  \hspace{1cm} (1.5)

where $\Delta t$ is the stepsize, given by $\Delta t = T/N$ for some integer $N \geq 1$, $t_{n} = n \Delta t$, and $U_{h}^{n}$ is an approximate value of $U_{h}(t_{n})$. The scheme (1.5) is obtained by applying the backward Euler formula to the linear part of (1.4) and the forward Euler formula to the nonlinear part. This type of scheme is called implicit-explicit (IMEX) scheme, or semi-implicit scheme.

The scheme (1.5) is of first order in the sense of order of convergence. There are two ways of improving (1.5) in accuracy: one is along the idea of linear multistep methods [1, 3, 12]; the other is along the idea of Runge-Kutta (RK) methods [2, 6, 8]. We follow the latter approach.

Let us consider a pair of two RK methods defined by the arrays

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The left formula determines a diagonally implicit (semi-implicit) RK method, the right formula an explicit RK method. As usual, we assume that

$$c_{i} = \sum_{j=1}^{i} a_{ij} + \sum_{j=1}^{i-1} \hat{a}_{ij}, \hspace{1cm} 0 \leq c_{i} \leq 1, \hspace{1cm} i = 2, 3, \ldots, s.$$  \hspace{1cm} (1.7)

By applying the left formula to the linear part of (1.4) and the right formula to the nonlinear part, we obtain the following scheme for the initial-boundary value problem (1.1)–(1.2):

$$U_{h,n}^{(i)} = U_{h}^{n} + \Delta t \sum_{j=1}^{i} a_{ij} \left( L_{h} U_{h,n}^{(j)} + \varphi_{h}(t_{n} + c_{j} \Delta t) \right) + \Delta t \sum_{j=1}^{i-1} \hat{a}_{ij} g_{h}(t_{n} + c_{j} \Delta t, U_{h,n}^{(j)}) \hspace{1cm} i = 1, 2, \ldots, s,$$  \hspace{1cm} (1.8)

$$U_{h}^{n+1} = U_{h}^{n} + \Delta t \sum_{i=1}^{s} b_{i} \left( L_{h} U_{h,n}^{(i)} + \varphi_{h}(t_{n} + c_{i} \Delta t) \right) + \Delta t \sum_{i=1}^{s} \hat{b}_{i} g_{h}(t_{n} + c_{i} \Delta t, U_{h,n}^{(i)}).$$  \hspace{1cm} (1.9)
Here, $U_{h}^{0}$ is given by $U_{h}^{0} = [u^{0}(x)]_{x \in \Omega_{h}}$.

The main purpose of the present paper is to clarify the convergence property of the scheme (1.8)–(1.9), especially from a viewpoint of the $B$-convergence theory [7]. The concept of $B$-convergence is closely related to the so-called order reduction phenomena, which were first pointed out and studied by Verwer [15] in the PDE context (see also [13, 14]).

2. Main theorem

For $C^{m}$-valued functions on $\Omega_{h}$ we define an inner product by

$$
\langle U, V \rangle_{h} = h^{d} \sum_{x \in \Omega_{h}} \overline{U}(x)^{T} V(x),
$$

(2.1)

and let $\| \cdot \|_{h}$ denote the corresponding norm. We also put

$$
\alpha_{h}(t) = u_{h}'(t) - L_{h}u_{h} - \varphi_{h}(t) - g_{h}(t, u_{h}),
$$

(2.2)

where $u_{h}(t) = [u(t, x)]_{x \in \Omega_{h}}$, and consider the following conditions concerning the problem (1.1)–(1.3) and the MOL approximation (1.4)

(A1) The exact solution $u(t, x)$ is of class $C^{3}$ with respect to $t$; $g(t, x, u)$ is of class $C^{2}$ with respect to $t, u$ and the functions

$$
g, \frac{\partial g}{\partial t}, \frac{\partial g}{\partial u_{i}}, \frac{\partial^{2} g}{\partial t^{2}}, \frac{\partial^{2} g}{\partial t \partial u_{i}}, \frac{\partial^{2} g}{\partial u_{i} \partial u_{j}}$$

are bounded for $(t, x, u) \in [0, T] \times \Omega \times \mathbb{R}^{m}$.

(A2) For any $C^{m}$-valued function $U$ on $\Omega_{h}$, $Re\langle U, L_{h}U \rangle_{h} \leq 0$.

(A3) $\| \alpha_{h}(t) \|_{h} \to 0$ as $h \to 0$.

Moreover, we write

$$
A = (a_{ij})_{1 \leq i, j \leq s}, \quad b = [b_{1}, b_{2}, \ldots, b_{s}]^{T},
$$

$$
\hat{A} = (\hat{a}_{ij})_{1 \leq i, j \leq s}, \quad \hat{b} = [\hat{b}_{1}, \hat{b}_{2}, \ldots, \hat{b}_{s}]^{T},
$$

and consider the following conditions concerning the RK pair (1.6)

(B1) The partitioned RK method (1.6) is of second order, i.e., the parameters $b_{i}, \hat{b}_{i}, c_{i}$ satisfy

$$
\sum_{i=1}^{s} b_{i} = 1, \quad \sum_{i=1}^{s} b_{i} c_{i} = 1/2, \quad \sum_{i=1}^{s} \hat{b}_{i} = 1, \quad \sum_{i=1}^{s} \hat{b}_{i} c_{i} = 1/2.
$$
The diagonally implicit RK method is $A$-stable, $ASI$-stable, and $AS$-stable, i.e., the stability function $r(z) = 1 + zb^{T}(I_{s} - zA)^{-1}1$, $1 = [1, 1, \ldots, 1]^{T}$, satisfies

$$|r(z)| \leq 1 \text{ for any } z \in \mathcal{C}_{-},$$

and each component of $(I_{s} - zA)^{-1}$ and $zb^{T}(I_{s} - zA)^{-1}$ is bounded on $\mathcal{C}_{-}$, where $\mathcal{C}_{-} = \{z \in \mathcal{C} : \text{Re } z < 0\}$.

The rational functions

$$\phi(z) = \frac{b^{T}(I_{s} - zA)^{-1}\gamma}{b^{T}(I_{s} - zA)^{-1}1}, \quad \hat{\phi}(z) = \frac{b^{T}(I_{s} - zA)^{-1}\hat{\gamma}}{b^{T}(I_{s} - zA)^{-1}1} \quad (2.3)$$

are bounded on $\mathcal{C}_{-}$, where

$$\gamma = [\gamma_{1}, \gamma_{2}, \ldots, \gamma_{s}]^{T}, \quad \hat{\gamma} = [\hat{\gamma}_{1}, \hat{\gamma}_{2}, \ldots, \hat{\gamma}_{s}]^{T},$$

$$\gamma_{i} = \frac{c_{i}^{2}}{2} - \sum_{j=1}^{i}a_{ij}c_{j}, \quad \hat{\gamma}_{i} = \hat{a}_{ij}c_{j} - \sum_{j=1}^{i-1}\hat{a}_{ij}c_{j}.$$ 

**Theorem 2.1** Assume that $(A_{1})$--$(A_{3})$ and $(B_{1})$--$(B_{3})$ are satisfied. Then, there are positive numbers $h_{0}, \Delta t_{0}, C$ such that

$$\max_{1 \leq n \leq N} \|U_{h}^{n} - u_{h}(t_{n})\|_{h} \leq C(\Delta t^{2} + \max_{0 \leq t \leq T}\|\alpha_{h}(t)\|_{h}) \quad (2.4)$$

holds for any $h \leq h_{0}$ and $\Delta t \leq \Delta t_{0}$.

The proof is carried out by a similar argument as in the proof of Theorem 3.3 [5], on the based of the following lemma (see, e.g., [10], IV.11).

**Lemma 2.2 (Theorem of von Neumann)** Let $\psi(z)$ be a rational function which has no pole in $\mathcal{C}_{-}$, and assume that $L_{h}$ satisfies $(A_{2})$. Then, we have

$$\|\psi(\Delta tL_{h})\|_{h} \leq \sup_{\text{Re } z \leq 0} |\psi(z)|. \quad (2.5)$$

**Proof of Theorem 2.1.** Put $t_{n,i} = t_{n} + c_{i}\Delta t$ and define $r_{h,n}^{(i)}, \rho_{h,n}$ by

$$u_{h}(t_{n,i}) = u_{h}(t_{n}) + \Delta t \sum_{j=1}^{i}a_{ij}(L_{h}u_{h}(t_{n,j}) + \varphi_{h}(t_{n,j})) + \Delta t \sum_{j=1}^{i-1}\hat{a}_{ij}g_{h}(t_{n,j}, u_{h}(t_{n,j})) + r_{h,n}^{(i)} \quad (2.6)$$

$$u_{h}(t_{n+1}) = u_{h}(t_{n}) + \Delta t \sum_{i=1}^{s}b_{i}(L_{h}u_{h}(t_{n,i}) + \varphi_{h}(t_{n,i})) + \Delta t \sum_{i=1}^{s}\hat{b}_{i}g_{h}(t_{n,i}, u_{h}(t_{n,i})) + \rho_{h,n} \quad (2.7)$$
Then, it follows from (2.2) and (1.7) that

\[ r_{h,n}^{(i)} = u_h(t_{n,i}) - u_h(t_n) - \Delta t \sum_{j=1}^{i} a_{ij} \left[ u_h'(t_{n,j}) - g_h(t_{n,j}, u_h(t_{n,j})) - \alpha_h(t_{n,j}) \right] \\
- \Delta t \sum_{j=1}^{i-1} a_{ij} g_h(t_{n,j}, u_h(t_{n,j})) \]

\[ = \Delta t^2 \gamma_i u_h''(t_n) + \Delta t^2 \gamma_i g_h^{(1)}(t_n, u_h(t_n)) + \Delta t \sum_{j=1}^{i} a_{ij} \alpha_h(t_{n,j}) + \mathcal{O}(\Delta t^3), \]  

(2.8)

where

\[ g_h^{(1)}(t, u_h(t)) = \frac{\partial g_h}{\partial t}(t, u_h(t)) + \frac{\partial g_h}{\partial U}(t, u_h(t)) u_h'(t) \]

Similarly, it follows from (2.2) and (B1) that

\[ \rho_{h,n} = \Delta t \sum_{i=1}^{s} b_i \alpha_h(t_{n,i}) + \mathcal{O}(\Delta t^3). \]  

(2.9)

On the other hand, (2.6), (2.7), (1.8), (1.9) imply

\[ \delta_{h,n}^{(i)} = \epsilon_h^n + \Delta t \sum_{j=1}^{i} a_{ij} L_h \delta_{h,n}^{(j)} + \Delta t \sum_{j=1}^{i-1} \hat{a}_{ij} J_{h,n}^{(j)} \delta_{h,n}^{(j)} + \epsilon_{h,nt}^{(i)}, \]

\[ \epsilon_h^{n+1} = \epsilon_h^n + \Delta t \sum_{i=1}^{s} b_i L_h \delta_{h,n}^{(i)} + \Delta t \sum_{i=1}^{s} \hat{b}_i J_{h,n}^{(i)} \delta_{h,n}^{(i)} + \rho_{h,n}, \]

where

\[ \delta_{h,n}^{(i)} = u_h(t_{n,i}) - U_{h,n}^{(i)}, \quad \epsilon_h^n = u_h(t_n) - U_h^n, \]

\[ J_{h,n}^{(i)} = \int_0^1 \frac{\partial g_h}{\partial U}(t_{n,i}, (1-\theta)U_{h,n}^{(i)} + \theta u_h(t_{n,i})) d\theta. \]

Eliminating \( \delta_{h,n}^{(i)} \), we get

\[ \epsilon_{h,nt}^{n+1} = \left[ I + (b^T Z + \hat{b}^T W_n)(I - AZ - \overline{A} W_n)^{-1}(1 \otimes I) \right] \epsilon_h^n \\
+ (b^T Z + \hat{b}^T W_n)(I - AZ - \overline{A} W_n)^{-1} r_{h,n} + \rho_{h,n}, \]  

(2.10)

where \( A = A \otimes I, \overline{A} = \hat{A} \otimes I, b = b \otimes I, \hat{b} = \hat{b} \otimes I, I = I_s \otimes I, \)

\[ Z = \Delta t (1 \otimes L_h), \quad W_n = \Delta t \left[ (J_{h,n}^{(1)})^T, (J_{h,n}^{(2)})^T, \ldots, (J_{h,n}^{(s)})^T \right]^T, \]

\[ r_{h,n} = \left[ (r_{h,n}^{(1)})^T, (r_{h,n}^{(2)})^T, \ldots, (r_{h,n}^{(s)})^T \right]^T. \]

Moreover, letting

\[ \epsilon_h^n = \epsilon_h^n + \Delta t^2 v_h^n, \quad v_h^n = \phi(\Delta t L_h) u_h''(t_n) + \hat{\phi}(\Delta t L_h) g_h^{(1)}(t_n, u_h(t_n)), \]
we can rewrite (2.10) as

$$\hat{c}_{h}^{n+1} = [I + (b^{T}Z + \hat{b}^{T}W_{n})(I - AZ - \hat{A}W_{n})^{-1}(1 \otimes I)]\hat{c}_{h}^{n} + (b^{T}Z + \hat{b}^{T}W_{n})(I - AZ - \hat{A}W_{n})^{-1}\hat{r}_{h,n} + \hat{\rho}_{h,n},$$

where

\[
\hat{r}_{h,n} = r_{h,n} - \Delta t^{2}\gamma \otimes u_{h}'(t_{n}) - \Delta t^{2}\hat{\gamma} \otimes g_{h}^{(1)}(t_{n}, u_{h}(t_{n})),
\]

\[
\hat{\rho}_{h,n} = \rho_{h,n} + \Delta t^{2}(v_{h}^{n+1} - v_{h}^{n}) + \Delta t^{2}(b^{T}Z + \hat{b}^{T}W_{n})(I - AZ - \overline{A}W_{n})^{-1}w_{h,n},
\]

\[
w_{h,n} = \gamma \otimes u_{h}'(t_{n}) + \hat{\gamma} \otimes g_{h}^{(1)}(t_{n}, u_{h}(t_{n}))-1 \otimes v_{h}^{n}.
\]

By (2.9), we have

$$\hat{r}_{h,n}^{(i)} = \Delta t \sum_{j=1}^{i} a_{ij} \alpha_{h}(t_{n,j}) + O(\Delta t^{3}).$$

(2.12)

Moreover, \(b^{T}Z(I - AZ)^{-1}w_{h,n} = 0\), by the definitions of \(v_{h}^{n}\) and \(w_{h,n}\). Hence, it follows from

$$(I - AZ - \hat{A}W_{n})^{-1} = (I - AZ)^{-1} + (I - AZ)^{-1}\hat{A}W_{n}(I - AZ - \hat{A}W_{n})^{-1}$$

(2.13)

that

$$\hat{b}^{T}W_{n}(I - AZ - \hat{A}W_{n})^{-1}w_{h,n} = (b^{T}Z + \hat{b}^{T}W_{n})(I - AZ)^{-1}w_{h,n}$$

$$+(b^{T}Z + \hat{b}^{T}W_{n})(I - AZ)^{-1}\hat{A}W_{n}(I - AZ - \hat{A}W_{n})^{-1}w_{h,n}.$$  

By making use of Lemma 2.2 it is shown that this value is of \(O(\Delta t)\) by ASI-stability and AS-stability of the implicit RK method, which, together with, \(v_{h}^{n+1} - v_{h}^{n} = O(\Delta t)\), implies

$$\hat{\rho}_{h,n} = \Delta t \sum_{i=1}^{s} b_{i} \alpha_{h}(t_{n,i}) + O(\Delta t^{3}).$$

(2.14)

It follows from (2.11), (2.12), (2.14) that there exists \(\tilde{C}\) such that

$$\| \varepsilon_{h}^{n} \|_{h} \leq \tilde{C}(\Delta t^{2} + \max_{0 \leq t \leq T} \| \alpha_{h}(t) \|_{h})$$

(2.15)

holds for sufficiently small \(h\) and \(\Delta t\). This is also verified on the basis of Lemma 2.2. Therefore, (2.4) holds. \(\square\)
3. Numerical examples

Consider the simple model problem

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(t, x, u), \quad t \geq 0, \quad 0 \leq x \leq 1, \tag{3.1} \]

\[ g(t, x, u) = \frac{\pi^2}{2} u - u^2 + e^{-\pi^2 t} \cos^2(\pi x), \]

\[ u(t, 0) = e^{-\pi^2 t/2}, \quad u(t, 1) = -e^{-\pi^2 t/2}, \quad t \geq 0, \tag{3.2} \]

\[ u(0, x) = \cos(\pi x), \quad 0 \leq x \leq 1, \tag{3.3} \]

whose exact solution is

\[ u(t, x) = e^{-\pi^2 t} \cos(\pi x). \]

Moreover, consider the grid

\[ 0 = x_0 < \cdots < x_j = jh < \cdots < x_M = 1, \quad h = 1/M, \]

and an MOL approximation determined by

\[ \frac{u'_{j-1} + 10u'_j + u'_{j+1}}{12} = \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} \]

\[ + \frac{g(t, x_{j-1}, u_{j-1}) + 10g(t, x_j, u_j) + g(t, x_{j+1}, u_{j+1})}{12}, \quad j = 1, 2, \ldots, M - 1, \tag{3.4} \]

where \( M \) is a positive integer and \( u_j(t) \) is an approximation of \( u(t, x_j) \). The functions \( u_0(t) \) and \( u_M(t) \) are given by

\[ u_0(t) = e^{-\pi^2 t/2}, \quad u_M(t) = -e^{-\pi^2 t/2}, \]

corresponding to (3.2). Simple computation shows that

\[ \alpha_h(t) = O(h^4) \tag{3.5} \]

holds for (3.4).

One of the simplest RK pairs which satisfy (B1)–(B3) is the pair of the trapezoidal rule and Heun’s method (a modification of the Crank-Nicolson scheme),

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1/2 & 1/2 & 1 & 1 & 0 \\
1/2 & 1/2 & 1/2 & 1/2 & 1/2
\end{pmatrix}
\]

Clearly, (B1) is satisfied, and it follows from

\[ (I_2 - zA)^{-1} = \frac{1}{1 - z/2} \begin{bmatrix} 1 - z/2 & 0 \\ z/2 & 1 \end{bmatrix}, \quad zb^T(I_2 - zA)^{-1} = \frac{z}{1 - z/2} \begin{bmatrix} 1/2, 1/2 \end{bmatrix}, \]
\[ r(z) = 1 + zb^T(I_2 - zA)^{-1}1 = \frac{1+z/2}{1-z/2}, \]

that the implicit method satisfies \((B_2)\). In addition, \(\gamma = [0, 0]^T\) and \(\hat{\gamma} = [0, 1/2]^T\).

Hence,

\[ \phi(z) = \frac{b^T(I_2 - zA)^{-1}\gamma}{b^T(I_2 - zA)^{-1}1} = 0, \quad \hat{\phi}(z) = \frac{b^T(I_2 - zA)^{-1}\hat{\gamma}}{b^T(I_2 - zA)^{-1}1} = 1/4, \]

and \((B_3)\) is satisfied.

The RK pair

\[
\begin{array}{ccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 \\
\alpha & 0 & \alpha & 0 & \alpha & 0 \\
1 - \alpha & 0 & 1 - 2\alpha & \alpha & 0 & 1 - \alpha \\
\hline
0 & 1/2 & 1/2 & 0 & 0 & 1/2 \\
\end{array}
\]

\(\alpha = \frac{3 + \sqrt{3}}{6}\) \(\text{ (3.7)}\)

also satisfies \((B_1)-(B_3)\). This pair, which was proposed by Ascher, Ruuth and Spiteri [2], determines a third order partitioned RK method for ODEs. In particular, \((B_1)\) is satisfied. The conditions \((B_2)\) and \((B_3)\) follow from

\[
(I_3 - zA)^{-1} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 - \frac{z}{1 - \alpha z} & 0 \\
0 & 0 & 1 - \frac{z}{1 - \alpha z}
\end{bmatrix},
\]

\[
z b^T(I_3 - zA)^{-1} = \frac{z}{2} \begin{bmatrix}
0 & 1 - \frac{3\alpha - 1}{1 - \alpha z} & 1 \\
0 & 0 & 1 - \alpha z
\end{bmatrix},
\]

\[
r(z) = \frac{1 - (2\alpha - 1)z - (\alpha - 1/3)z^2}{(1 - \alpha z)^2},
\]

\[
\phi(z) = \frac{\alpha^2}{2} \frac{(2\alpha - 1)z}{2 + (1 - 4\alpha)z}, \quad \hat{\phi}(z) = -\frac{\alpha^2(2\alpha - 1)z}{2 + (1 - 4\alpha)z}.
\]

We apply the RK pairs (3.6) and (3.7) to the MOL approximation (3.4), and integrate it from \(t = 0\) to \(t = 1\), with various gridsizes and stepsizes of the form

\[
\Delta t = h = \frac{1}{M}. \quad \text{(3.8)}
\]

Table 1 shows the values

\[-\log_2 \varepsilon_M, \quad \varepsilon_M = \max_{1 \leq n \leq M} \left( \max_{1 \leq j \leq M} \left| u(t_n, x_j) - u^n_j \right| \right).\]

It is observed that \(\varepsilon_M\) is of \(O(\Delta t^2)\) for each method. Noting (3.5) and (3.8), we can consider the result for (3.7) presents an order reduction phenomenon, i.e., the error of a “third order” method behaves as \(O(\Delta t^2)\).
Table 1. Numerical results for the model problem (3.1)–(3.3)

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<tr>
<td>Method (3.6)</td>
<td>3.63</td>
<td>5.09</td>
<td>6.79</td>
<td>8.64</td>
<td>10.56</td>
<td>12.52</td>
</tr>
<tr>
<td>Method (3.7)</td>
<td>5.60</td>
<td>7.42</td>
<td>9.32</td>
<td>11.25</td>
<td>13.19</td>
<td>15.16</td>
</tr>
</tbody>
</table>

Fig. 1 shows a numerical result concerning the "soliton solution"

$$w(t, x) = \sqrt{2\alpha} \exp\left[ i \left\{ \frac{c}{2} x - \left( \frac{c^{2}}{4} - \alpha \right) t \right\} \right] \text{sech} \left[ \alpha (x - ct) \right]$$  \hspace{2cm} (3.9)

to the simple nonlinear Schrödinger equation

$$\frac{\partial w}{\partial t} = i \frac{\partial^{2} w}{\partial x^{2}} + i \mid w \mid^{2} w.$$  \hspace{2cm} (3.10)

The "1st order scheme" indicates the method (1.5), and the "2nd order scheme" indicates the method (3.6). The values $\alpha = 0.5$, $c = 1$, $\Delta x = 0.2$, $\Delta t = 0.005$ are used for the computation.

Fig. 1. Numerical solutions of the nonlinear Schrödinger equation (3.10).

Fig. 2 shows a stationary solution to the equation (Brusselator)

$$\begin{cases}
\frac{\partial u}{\partial t} = D_U \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \alpha - (\beta + 1)u + u^2 v \\
\frac{\partial v}{\partial t} = D_V \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \beta u - u^2 v
\end{cases} \hspace{2cm} (3.11)

0 \leq x \leq 4, \ 0 \leq y \leq 4, \ D_U = 0.02, \ D_V = 1, \ \alpha = 1, \ \beta = 1.8,

under the Neumann boundary condition, obtained by the method (3.6). These figures suggest that the method (3.6) is useful for some problems.
Fig. 2. Stationary solution to the equation (3.11).

References


