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GLOBAL EXISTENCE OF RADIALLY SYMMETRIC SOLUTIONS OF THE ISENTROPIC COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH VACUUM

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1. INTRODUCTION

We consider the isentropic compressible Navier-Stokes equations in $(0,\infty)\times\Omega$

$$(\rho\mathbf{u})_t+\mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u})-\mu\Delta\mathbf{u}-(\lambda+\mu)\nabla\mathrm{div}\,\mathbf{u}+\nabla p=\rho\mathbf{f}, \tag{1.1}$$

$$\rho_t+\mathrm{div}(\rho\mathbf{u})=0,\quad p=A\rho^\gamma\qquad (A>0,\ \gamma>1), \tag{1.2}$$

where Ω is a bounded annulus in $\mathbf{R}^n$ ($n\ge 1$) and the given data are radially symmetric. More precisely, the domain Ω and the external force f are given by

$$\Omega=\{\mathbf{x}\in\mathbf{R}^n : a<|\mathbf{x}|<b\},\quad \mathbf{f}(t,\mathbf{x})=f(t,|\mathbf{x}|\,)\frac{\mathbf{x}}{|\mathbf{x}|}$$

for some constants a,b with $0<a<b<\infty$, and the initial and boundary conditions are imposed as follows:

$$(\rho,\,\mathbf{u})|_{t=0}=(\rho_0,\,\mathbf{u}_0)\ \text{ in }\Omega\quad ;\quad \mathbf{u}=\mathbf{0}\text{ on }(0,\infty)\times\partial\Omega, \tag{1.3}$$

where

$$\rho_0(\mathbf{x})=\rho_0(|\mathbf{x}|\,)\ge 0,\quad \mathbf{u}_0(\mathbf{x})=u_0(|\mathbf{x}|\,)\frac{\mathbf{x}}{|\mathbf{x}|}\quad\text{for}\quad \mathbf{x}\in\Omega. \tag{1.4}$$

Here ρ, $\mathbf{u}$ and p denote the unknown density, velocity and pressure, respectively. The viscosity constants μ and λ are assumed to satisfy the usual physical requirements $\mu>0$, $2\mu+n\lambda\ge 0$.

The main concern of this note is to study global existence of radially symmetric solutions to the initial boundary value problem (1.1)–(1.3). The first existence result was proved by D. Hoff [4]; he proved global existence of radially symmetric weak solutions to the problem (1.1)–(1.3) with the strictly positive initial densities. Then it was extended by S. Jiang and P. Zhang [6] to the Cauchy problem with general nonnegative initial densities. Roughly speaking, they proved the global existence of radially symmetric weak solutions under the regularity assumption that $0\le\rho_0\in L^\gamma(\mathbf{R}^n)$, $\sqrt{\rho_0}\mathbf{u}_0\in L^2(\mathbf{R}^n)$ and $\mathbf{f}=0$, where $n=2$ or 3.

In this note, we prove global existence and uniqueness of radially symmetric strong solutions with nonnegative densities.

Theorem 1.1. *Assume that the radially symmetric data $\rho_0,\mathbf{u}_0,\mathbf{f}$ satisfy the regularity condition*

$$0\le\rho_0\in H^1,\quad \mathbf{u}_0\in H_0^1\cap H^2,\quad \mathbf{f},\,\nabla\mathbf{f},\,\mathbf{f}_t\in L^2_{loc}(0,\infty;L^2). \tag{1.5}$$

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Then there exists a radially symmetric strong solution $(\rho, \mathbf{u})$ *to the initial boundary value problem* (1.1)–(1.3) *satisfying the regularity*

$$\begin{aligned} \rho &\in C([0,\infty);H^1), \quad \mathbf{u} \in C([0,\infty);H_0^1 \cap H^2), \\ \rho_t &\in C([0,\infty);L^2), \quad \mathbf{u}_t \in L^2_{loc}(0,\infty;H_0^1), \quad \sqrt{\rho}\mathbf{u}_t \in L^\infty_{loc}(0,\infty;L^2), \end{aligned} \tag{1.6}$$

if and only if the initial data $(\rho_0, \mathbf{u}_0)$ *satisfy the compatibility condition*

$$-\mu\Delta\mathbf{u}_0 - (\lambda+\mu)\nabla \mathrm{div}\, \mathbf{u}_0 + \nabla(A\rho_0^\gamma) = \rho_0^{\frac{1}{2}}\,\mathbf{g} \tag{1.7}$$

for some radially symmetric $\mathbf{g} \in L^2$. *In this case, the initial condition is satisfied in the following sense:*

$$|\rho(t) - \rho_0|_{H^1} + |\mathbf{u}(t) - \mathbf{u}_0|_{H^2} \to 0 \quad as \quad t \to 0. \tag{1.8}$$

Here we used the simplified notations for the standard Sobolev spaces $L^q = L^q(\Omega)$ and $H^k = W^{k,2}(\Omega)$, etc.

The compatibility condition (1.7) has been considered by Y. Cho, H. J. Choe and H. Kim [1], H.J. Choe and H. Kim [2] and R. Salvi and I. Straškraba [11] to prove local existence of a unique strong solution with nonnegative density. Hence our result on the global existence of strong solutions is an extension of the previous local ones in the case of radially symmetric data.

Finally, we remark that for the Navier-Stokes equations of compressible heat-conducting gases, it is still an open problem to prove existence of strong solutions with nonnegative densities. In the case of positive initial densities, the existence of strong solutions has been well-known and in particular, S. Jiang [5] and V.B. Nikolaev [10] proved global existence of radially symmetric strong solutions in annular domains.

2. A PRIORI ESTIMATES

In this section, we derive various a priori estimates for radially symmetric solutions of the Navier-Stokes equations (1.1) and (1.2), which are independent of lower bounds of the initial density.

To construct radially symmetric solutions, we first consider the following initial boundary value problem in $(0,\infty) \times (a,b)$:

$$\rho_t + (\rho u)_r + m\frac{\rho u}{r} = 0, \tag{2.1}$$

$$(\rho u)_t + (\rho u^2)_r + m\frac{\rho u^2}{r} - \nu\left(u_r + m\frac{u}{r}\right)_r + p_r = \rho f, \tag{2.2}$$

$$\rho(0,r) = \rho_0(r), \quad u(0,r) = u_0(r), \qquad u(t,a) = u(t,b) = 0 \tag{2.3}$$

where $p = A\rho^\gamma$, $\nu = \lambda + 2\mu > 0$ and $m = n - 1 \geq 0$.

Using a standard technique(for instance, a fixed point argument), we can prove the following local existence result.

Theorem 2.1. *Assume that*

$$\rho_0 \in H^2(a,b), \;\; u_0 \in H_0^1(a,b) \cap H^2(a,b), \;\; f, \; f_r, \; f_t \in L^2_{loc}(0,\infty;L^2(a,b))$$

and

$$\rho_0 \geq \varepsilon \quad on \;\; [a,b] \quad for\ some\ constant \quad \varepsilon > 0.$$

Then there exist a small time $T > 0$ and a unique strong solution (ρ, u) to the initial boundary value problem (2.1)–(2.3) *such that*

$$
\begin{aligned}
&\rho \in C([0,T]; H^2(a,b)), \quad \rho_t \in C([0,T]; H^1(a,b)),\\
&u \in C([0,T]; H_0^1(a,b) \cap H^2(a,b)) \cap L^2(0,T; H^3(a,b)),\\
&u_t \in C([0,T]; L^2(a,b)) \cap L^2(0,T; H_0^1(a,b)),\\
&\qquad \rho > 0 \ \ on \ \ [0,T] \times [a,b].
\end{aligned}
\tag{2.4}
$$

Remark 2.2. *In fact, the strong solution exists globally in time, as is proved later.*

Let (ρ, u) be a strong solution to the problem (2.1)–(2.3) satisfying the regularity (2.4), and let us define

$$\rho(t,\mathbf{x}) = \rho(t, |\mathbf{x}|) \quad \text{and} \quad \mathbf{u}(t,\mathbf{x}) = u(t, |\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|}.$$

Then a direct calculation shows that

$$\Delta\mathbf{u} = \nabla\mathrm{div}\,\mathbf{u} = \left(u_r + m\frac{u}{r}\right)_r \frac{\mathbf{x}}{r} \quad \text{with} \quad r = |\mathbf{x}|. \tag{2.5}$$

Thanks to this identity, we can easily show that $(\rho, \mathbf{u})$ is a radially symmetric strong solution to the original problem (1.1)–(1.3). From now on, we will derive some a priori estimates for $(\rho, \mathbf{u})$, independent of $\varepsilon = \inf \rho_0 > 0$.

To begin with, we recall the following elementary result (the conservation of mass and energy inequality).

Lemma 2.3.

$$\sup_{0\le t\le T} \left(|\sqrt{\rho}\mathbf{u}(t)|_{L^2}^2 + |\rho(t)|_{L^1} + |p(t)|_{L^1}\right) + \int_0^T |\nabla\mathbf{u}|_{L^2}^2\, dt \le C. \tag{2.6}$$

Throughout this paper, we denote by C a generic positive constant depending only on ν, a, T and the norms of the data, but independent of $\varepsilon = \inf \rho_0$.

Next, we prove the boundedness of the density, which is one of the most important estimates in this paper. Following the arguments in [7], we prove

Lemma 2.4.

$$\sup_{0\le t\le T} |\rho(t)|_{L^\infty} \le C. \tag{2.7}$$

Proof. We introduce the *Lagrangian mass coordinates* (t, y), defined by

$$t = t \quad \text{and} \quad y = \int_a^r \rho(t,r) r^m\, dr$$

Then since

$$\frac{\partial(t,y)}{\partial(t,r)} = \begin{pmatrix} 1 & 0 \\ -\rho u r^m & \rho r^m \end{pmatrix} \quad \text{and} \quad \frac{\partial(t,r)}{\partial(t,y)} = \begin{pmatrix} 1 & 0 \\ u & (\rho r^m)^{-1} \end{pmatrix},$$

the problem (2.1)–(2.3) can be rewritten in Lagrangian coordinates as

$$
\begin{cases}
\rho_t + \rho^2 (r^m u)_y = 0,\\
r^{-m} u_t - \nu \left(\rho (r^m u)_y\right)_y + p_y = r^{-m} f(t,r),\\
r^n = a^n + n\displaystyle\int_0^y \frac{1}{\rho(t,z)}\, dz,\\
\rho(0,y) = \rho_0(y), \ \ u(0,y) = u_0(y); \quad u(t,0) = u(t,Y) = 0,
\end{cases}
\tag{2.8}
$$

where $0 \le t \le T$, $0 \le y \le Y = \int_a^b \rho_0(r) r^m \, dr$ and $p = p(t,y) = A\rho(t,y)^\gamma$. Note also that $Y = \int_a^b \rho(t,r) r^m \, dr$ for all $t \in [0,T]$ (conservation of mass).

Now we have only to show that $\rho(t,y) \le C$ for $0 \le t \le T$ and $0 \le y \le Y$. To begin with, we observe from (2.8) that

$$\begin{aligned}
\nu(\log\rho)_{ty} = \nu\left(\frac{\rho_t}{\rho}\right)_y &= -\nu\left(\rho(r^m u)_y\right)_y = -r^{-m}u_t - p_y + r^{-m} f \\
&= -\left(r^{-m}u\right)_t - p_y + r^{-m}\left(f - m\frac{u^2}{r}\right).
\end{aligned}$$

Thus, integrating over $(0,t)\times(0,y)$, we deduce that

$$\begin{aligned}
\nu\log\frac{\rho(t,y)}{\rho(t,0)} = \nu\log\frac{\rho_0(y)}{\rho_0(0)} &+ \int_0^y \left((r^{-m}u)(0,z) - (r^{-m}u)(t,z)\right) dz \\
&+ \int_0^t \left(p(s,0) - p(s,y)\right) ds + \int_0^t\int_0^y r^{-m}\left(f - m\frac{u^2}{r}\right) dz ds
\end{aligned}$$

and

$$\begin{aligned}
\frac{\rho(t,y)}{\rho(t,0)} &= \frac{\rho_0(y)}{\rho_0(0)} \exp\left(\frac{1}{\nu}\int_0^y \left((r^{-m}u)(0,z) - (r^{-m}u)(t,z)\right) dz\right) \\
&\times \exp\left(\frac{1}{\nu}\int_0^t \left(p(s,0) - p(s,y)\right) ds\right) \exp\left(\frac{1}{\nu}\int_0^t\int_0^y r^{-m}\left(f - m\frac{u^2}{r}\right) dz ds\right).
\end{aligned}$$

From this identity, we derive a representation formula for ρ:

$$\rho(t,y) = P(t)Q(t,y)\exp\left(-\frac{1}{\nu}\int_0^t p(s,y)\, ds\right), \tag{2.9}$$

where

$$P(t) = \frac{\rho(t,0)}{\rho_0(0)} \exp\left(\frac{1}{\nu}\int_0^t p(s,0)\, ds\right)$$

and

$$\begin{aligned}
Q(t,y) = \rho_0(y) &\exp\left(\frac{1}{\nu}\int_0^y \left((r^{-m}u)(0,z) - (r^{-m}u)(t,z)\right) dz\right) \\
&\times \exp\left(\frac{1}{\nu}\int_0^t\int_0^y r^{-m}\left(f - m\frac{u^2}{r}\right) dz ds\right).
\end{aligned}$$

Moreover, ρ can be represented only in terms of $P(t)$ and $Q(t,y)$. Since $p = A\rho^\gamma$, it follows from (2.9) that

$$\begin{aligned}
\frac{d}{dt}\exp\left(\frac{\gamma}{\nu}\int_0^t p(s,y)\, ds\right) &= \frac{A\gamma}{\nu}\rho(t,y)^\gamma \exp\left(\frac{\gamma}{\nu}\int_0^t p(s,y)\, ds\right) \\
&= \frac{A\gamma}{\nu}\{P(t)Q(t,y)\}^\gamma
\end{aligned}$$

and thus

$$\exp\left(\frac{1}{\nu}\int_0^t p(s,y)\, ds\right) = \left[1 + \frac{A\gamma}{\nu}\int_0^t \{P(s)Q(s,y)\}^\gamma ds\right]^{\frac{1}{\gamma}}.$$

Therefore, substituting this into (2.9), we obtain

$$\rho(t,y) = \frac{P(t)Q(t,y)}{\left[1 + \frac{A\gamma}{\nu}\int_0^t \{P(s)Q(s,y)\}^\gamma ds\right]^{\frac{1}{\gamma}}}. \tag{2.10}$$

To prove the boundedness of ρ, it thus remains to estimate $P(t)$ and $Q(t,y)$. First, converting back into the Eulerian coordinates and using the previous lemma, we have

$$\begin{aligned}\int_0^Y r^{-m}|u|\,dy &= \int_a^b \rho|u|\,dr \le \frac{1}{a^m}\int_a^b \rho|u|r^m\,dr\\ &\le \frac{1}{a^m}\int \rho|\mathbf{u}|\,dx \le C \quad \text{for} \quad 0\le t\le T\end{aligned}$$

and

$$\begin{aligned}\int_0^T\int_0^Y r^{-m}\left(|f|+m\frac{|u|^2}{r}\right)dydt &= \int_0^T\int_a^b\left(\rho|f|+m\frac{\rho|u|^2}{r}\right)drdt\\ &\le \frac{1}{a^m}\int_0^T\int\left(\rho|\mathbf{f}|+\frac{m}{a}\rho|\mathbf{u}|^2\right)dxdt \le C.\end{aligned}$$

Hence it follows from the definition of $Q(t,y)$ that

$$\left|\log\frac{Q(t,y)}{\rho_0(y)}\right| \le C,$$

or equivalently

$$C^{-1}\rho_0(y) \le Q(t,y) \le C\,\rho_0(y). \tag{2.11}$$

Next, to estimate $P(t)$, observe that

$$\int_0^Y \frac{1}{\rho(t,y)}\,dy = \int_a^b r^m\,dr = \frac{b^n-a^n}{n}.$$

Then we deduce from (2.10) and (2.11) that

$$\begin{aligned}\frac{b^n-a^n}{n}P(t) &= \int_0^Y \frac{P(t)}{\rho(t,y)}\,dy = \int_0^Y \frac{\left[1+\frac{A\gamma}{\nu}\int_0^t\{P(s)Q(s,y)\}^\gamma\,ds\right]^{\frac{1}{\gamma}}}{Q(t,y)}\,dy\\ &\le \int_0^Y \frac{1}{Q(t,y)}\,dy + \left(\frac{A\gamma}{\nu}\right)^{\frac{1}{\gamma}}\int_0^Y\left[\int_0^t\left(P(s)\frac{Q(s,y)}{Q(t,y)}\right)^\gamma ds\right]^{\frac{1}{\gamma}}dy\\ &\le C\frac{b^n-a^n}{n} + C\left(\int_0^t P(s)^\gamma\,ds\right)^{\frac{1}{\gamma}}.\end{aligned}$$

Therefore, dividing both sides by $\frac{b^n-a^n}{n}$, taking the γ-th power and then using Grownall's inequality, we deduce that

$$P(t) \le C\exp\left(\frac{C}{(b^n-a^n)^\gamma}\right) \quad \text{for} \quad 0\le t\le T. \tag{2.12}$$

Combining (2.10), (2.11) and (2.12), we complete the proof of Lemma 2.4. □

To obtain further estimates, we make use of the following versions of Sobolev inequalities for radially symmetric functions:

$$|\rho|_{L^\infty} \le C|\rho|_{H^1}, \quad |\mathbf{f}|_{L^\infty} \le C|\mathbf{f}|_{H^1} \quad \text{and} \quad |\mathbf{u}|_{L^\infty} \le C|\nabla\mathbf{u}|_{L^2}. \tag{2.13}$$

Moreover, we need to introduce the *effective viscous flux* $G = \nu\,\mathrm{div}\,\mathbf{u} - p$.

Lemma 2.5.

$$\sup_{0\le t\le T}\left(|\mathbf{u}(t)|_{L^\infty} + |\mathbf{u}(t)|_{H_0^1}\right) + \int_0^T\left(|\sqrt{\rho}\mathbf{u}_t(t)|_{L^2}^2 + |G(t)|_{H^1}^2\right)dt \le C. \tag{2.14}$$

Proof. In view of the continuity equation (1.2) and the identity (2.5), the momentum equation (1.1) can be rewritten as

$$\rho\mathbf{u}_t + \rho\mathbf{u}\cdot\nabla\mathbf{u} - \nu\nabla\mathrm{div}\,\mathbf{u} + \nabla p = \rho\mathbf{f}.$$

Multiplying this by $\mathbf{u}_t$, integrating (by parts) over Ω and using Young's inequality, we have

$$\begin{aligned}\frac{1}{2}\int\rho|\mathbf{u}_t|^2\,dx+&\frac{d}{dt}\int\frac{\nu}{2}(\mathrm{div}\,\mathbf{u})^2\,dx\\ &\le\int\rho|\mathbf{f}|^2\,dx+\int\rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2\,dx+\int p\,\mathrm{div}\,\mathbf{u}_t\,dx.\end{aligned}\tag{2.15}$$

Using the continuity equation (1.2), we obtain

$$\begin{aligned}\int p\,\mathrm{div}\,\mathbf{u}_t\,dx &= \frac{d}{dt}\int p\,\mathrm{div}\,\mathbf{u}\,dx + \int(\mathrm{div}(p\,\mathbf{u}) + (\gamma-1)p\mathrm{div}\,\mathbf{u})\mathrm{div}\,\mathbf{u}\,dx\\ &= \frac{d}{dt}\int p\,\mathrm{div}\,\mathbf{u}\,dx - \int p\,\mathbf{u}\cdot\nabla\mathrm{div}\,\mathbf{u}\,dx + (\gamma-1)\int p\,(\mathrm{div}\,\mathbf{u})^2\,dx\\ &= \frac{d}{dt}\int p\,\mathrm{div}\,\mathbf{u}\,dx + \frac{4\gamma-3}{2\nu}\int p^2\,\mathrm{div}\,\mathbf{u}\,dx\\ &\qquad+\frac{\gamma-1}{\nu^2}\int p(G^2-p^2)\,dx - \frac{1}{\nu}\int p\,\mathbf{u}\cdot\nabla G\,dx\\ &= \frac{d}{dt}\int p\,\mathrm{div}\,\mathbf{u}\,dx - \frac{d}{dt}\int\frac{4\gamma-3}{2\nu(2\gamma-1)}p^2\,dx\\ &\qquad+\frac{\gamma-1}{\nu^2}\int p(G^2-p^2)\,dx - \frac{1}{\nu}\int p\,\mathbf{u}\cdot\nabla G\,dx.\end{aligned}$$

Substituting this identity into (2.15), integrating over $(0,t)$ and using the obvious inequality

$$\nu\frac{2\gamma-2}{4\gamma-3}(\mathrm{div}\,\mathbf{u})^2 \;\le\; \nu(\mathrm{div}\,\mathbf{u})^2 - 2p\,(\mathrm{div}\,\mathbf{u}) + \frac{4\gamma-3}{\nu(2\gamma-1)}p^2,$$

we derive

$$\begin{aligned}\int_0^t\int\rho|\mathbf{u}_t|^2\,dxds &+ \int|\mathrm{div}\,\mathbf{u}(t)|^2\,dx\\ &\le C + C\int_0^t\int\left(\rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2 + p\,G^2 + p|\mathbf{u}||\nabla G|\right)\,dxds.\end{aligned}\tag{2.16}$$

We estimate each term of the right hand side of (2.16). By virtue of the estimates (2.6), (2.7) and (2.13), we have

$$\int_0^t\int\rho|\mathbf{u}|^2|\nabla\mathbf{u}|^2\,dxds \le \int_0^t|\rho|_{L^\infty}|\mathbf{u}|^2_{L^\infty}|\nabla\mathbf{u}|^2_{L^2}\,ds \le C\int_0^t|\nabla\mathbf{u}|^4_{L^2}\,ds$$

and

$$\int_0^t\int p\,G^2\,dxds \le C\int_0^t\int p(|\nabla\mathbf{u}|^2 + p^2)\,dxds \le C.$$

Using the identity

$$\nabla G = \rho\mathbf{u}_t + \rho\mathbf{u}\cdot\nabla\mathbf{u} - \rho\mathbf{f}\tag{2.17}$$

together with (2.6) and (2.7), we also have

$$C\int_0^t\int p|\mathbf{u}||\nabla G|\,dxds \le C\int_0^t |\rho|_{L^\infty}^{\gamma-\frac{1}{2}}|\sqrt{\rho}\mathbf{u}|_{L^2}|\nabla G|_{L^2}\,ds$$
$$\le C\int_0^t (|\rho\mathbf{u}_t|_{L^2}+|\rho\mathbf{u}\cdot\nabla\mathbf{u}|_{L^2}+|\rho\mathbf{f}|_{L^2})\,ds$$
$$\le C+\frac{1}{2}\int_0^t|\sqrt{\rho}\mathbf{u}_t|_{L^2}^2\,ds.$$

Substituting these estimates into (2.16) and recalling that $|\operatorname{div}\mathbf{u}|_{L^2}=|\nabla\mathbf{u}|_{L^2}$, we finally obtain

$$\int_0^t|\sqrt{\rho}\mathbf{u}_t|_{L^2}^2\,ds+|\nabla\mathbf{u}(t)|_{L^2}^2\le C+C\int_0^t|\nabla\mathbf{u}|_{L^2}^4\,ds.$$

Since $\int_0^T|\nabla\mathbf{u}|_{L^2}^2\,ds\le C$, it follows from Gronwall's lemma that

$$\int_0^T|\sqrt{\rho}\mathbf{u}_t|_{L^2}^2\,ds+\sup_{0\le t\le T}|\nabla\mathbf{u}|_{L^2}^2\le C.$$

Then utilizing (2.13) and (2.17), we complete the proof of Lemma 2.5. □

Lemma 2.6.

$$\sup_{0\le t\le T}|\nabla\rho(t)|_{L^2}+\int_0^T\left(|\nabla\mathbf{u}(t)|_{L^\infty}^2+|\mathbf{u}(t)|_{H^2}^2\right)dt\le C. \tag{2.18}$$

Proof. First, since G is a radially symmetric scalar function, we can apply Sobolev inequality (2.13) and use the estimate (2.14) to obtain

$$\int_0^T|G|_{L^\infty}^2\,dt\le C\int_0^T|G|_{H^1}^2\,dt\le C. \tag{2.19}$$

A simple calculation shows that

$$|\nabla\mathbf{u}|^2=u_r^2+m\frac{u^2}{r^2}\le 2\left(u_r+m\frac{u}{r}\right)^2+m(2m+1)\frac{u^2}{r^2}$$
$$\le 2(\operatorname{div}\mathbf{u})^2+m(2m+1)\frac{u^2}{a^2}\le C(G^2+p^2+|\mathbf{u}|^2).$$

Hence it follows from the estimates (2.7), (2.14) and (2.19) that

$$\int_0^T|\nabla\mathbf{u}|_{L^\infty}^2\,dt\le C\int_0^T\left(|G|_{L^\infty}^2+|p|_{L^\infty}^2+|\mathbf{u}|_{L^\infty}^2\right)\,dt\le C. \tag{2.20}$$

To obtain the estimate for $\nabla\rho$, we differentiate the continuity equation

$$\rho_t+\mathbf{u}\cdot\nabla\rho+\rho\operatorname{div}\mathbf{u}=0 \tag{2.21}$$

with respect to x_j and obtain

$$(\rho_{x_j})_t+\mathbf{u}_{x_j}\cdot\nabla\rho+\mathbf{u}\cdot\nabla\rho_{x_j}+\rho_{x_j}\operatorname{div}\mathbf{u}+\rho\operatorname{div}\mathbf{u}_{x_j}=0.$$

Then multiplying this equation by ρ_{x_j}, integrating over Ω and summing over j, we deduce that

$$\frac{d}{dt}\int|\nabla\rho|^2\,dx\le C\int|\nabla\mathbf{u}||\nabla\rho|^2+\rho|\nabla\rho||\nabla\operatorname{div}\mathbf{u}|\,dx$$
$$\le C\int|\nabla G|^2\,dx+C(|\nabla\mathbf{u}|_{L^\infty}+1)\int|\nabla\rho|^2\,dx$$

Thanks to the estimates (2.14) and (2.20), we thus obtain

$$\sup_{0\le t\le T} |\nabla\rho|_{L^2} \le C.$$

Finally, in view of the well-known elliptic regularity estimate and the identity (2.5), we obtain

$$\begin{aligned}\int_0^T |\nabla^2\mathbf{u}|_{L^2}^2\,dt &\le C_0\int_0^T \left(|\Delta\mathbf{u}|_{L^2}^2 + |\nabla\mathbf{u}|_{L^2}^2\right)\,dt \le C\int_0^T \left(|\nabla\mathrm{div}\,\mathbf{u}|_{L^2}^2 + 1\right)\,dt \\ &\le C\int_0^T \left(|\nabla G|_{L^2}^2 + |\nabla p|_{L^2}^2 + 1\right)\,dt \le C.\end{aligned}$$

This completes the proof of Lemma 2.6. □

Now we prove the key estimate.

Lemma 2.7.

$$\sup_{0\le t\le T}\left(|\sqrt{\rho}\mathbf{u}_t(t)|_{L^2} + |\mathbf{u}(t)|_{H^2}\right) + \int_0^T \left(|\mathbf{u}_t(t)|_{H_0^1}^2 + |G(t)|_{H^2}^2\right)dt \le C_0 \tag{2.22}$$

for some C_0 depending only on $\mathcal{C}(\rho_0,\mathbf{u}_0)$ as well as the parameters of C. Here the functional $\mathcal{C}$ is defined

$$\mathcal{C}(\rho_0,\mathbf{u}_0) = \int \rho_0^{-1}|\mu\Delta\mathbf{u}_0 + (\lambda+\mu)\nabla\mathrm{div}\,\mathbf{u}_0 - \nabla(A\rho_0^\gamma)|^2\,dx. \tag{2.23}$$

Proof. To begin with, rewrite the momentum equation (1.1) as

$$\rho\mathbf{u}_t + \rho\mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \rho\mathbf{f}. \tag{2.24}$$

If we differentiate this with respect to time, then

$$\rho\mathbf{u}_{tt} + \rho\mathbf{u}\cdot\nabla\mathbf{u}_t - \nu\Delta\mathbf{u}_t + \nabla p_t = (\rho\mathbf{f})_t - \rho_t(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) - \rho\,\mathbf{u}_t\cdot\nabla\mathbf{u}$$

and thus by virtue of the continuity equation, we obtain

$$\begin{aligned}&\frac{1}{2}\left(\rho|\,\mathbf{u}_t|^2\right)_t + \frac{1}{2}\mathrm{div}\left(\rho\,\mathbf{u}|\,\mathbf{u}_t|^2\right) - \nu\Delta\mathbf{u}_t\cdot\mathbf{u}_t + \nabla p_t\cdot\mathbf{u}_t \\ &\quad = \mathrm{div}\,(\rho\mathbf{u})(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{f})\cdot\mathbf{u}_t - \rho(\mathbf{u}_t\cdot\nabla\mathbf{u})\cdot\mathbf{u}_t + \rho\,\mathbf{f}_t\cdot\mathbf{u}_t.\end{aligned}$$

Hence integrating over Ω, we obtain

$$\begin{aligned}&\frac{d}{dt}\int\frac{1}{2}\rho|\,\mathbf{u}_t|^2\,dx + \nu\int|\nabla\mathbf{u}_t|^2\,dx - \int p_t\,\mathrm{div}\,\mathbf{u}_t\,dx \\ &= \int \rho\mathbf{u}\cdot\nabla((\mathbf{f} - \mathbf{u}_t - \mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u}_t) - \rho(\mathbf{u}_t\cdot\nabla\mathbf{u})\cdot\mathbf{u}_t + \rho\,\mathbf{f}_t\cdot\mathbf{u}_t\ \,dx.\end{aligned} \tag{2.25}$$

This identity can be proved rigorously by means of a standard regularization technique. For a simple proof, see Y. Cho, H.J. Choe and H. Kim [1]. Using the continuity equation again, we have

$$
\begin{aligned}
&-\int p_t \operatorname{div} \mathbf{u}_t \, dx \\
&= \int (\nabla p \cdot \mathbf{u} + \gamma p \operatorname{div} \mathbf{u}) \operatorname{div} \mathbf{u}_t \, dx \\
&= \int \nabla p \cdot (\mathbf{u} \operatorname{div} \mathbf{u}_t) \, dx + \frac{d}{dt} \int \frac{\gamma}{2} p \, (\operatorname{div} \mathbf{u})^2 \, dx - \frac{\gamma}{2} \int p_t \, (\operatorname{div} \mathbf{u})^2 \, dx \\
&= \frac{d}{dt} \int \frac{\gamma}{2} p \, (\operatorname{div} \mathbf{u})^2 \, dx + \int \nabla p \cdot (\mathbf{u} \operatorname{div} \mathbf{u}_t) \, dx \\
&\qquad + \frac{\gamma}{2} \int -p \, \mathbf{u} \cdot \nabla (\operatorname{div} \mathbf{u})^2 + (\gamma - 1) p \, (\operatorname{div} \mathbf{u})^3 \, dx.
\end{aligned}
$$

Substituting this identity into (2.25), we deduce that

$$
\begin{aligned}
&\frac{d}{dt} \int \frac{1}{2} \rho |\mathbf{u}_t|^2 + \frac{\gamma}{2} p \, (\operatorname{div} \mathbf{u})^2 \, dx + \nu \int |\nabla \mathbf{u}_t|^2 \, dx \\
&\le \int 2\rho |\mathbf{u}||\mathbf{u}_t||\nabla \mathbf{u}_t| + \rho |\mathbf{u}||\mathbf{u}_t||\nabla \mathbf{u}|^2 + \rho |\mathbf{u}|^2 |\mathbf{u}_t||\nabla^2 \mathbf{u}| + \rho |\mathbf{u}|^2 |\nabla \mathbf{u}||\nabla \mathbf{u}_t| \\
&\quad + \rho |\mathbf{u}_t|^2 |\nabla \mathbf{u}| + |\nabla p||\mathbf{u}||\nabla \mathbf{u}_t| + \gamma \, p |\mathbf{u}||\nabla \mathbf{u}||\nabla^2 \mathbf{u}| + \gamma^2 \, p |\nabla \mathbf{u}|^3 \\
&\quad + \rho |\mathbf{u}||\mathbf{u}_t||\nabla \mathbf{f}| + \rho |\mathbf{u}||\mathbf{f}||\nabla \mathbf{u}_t| + \rho |\mathbf{u}_t||\mathbf{f}_t| \; dx.
\end{aligned}
$$

Using the previous lemmas and Young's inequality, we can easily show that

$$
\begin{aligned}
&\frac{d}{dt} \int \rho |\,\mathbf{u}_t|^2 + \gamma p \, (\operatorname{div} \mathbf{u})^2 \, dx + \int |\nabla \mathbf{u}_t|^2 \, dx \\
&\le C(1 + |\nabla \mathbf{u}|_{L^\infty}^2 + |\nabla^2 \mathbf{u}|_{L^2}^2 + |\,\mathbf{f}_t|_{L^2}^2 + |\nabla \mathbf{f}|_{L^2}^2) + C |\nabla \mathbf{u}|_{L^\infty} \int \frac{1}{2} \rho |\mathbf{u}_t|^2 \, dx.
\end{aligned}
$$

Then integrating over $(\tau, t) \subset\subset (0, T)$ and using the lemmas again, we obtain

$$
\begin{aligned}
&|\sqrt{\rho} \mathbf{u}_t(t)|_{L^2}^2 + \int_\tau^t |\nabla \mathbf{u}_t|_{L^2}^2 \, ds \\
&\qquad \le C + |\sqrt{\rho} \mathbf{u}_t(\tau)|_{L^2}^2 + C \int_0^t |\nabla \mathbf{u}|_{L^\infty} |\sqrt{\rho} \mathbf{u}_t|_{L^2}^2 \, ds.
\end{aligned}
\tag{2.26}
$$

On the other hand, we can deduce from the momentum equation (2.24) that

$$
\begin{aligned}
|\sqrt{\rho} \mathbf{u}_t(\tau)|_{L^2}^2 &\le |\sqrt{\rho} (\mathbf{u} \cdot \nabla \mathbf{u} - \mathbf{f})(\tau)|_{L^2}^2 + |(\sqrt{\rho})^{-1} (\nu \Delta \mathbf{u} - \nabla p)(\tau)|_{L^2}^2 \\
&\to |\sqrt{\rho_0} (\mathbf{u}_0 \cdot \nabla \mathbf{u}_0 - \mathbf{f}(0))|_{L^2}^2 + \mathcal{C}(\rho_0, \mathbf{u}_0) \le C \quad \text{as} \quad \tau \to 0,
\end{aligned}
$$

where $\mathcal{C}(\rho_0, \mathbf{u}_0)$ was defined in (2.23). Therefore, letting $\tau \to +0$ in (2.26), we conclude that

$$
|\sqrt{\rho} \mathbf{u}_t(t)|_{L^2}^2 + \int_0^t |\nabla \mathbf{u}_t|_{L^2}^2 \, ds \le C_0 + C_0 \int_0^t |\nabla \mathbf{u}|_{L^\infty} |\sqrt{\rho} \mathbf{u}_t|_{L^2}^2 \, ds.
$$

Now since $\int_0^T |\nabla \mathbf{u}|_{L^\infty}^2 \, dt \le C$, we can apply Gronwall's lemma to obtain

$$
\sup_{0 \le t \le T} |\sqrt{\rho} \mathbf{u}_t(t)|_{L^2}^2 + \int_0^T |\mathbf{u}_t(t)|_{H_0^1}^2 \, ds \le C_0.
$$

The remaining estimates for $|\mathbf{u}|_{H^2}$ and $|\nabla^2 G|_{L^2}$ can be easily derived from this estimate and the previous lemmas by using elliptic regularity estimates on the momentum equation. This completes the proof of the lemma. □

Lemma 2.8.

$$\sup_{0\le t\le T}(|\rho(t)|_{H^2}+|\rho_t(t)|_{H^1})+\int_0^T|\mathbf{u}(t)|^2_{H^3}\,dt\le C(\varepsilon) \tag{2.27}$$

for some $C(\varepsilon)$ depending only on ε and the parameters of C_0.

Proof. If we take the differential operator ∇^2 to the continuity equation (2.21), multiply by $\nabla^2\rho$ and then integrate over Ω, we get

$$\frac{d}{dt}\int|\nabla^2\rho|^2\,dx\le C_0\int|\nabla\mathbf{u}||\nabla^2\rho|^2+|\nabla^2\mathbf{u}||\nabla\rho||\nabla^2\rho|+\rho|\nabla^2\mathrm{div}\,\mathbf{u}||\nabla^2\rho|\,dx.$$

Using the previous lemmas and Sobolev inequality (2.13), we have

$$\begin{aligned}\frac{d}{dt}|\nabla^2\rho|^2_{L^2}&\le C\left[\,|\nabla\mathbf{u}|_{H^1}|\nabla\rho|^2_{H^1}+\left(|\nabla^2G|_{L^2}+|\nabla^2p|_{L^2}\right)|\nabla^2\rho|_{L^2}\,\right]\\&\le C\left(|\rho^{\gamma-2}|_{L^\infty}+1\right)|\nabla\rho|^2_{H^1}+C|G|^2_{H^2}\end{aligned}$$

and thus

$$|\rho(t)|^2_{H^2}\le C\left(1+|\nabla^2\rho_0|^2_{L^2}\right)+C\int_0^t\left(|\rho^{\gamma-2}|_{L^\infty}+1\right)|\rho|^2_{H^2}\,ds.$$

Note that the continuity equation (2.21) yields

$$\inf\rho(t)\ge(\inf\rho_0)\exp\left(-\int_0^t|\mathrm{div}\,\mathbf{u}|_{L^\infty}\,ds\right)\ge\varepsilon e^{-Ct}.$$

Then we can easily show that $|\rho^{\gamma-2}|_{L^\infty}\le C(\varepsilon)$. Therefore, in view of Gronwall's inequality, we get the desired estimate for ρ. The estimate for ρ_t follows from this estimate by using the continuity equation. Finally, using an elliptic regularity estimate, we can obtain the estimate for $\mathbf{u}$. This completes the proof of the lemma. □

Combining Theorem 2.1 and all the lemmas in this section, we conclude that the solutions of Theorem 2.1 exist globally in time.

Theorem 2.9. *If the data (ρ_0,u_0,f) satisfy the hypotheses of Theorem 2.1, then there exists a unique global strong solution (ρ,u) to the initial boundary value problem* (2.1)–(2.3), *which satisfies* (2.4) *for each $T>0$.*

3. Proof of Theorem 1.1

We first prove the necessity of the compatibility condition (1.7), an easy part of the theorem. Let (ρ,u) be a strong solution to the problem (1.1)–(1.3) satisfying (1.6) and (1.8). Since $\sqrt{\rho}\mathbf{u}_t\in L^\infty_{loc}(0,\infty;L^2)$, we can find a sequence $\{t_k\}$, $t_k\to0$, such that $\{\sqrt{\rho}\mathbf{u}_t(t_k)\}$ converges weakly in L^2. Therefore, letting $t_k\to0$ in the momentum equation (1.1), we obtain

$$-\mu\Delta\mathbf{u}(0)-(\lambda+\mu)\nabla\mathrm{div}\,\mathbf{u}(0)+\nabla(A\rho(0)^\gamma)=\rho(0)^{\frac12}\tilde{\mathbf{g}} \tag{3.1}$$

for some $\tilde{\mathbf{g}}\in L^2$. Since $\rho(0)=\rho_0$ and $\mathbf{u}(0)=\mathbf{u}_0$, this proves the necessity of the condition (1.7).

To prove the converse, let $(\rho_0,\mathbf{u}_0,\mathbf{f})$ be a given data satisfying the conditions (1.5) and (1.7). To begin with, we construct a sequence $\rho_0^\varepsilon\in H^2(a,b)$ of smooth radial functions such that

$$0<\varepsilon\le\rho_0^\varepsilon,\quad\rho_0^\varepsilon\to\rho_0\ \text{in}\ H^1(a,b)\quad\text{and}\quad|\rho_0^\varepsilon|_{H^1(\Omega)}\le C,$$

where $\rho_0^\varepsilon(\mathbf{x}) = \rho_0^\varepsilon(|\mathbf{x}|)$ for $\mathbf{x} \in \Omega$, and let $u_0^\varepsilon \in H_0^1(a,b) \cap H^2(a,b)$ be the solution to the boundary value problem

$$-\nu\left((u_0^\varepsilon)_r + m\frac{u_0^\varepsilon}{r}\right)_r + (A\rho_0^\varepsilon)_r = (\rho_0^\varepsilon)^{\frac{1}{2}}g, \quad a < r < b.$$

Then, let $(\rho^\varepsilon, u^\varepsilon)$ be the strong solution in $(0,\infty) \times (a,b)$ to the radial problem (2.1)–(2.3) with the initial data $(\rho_0^\varepsilon, u_0^\varepsilon)$. As shown in the last section, if we define

$$\rho^\varepsilon(t,\mathbf{x}) = \rho^\varepsilon(t,|\mathbf{x}|) \quad \text{and} \quad \mathbf{u}^\varepsilon(t,\mathbf{x}) = u^\varepsilon(t,|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|},$$

then $(\rho^\varepsilon, \mathbf{u}^\varepsilon)$ is a global radially symmetric strong solution to the problem (1.1)–(1.3) with the initial data $(\rho_0^\varepsilon, \mathbf{u}_0^\varepsilon)$, where $\mathbf{u}_0^\varepsilon(\mathbf{x}) = u_0^\varepsilon(|\mathbf{x}|)\ (\mathbf{x}/|\mathbf{x}|)$.

Note that the regularized initial data $(\rho_0^\varepsilon, \mathbf{u}_0^\varepsilon)$ satisfy the same compatibility condition as (1.7) of $(\rho_0, \mathbf{u}_0)$:

$$-\mu\Delta\mathbf{u}_0^\varepsilon - (\lambda+\mu)\nabla\mathrm{div}\,\mathbf{u}_0^\varepsilon + \nabla(A(\rho_0^\varepsilon)^\gamma) = (\rho_0^\varepsilon)^{\frac{1}{2}}\,\mathbf{g}.$$

In particular, it follows from the elliptic regularity estimate that $\mathbf{u}_0^\varepsilon \to \mathbf{u}_0$ in H^2 as $\varepsilon \to 0$ since $\rho_0^\varepsilon \to \rho_0$ in $L^\infty \cap H^1$ as $\varepsilon \to 0$. Therefore, using Lemma 2.3 to Lemma 2.7, we conclude that $(\rho^\varepsilon, \mathbf{u}^\varepsilon)$ satisfies the following uniform estimate: for each $0 < T < \infty$,

$$\sup_{0\le t\le T}\left(|\rho^\varepsilon|_{H^1} + |\mathbf{u}^\varepsilon|_{H_0^1\cap H^2} + |\sqrt{\rho^\varepsilon}\mathbf{u}^\varepsilon|_{L^2}\right) + \int_0^T |\mathbf{u}_t^\varepsilon|_{H_0^1}^2\,dt \le C_0(T).$$

Now it can be easily shown that a subsequence of approximate solutions $(\rho^\varepsilon, \mathbf{u}^\varepsilon)$ converges, in a weak sense, to a radially symmetric strong solution $(\rho, \mathbf{u})$ satisfying the regularity (1.6) except the continuity.

We first prove the continuity of ρ. From the continuity equation (1.2), it follows that $\rho_t \in L^\infty_{loc}(0,\infty;L^2)$. Hence the well-known embedding result shows that $\rho \in C([0,\infty);L^2)$. Then we deduce that $\rho \in C([0,\infty);H^1-weak)$, that is, ρ is weakly continuous with values in H^1. For a proof, we refer to Chapter 3 in R. Teman [12]. It thus remains to show that $\nabla\rho \in C([0,\infty);L^2)$. Note that the linear transport equation (1.2) is invariant under the translation and reflection. Hence it suffice to show that

$$\lim_{t\to+0}|\nabla\rho(t) - \nabla\rho(0)|_{L^2} = 0. \tag{3.2}$$

To show this, we differentiate (1.2) with respect to x_j, multiply by ρ_{x_j} and integrate over Ω. Then summing over j, we obtain

$$\frac{d}{dt}\int|\nabla\rho|^2\,dx \le C\int|\nabla\mathbf{u}||\nabla\rho|^2 + \rho|\nabla\rho||\nabla\mathrm{div}\,\mathbf{u}|\,dx.$$

In view of Sobolev inequality (2.13) and the regularity of ρ, we deduce

$$\frac{d}{dt}\int|\nabla\rho|^2\,dx \le C|\nabla\mathbf{u}|_{H^1}|\rho|_{H^1}^2 \le C|\nabla\mathbf{u}|_{H^1}$$

and thus

$$|\nabla\rho(t)|_{L^2}^2 \le |\nabla\rho(0)|_{L^2} + C\int_0^t|\nabla\mathbf{u}(s)|_{H^1}\,ds. \tag{3.3}$$

This inequality can be proved rigorously by using a standard regularization technique. Now letting $t \to +0$ in the inequality (3.3), we deduce that

$$\limsup_{t\to+0}|\nabla\rho(t)|_{L^2}^2 \le |\nabla\rho(0)|_{L^2}. \tag{3.4}$$

The strong convergence (3.2) follows from (3.4) and the weak continuity of ρ in H^1. This completes the proof of the continuity of ρ.

Next we prove the continuity of $\mathbf{u}$. The weak-type continuity of $\mathbf{u}$ follows from the standard embedding results: $\mathbf{u} \in C([0,\infty); H_0^1) \cap C([0,\infty); H^2 - weak)$. Hence it remains to prove the strong continuity of $\mathbf{u}$ in H^2. We first prove the continuity of $\rho\mathbf{u}_t$ in L^2. From the momentum equation (2.24), we easily deduce that $(\rho\mathbf{u}_t)_t \in L^2_{loc}(0,\infty; H^{-1})$, where H^{-1} is the dual space of H_0^1. Then since $\rho\mathbf{u}_t \in L^2_{loc}(0,\infty; H_0^1)$, it follows from a standard embedding result that $\rho\mathbf{u}_t \in C([0,\infty); L^2)$. Therefore, we can conclude that for each $t \in [0,\infty)$, $\mathbf{u} = \mathbf{u}(t) \in H_0^1 \cap H^2$ is a solution of the elliptic system

$$\nu\Delta\mathbf{u} = \rho\mathbf{u}_t + \rho\mathbf{u}\cdot\nabla\mathbf{u} + \nabla(A\rho^\gamma) - \rho\mathbf{f}.$$

Now it is not difficult to show that $\mathbf{u} \in C([0,\infty); H^2)$. Recall from the elliptic estimate that for $s, t \geq 0$,

$$\begin{aligned}
&|\mathbf{u}(t) - \mathbf{u}(s)|_{H^2} \\
&\leq C|\rho\mathbf{u}\cdot\nabla\mathbf{u}(t) - \rho\mathbf{u}\cdot\nabla\mathbf{u}(s)|_{L^2} + C|\nabla(A\rho^\gamma)(t) - \nabla(A\rho^\gamma)(s)|_{L^2} \\
&\quad + C\left(|\rho\mathbf{u}_t(t) - \rho\mathbf{u}_t(s)|_{L^2} + |\rho\mathbf{f}(t) - \rho\mathbf{f}(s)|_{L^2} + |\mathbf{u}(t) - \mathbf{u}(s)|_{H_0^1}\right).
\end{aligned} \tag{3.5}$$

Using Sobolev inequality together with the regularity of $(\rho, \mathbf{u})$, we obtain

$$\begin{aligned}
&|\rho\mathbf{u}\cdot\nabla\mathbf{u}(t) - \rho\mathbf{u}\cdot\nabla\mathbf{u}(s)|_{L^2} \\
&\leq C|(\rho(t) - \rho(s))\mathbf{u}(t)\cdot\nabla\mathbf{u}(t)|_{L^2} + C|\rho(s)(\mathbf{u}(t) - \mathbf{u}(s))\cdot\nabla\mathbf{u}(t)|_{L^2} \\
&\quad + C|\rho(s)\mathbf{u}(s)\cdot(\nabla\mathbf{u}(t) - \nabla\mathbf{u}(s))|_{L^2} \\
&\leq C|\rho(t) - \rho(s)|_{L^\infty}|\nabla\mathbf{u}(t)|_{L^2}^2 + C|\rho(s)|_{L^\infty}|\nabla(\mathbf{u}(t) - \mathbf{u}(s))|_{L^2}|\nabla\mathbf{u}(t)|_{L^2} \\
&\quad + C|\rho(s)|_{L^\infty}|\nabla\mathbf{u}(s)|_{L^2}|\nabla(\mathbf{u}(t) - \mathbf{u}(s))|_{L^2} \\
&\leq C(|\rho(t) - \rho(s)|_{L^\infty} + |\mathbf{u}(t) - \mathbf{u}(s)|_{H_0^1})
\end{aligned}$$

and

$$\begin{aligned}
&|\nabla(\rho^\gamma)(t) - \nabla(\rho^\gamma)(s)|_{L^2} \\
&\leq C|\left(\rho^{\gamma-1}(t) - \rho^{\gamma-1}(s)\right)\nabla\rho(t)|_{L^2} + C|\rho^{\gamma-1}\left(\nabla\rho(t) - \nabla\rho(s)\right)|_{L^2} \\
&\leq C\left(|\rho^{\gamma-1}(t) - \rho^{\gamma-1}(s)|_{L^\infty} + |\nabla\rho(t) - \nabla\rho(s)|_{L^2}\right).
\end{aligned}$$

Substituting these results into (3.5), we conclude that $|\mathbf{u}(t) - \mathbf{u}(s)|_{D^2} \leq \Theta(t,s)$ for some function $\Theta(t,s)$ such that $\lim_{t\to s}\Theta(t,s) = 0$.

We have proved the existence of a radially symmetric strong solution $(\rho, \mathbf{u})$ satisfying the regularity (1.6). Hence to complete the proof of the sufficiency, it remains to prove the convergence property (1.8) of $(\rho, \mathbf{u})$ as $t \to 0$. Now we show that

$$\rho(0) = \rho_0 \quad \text{and} \quad \mathbf{u}(0) = \mathbf{u}_0 \quad \text{in} \quad \Omega, \tag{3.6}$$

which is equivalent to (1.8) because of the continuity of $(\rho, \mathbf{u})$. The first identity in (3.6) follows easily from the weak formulation of the continuity equation (1.2). But from the momentum equation (1.1), we deduce only that $(\rho\mathbf{u})(0) = \rho_0\mathbf{u}_0$ in Ω. Hence we have to show that $\mathbf{u}(0) = \mathbf{u}_0$ in the set $\Omega_0 = \{\mathbf{x} \in \Omega : \rho_0(\mathbf{x}) = 0\}$. Define $\mathbf{w} = \mathbf{u}(0) - \mathbf{u}_0$. Then since $(\rho(0), \mathbf{u}(0))$ also satisfies the condition (3.1) for some $\tilde{\mathbf{g}} \in L^2$, we find that the radial part w of $\mathbf{w}$ satisfies

$$-\nu\left(w_r + m\frac{w}{r}\right)_r = 0 \quad \text{in} \quad V, \tag{3.7}$$

where $V = int\{r \in (a,b) : \rho_0(r) = 0\}$. It is clear that $w \in H_0^1(V) \cap H^2(V)$. Moreover, since V is a countable union of open intervals, we easily prove that $w = 0$ in V, that is, $\mathbf{u}(0) = \mathbf{u}_0$ in the set Ω_0. Therefore, the proof of Theorem 1.1 has been completed.

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